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Modeling Complexity in Mathematics Education

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25 Modeling Complexity in Mathematics Education

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What is Needed?

There is no debate; the fundamental issues confronting mathematics education are so complex that significant help is needed when trying to model them. The underlying complexity of this area probably is what motivated some experts both at the workshop on The Modeling of the Modeling of Mathematics Education, held in Santa Fe, New Mexico in December 2001, and elsewhere to wonder whether help, advice, or at least insight might be found in the formal studies of complexity such as chaos, dynamical systems, fractals, and other forms of mathematics. As discussed here, maybe.

The nature of the general questions about mathematics education means that finding answers probably will require using some form of system thinking. The challenge is to find a way to capture the associated complexities. This is a necessary step whether a preferred research approach involves the theoretical, an assessment, or a statistical analysis. For instance, as we have learned from economics and other social sciences, data without theory is just data. Namely, a data analysis in the absence of a testable, carefully designed, and considered theory has limited value; rather than being informative, even replicated outcomes can be misleading and counterproductive. But although modeling and theory are crucial, it is not clear how to model the complexities central to education.

With this reality in mind, it is reasonable to explore whether modeling and theoretical insight for mathematics education can be gained by examining the source of the different forms of mathematical complexity that arise within systems. This is reasonable, so I describe new ways to address this approach. But let me caution against expecting help to come from the technical aspects of these mathematical advances. It will not happen. This is because modeling in mathematics education—actually, for most issues in most social sciences—is at such an early stage that it is premature to use these technically precise tools. So, while outlining how the conceptual aspects of these mathematical developments might offer help in structuring thought and analysis, I indicate briefly why those commonly heard terms such as sensitivity with respect to initial conditions, self-symmetry, and fractals are of minimal to zero value for modeling in this area—at least for now.

To be specific, motivated by the concerns that were raised and discussed actively during the above-mentioned workshop in Santa Fe in December 2001, I will discuss four themes:

1 Complexity. Conceptual help in handling the “system complexity” inherent in this area of mathematics education can come from chaos. This is not obvious, so to show that it is possible to transfer notions from dynamics to non-dynamic settings,
I use voting paradoxes to simplify the exposition and illustrate the ideas. I also raise doubts about currently used assessment procedures.

2 **Parts–whole.** A source of the complexity in mathematics education is the constant confrontation with “parts–whole” interactions; for example, a mathematics curriculum is divided into units—the parts—designed so that, it is hoped, all the students understand the material—the whole. Conversely, a large research project—the whole—may be divided into units—the parts—that are addressed by different groups. Central to modeling, then, is to understand “parts–whole” connections. The discussion here indicates, in a general way, what to emphasize; what to examine; and how to identify opportunities, pitfalls, and, maybe, new directions for mathematics education. One message is that beyond “the whole can be greater than the sum of the parts” adage, it can be that the “whole” is qualitatively and radically different from each part and the sum.

3 **Social norms.** A perplexing problem is to understand how to change the system—the “social norms.” Education is full of discouraging stories where, after investing considerable effort to introduce innovation and reform, the prevailing social norms defeat the lofty objectives and beat us back into a “much-the-same” environment. How can we model and address these pressing concerns? How can progress be made? I have no answers, but I can indicate ways to address these concerns. Some conclusions are discouraging, while others may provide insight into what needs to be done.

4 **Social movement.** Closely related to social norms is the pragmatic issue of whether educational reform can be promoted through a social movement. To make this a personal issue for the reader, restate it in terms of the modeling of mathematics education or another personal professional objective. How can you motivate researchers, teachers, and the community to sign on to your goals? Although I cannot answer this monumental concern, I can identify research that indicates what might be needed. This literature does provide some lessons.

**Fractals**

Mandelbrot sets and related pictures of fractals are powerful technical tools for understanding subtle aspects of dynamics; they also form intriguing art forms that can be displayed proudly. But other than serving as valued lessons in a mathematics classroom, it is doubtful whether they tell us anything about mathematics education. On the other hand, this topic does add excitement to the classroom.

To explain, let me describe a fractal in terms of the Chaos game (see Barnsley, 1988; Devaney, 1992). All you need is a die and an equilateral triangle, such as the one on the left in Figure 25.1, where the vertices are labeled A, B, C. To play the game, select an

![Figure 25.1 Sierpinski Triangle.](image-url)
initial point in the interior of the triangle; it does not matter where. Rolling the die determines how to move from a current position to the next point according to the following rules; the game continues forever:

- If a 1 or 2 comes up, move halfway from the current point toward vertex A.
- If a 3 or 4 comes up, move halfway toward vertex B.
- If a 5 or 6 comes up, move halfway toward vertex C.

Playing this game displays a fascinating phenomenon, the iterates rapidly start tracing out the Sierpinski triangle, whose construction is depicted in Figure 25.1. To understand this figure, divide the original equilateral triangle into the four smaller equilateral triangles as shown on the left side of Figure 25.1. Throw out the interior of the center triangle—the large shaded region in the first large triangle. Next, divide each of the remaining three unshaded smaller triangles in an identical manner. So, as shown in the second large triangle of Figure 25.1, each of the three remaining smaller triangles is divided into four even smaller equilateral triangles where the center (shaded) portion is dropped. Continue this process—forever. The boundary of what remains is the attractor of this dynamic; namely, the location of the points from the game move ever closer to the figure. Self-symmetry arises from the construction of dividing continually whatever equilateral triangle remains; it reflects a symmetry of this particular dynamic (going halfway toward one vertex or another).

How should this weird object, which clearly is more than one-dimensional, be classified? The final figure is too holey to be two-dimensional. A two-dimensional region, for instance, should contain the complete interior of at least one open ball, even if it is very small. But if this holey triangle contained such a ball, eventually the interior of the ball would be invaded by a triangle where the middle portion will be removed, making the ball incomplete. So, if the figure is not one- or two-dimensional, what is it? To describe these kinds of objects, mathematicians invented fractional dimensions: the so-called fractals. For instance, the fractal dimension of the Sierpinski triangle is ln(3)/ln(2) ≈ 1.58 . . . (e.g., Devaney, 1992). Fractals play such an important role in mathematical considerations that I encourage using them in the classroom (see Devaney, 1999). But, they play no natural role in understanding issues of mathematics education, so we can dismiss them safely from here on.

Chaotic Behavior

Whenever I ask a general audience for their sense of “chaos,” the typical response is “sensitivity with respect to initial conditions.” This familiar phrase is merely a technical description of a typical consequence of chaotic dynamics; it has nothing to do with the driving force causing complex behavior. On the other hand, this phrase, and the underlying source of the complexity of chaotic dynamics, might offer structure to understand the system complexity that arises in mathematics education.

Start by recalling some personally chaotic event. Typical responses I have received include “Life with my children”; “Being a PhD student, thanks to my advisor’s demands”; and even “Family gatherings after Uncle Fred has had too much to drink.” Surprisingly, these examples capture more accurately a sense of the force behind mathematical chaos than the sensitivity phrase.

To indicate what makes each example chaotic consider what can happen with children getting up in the morning. A typical story starts with several possible initial events such as, “When Torik and Heili get out of bed, they. . . .” Then, as anyone with children
understands readily, each possible event spawns several new possibilities ("Torik didn’t want to watch the same video as Heili, so he . . ."), and each of these generates another set of new events, and each of these. . . . The story goes on and on forever, creating a chaos tree of the form depicted in Figure 25.2. The complexity of the situation is manifested by the myriad possible paths with different consequences where it is not clear in advance whether the less traveled one, or something else, will be taken.

The different paths defined by the never-ending tree describe differing consequences of consequences of consequences of . . . With slight reflection, each of us can identify with this structure: just think of our own life experiences where if we had not been somewhere or done something at a particular time, our life most surely would have assumed a very different path and structure ("If I hadn’t met her at that banquet, I most surely would have married someone else, and . . ." or "Standing in an airport line, I happened to overhear Professor So-and-So describing the importance of . . ."). These are the “sensitivity-to-initial-conditions” events; a slight change could cause the dynamic to travel down a completely different chain of events. Examples also come from the classroom, ranging from discipline to insightful learning.

Important to the concerns of the chapters in this book about mathematics education—and, more generally, to the social sciences—is to understand how to analyze such complexity. A natural inclination is to modestly select a particular path and then follow its listing of cascading causes and effects. Be honest; this is precisely what we do in our research when we carve out a particular task. But, as we painfully learn, the analysis can be very difficult and involve a multitude of careful considerations. Such research most surely requires developing the necessary discipline to avoid becoming sidetracked by related issues or by the temptation to examine another path. But, with respect to modeling, emphasizing the particular can mask the appropriate general assumptions.

The surprising counterintuitive claim is that, at times, it may be simpler and more productive to tackle the far more ambitious objective of identifying everything that could possibly occur. Rather than following a single path, the more global perspective is to determine all the possible paths that can occur, along with their cascading consequences, and how the paths relate and interact with each other. This captures the underlying spirit of a technical approach (symbolic dynamics) used to describe chaotic behavior. There is an associated cost, but it may be minimal with respect to the potential gains.

This change in perspective mandates a change in analysis—a change that, by being forced to identify and understand the relationship and interaction among paths, we are required to emphasize general operative assumptions. This change in direction may assist in the modeling of mathematics education. For readers who recall the freshman calculus lesson about using Newton’s method to find a zero of a polynomial, see Saari (1995) for a description of this different perspective. In terms of playing a game of pool, an intuitive discussion starts on page 84 of Saari (2001a). Anyone interested in the mathematics of astronomy might be intrigued by the wild chaotic motion reported in Saari and Xia (1995). But to advance my intent of suggesting new ways to address the system complexities in mathematics education, this change in perspective is introduced.
with my favorite voting example. Voting is used because the details have been worked out to show how to transfer ideas from dynamics to non-dynamical settings. My hope is that this approach will provide a template to encourage readers to develop similar arguments for the modeling of mathematics education.

Suppose 15 people get together for a drink after a hard day at a conference. To save on cost, they agree to buy one beverage in bulk; to select the beverage of choice, they vote. Six prefer milk to wine to beer, five prefer beer to wine to milk, and four prefer wine to beer to milk. Using our usual voting method (“Let’s have a show of hands”), this group prefers milk (six votes) to beer (five votes) to wine (only four votes).

Do these people really prefer milk? By comparing these beverages pairwise, we discover that these voters prefer anything to milk. A landslide proportion of 60 percent prefer wine to milk by a 9 to 6 vote, and 60 percent prefer beer to milk by a 9 to 6 vote. Even more surprising, this same group prefers last place wine to anything; 60 percent prefer wine to milk by 9 to 6; and 66.7 percent of them prefer wine to beer by 10 to 5. So, it is arguable that wine, not milk, is their top choice. Anyone want beer? Beer is the top choice with a runoff, where wine is dropped at the first stage, and beer beats milk in the runoff.

What makes this example disturbing is that it violates long-held beliefs about decision procedures. More troubling, the example demonstrates that the election outcome can reflect more accurately the choice of a procedure, rather than what the information really means. This comment should raise red flags: If voting, which is a particularly simple aggregation procedure, can suffer serious problems, then what should we expect from the more complicated methods typically used in mathematics education? So, before moving on, it is worth reviewing what message this statement implies for, say, assessment procedures and even statistical methods. After all, we are far more comfortable and familiar with these centuries-old voting methods than with assessment and statistical methods. Yet, in spite of our acceptance of voting methods, rather than reflecting the data, we now know that an outcome can reflect which voting method has been adopted. Consequently, some “trusted” outcomes must be misleading and distorted. Similarly, we must worry whether the more complex assessment methods can distort the message in the data. As we are discovering, these fears are well founded.

The next natural question is to go beyond the specific example to determine what else can happen. Quite frankly, progress in this direction had been agonizingly slow. The reason is that, for over two centuries, a main approach used in voting theory required finding particular paradoxical examples of the above type; in our terms, this is equivalent to trying to create an example to establish that part of a particular path on the tree can occur. Notice that this approach is similar to what often is done in mathematics education. What slowed progress in voting theory, and in mathematics education, is that it can be very difficult to discover examples to demonstrate a particular peculiarity. To illustrate the challenge, try, if you can, to create a four-candidate example where the plurality rankings of each of the four triplets reverse the four-candidate plurality ranking, but the six pairwise rankings flip again to agree with the original plurality ranking. (Many examples exist.) That is to say, this standard approach of examining carefully each particular type of election effect is so complex and difficult that it has severely limited progress for this research area. My suspicion is that when a similar approach is used in the modeling of mathematics education, it also will be accompanied by an arduous analysis with limited conclusions. To achieve progress, we must explore how to simplify the analysis while expanding the conclusions.

The goal of the approach I developed, by modifying the conceptual framework of chaos, was to analyze everything that could happen with any possible method. In other
words, rather than searching for specific paradoxical examples, my goal was to identify all possible paradoxical effects that could occur with any example and any standard voting method. A big surprise was that this significantly more ambitious goal turned out to be technically easier to achieve! Relevant for this chapter is my suspicion that there are several issues in the modeling of mathematics education that can be addressed in a related manner. Consequently, to encourage exploration, the approach is described in detail.

To illustrate how this was done by using Figure 25.2, my initial stage consisted of all 13 possible election rankings that could occur with the plurality vote; this defines 13 branches. Six involve strict rankings, six involve one tie, and the last has a complete tie. The second set of branches, the consequences of the first stage, involved possible rankings of a particular pair, say beer and wine, that could accompany what happens at the first stage. There are three possible rankings (beer is better than wine, wine is better than beer, and a tie); the goal is to understand which election rankings of all three beverages—the 13 branches of the first stage—could be accompanied by which election rankings of the pair. In this analysis, throw away all the complications. If it takes more voters than participants in my Friday night poker club to create an example where a complete milk, wine, beer tie is accompanied by a “wine is preferred to beer” branch, that is perfectly okay. The goal, then, is to determine everything that can happen; refinements are left for later. The third stage of the branches is to understand what rankings of another pair, say milk and wine, can accompany the earlier branches. The fourth stage is to find what paths allow different beer and milk comparisons.

Incidentally, the conclusion is that all \(13 \times 3 \times 3 \times 3 = 351\) paths are possible. This means that you can list any ranking of the three beverages, any ranking for each of the three pairs, and you are assured that an example of voter preferences can be constructed where the specified listing defines the voters’ election outcomes for the different subsets of candidates. Notice the trade-off: although the approach allows us to describe everything that could possibly happen, we may not know how to construct an illustrating example for each behavior. (Thanks to subsequent research, I can do this now.)

The value of this approach becomes clear by learning that my approach analyzes what happens with any number of candidates. To make my point, it turns out that, with only five candidates, the number of admissible paths in the tree is far greater than \(6 \times 10^{27}\)—a number that exceeds the number of seconds since the big bang. This astronomical value is mentioned to make it clear that it is impossible to analyze each and every path. Instead, to make progress, indirect approaches must be developed that identify when a barrier to the existence of a branch occurred and when branches can be extended. In doing so, and this is an implied message for mathematics education, the new approach forces us to identify variables and assumptions that are basic for a general analysis, rather than for creating a particular example.

The flavor of this approach (motivated by chaotic dynamics) is given by my pool-playing example from Saari (2001a): the analysis emphasizes what it takes to go from one stage to the next. Suppose we want to strike the cue ball to hit the one-ball to hit the six-ball to. . . . A highly experienced and exceptionally talented player may know exactly where to strike the cue ball; this is equivalent to constructing one of the voting examples. But, instead of trying to understand how to make the precise shot, change the emphasis to determine whether a shot is possible. The new emphasis changes the analysis; we might find first all the ways to hit the cue ball so that the cue ball will hit the one-ball—somewhere. Then, scrutinize the one-ball, six-ball alignment to find all the ways to hit the one-ball so that it would hit the six-ball. A refinement is necessary as we need to determine where to hit the cue ball so that it hits the one-ball—not somewhere, but in
a specific region so that the one-ball will hit the six-ball. (This refinement causes the sensitivity-with-respect-to-initial-conditions comment.) Of course, when analyzing possible shots, it may be impossible for the one-ball to hit the six-ball because it is surrounded by others. Similarly in the voting analysis, when using a method known as the Borda Count (this is where 2, 1, 0 points are assigned, respectively, to a voter’s first-, second-, and third-ranked candidates), obstructions make it impossible for the Borda Count ranking to be the opposite of the pairwise rankings.

Carrying out this pool-playing illustration in terms of Figure 25.2, suppose we wish to analyze the path down the extreme right-hand side. Instead of a traditional “forward” reasoning of showing how to go from the initial stage to the second stage, then to the third, adopt the iterated, “inverse-function” nature of the pool illustration. That is, instead of looking forward, look backwards: examine what it takes to get somewhere. That is, start with the branch in the second stage and determine all the possible ways to get there from the initial stage. Next, select the specified branch in the third tier; use this same “looking backwards” approach to understand all the possible ways to get there from the second stage. A refinement is needed: we need to characterize all the ways to get from the initial stage not only to the indicated branch in the second stage, but also to those positions that lead to the desired third-stage consequences. While this sounds complicated, I’ll show a bit later how simple this can be.

The value of this approach is that it imposes structure upon the total complexity of a system. Rather than emphasizing specifics, this approach requires understanding how different effects are related. Think of this approach as a parts–whole study of examining how all the parts can be connected to create various wholes.

Can this approach be used in the modeling of mathematics education? At a very minimum, the tree format imposes a disciplined yet suggestive structure that will force us to consider the totality of events and to place added emphasis on how the events are related. The approach helps to identify and structure feedback and other notions when branches farther down a tree are constructed. The approach also helps solve the crucial problem in the modeling of mathematics education by identifying what assumptions should be made.

It is my sense that, at a minimum, a serious attempt to diagram the cascade of cause-and-effect behaviors will identify previously unexpected behavior and consequences, and it will raise new questions. My optimism is based on the fact that this approach has provided these kinds of rewards in other areas where these techniques have been used. For instance, knowing that we can dismiss a particular path means that a certain effect cannot be combined with others; this can be important information. (In the voting illustration, the fact that it is impossible to have the path where the Borda Count ranking is the opposite of the pairwise rankings is an important feature of this voting procedure.) Another advantage of addressing the totality of a tree structure is that it forces researchers to pose new kinds of questions, develop different methodologies, and, by necessity, worry about potential connections. The rewards could be worth the effort.

The Parts and the Whole

Understanding parts–whole conflicts provides a tool to determine which branches of a tree diagram can or cannot occur. But, before addressing these issues, let me digress by mentioning that my friend Erik is an avid fan of Lakers basketball to the extent that he will give 2:1 odds that the Lakers will beat San Antonio next season. In other words, if the Lakers win, Erik keeps the money you bet with him; should San Antonio win, Erik will return your money and give you $2 for each dollar you bet with him. Anni, another
good friend who happens to dislike the Lakers team, is equally as impulsive because she will give 2:1 odds that San Antonio will beat the Lakers. You happen to have $100 designated for household bills. What will you do?

Many people in mathematics education are described best as being “professional basketball-challenged.” As such, the usual answer is: “I would pay my bills. Anyway, I don’t follow basketball, and I don’t gamble.” This reasonable reaction reflects the reality that, even for a knowledgeable expert, each gamble carries a financial risk.

But the combination of the parts—the two gambles—creates a whole with a radically different characteristic than either part or even the sum of the parts. To explain, if you bet $50 with Erik and $50 with Anni, you are guaranteed a risk-free, $50 profit. To see why, you will lose $50 to either Erik or Anni, depending on which team wins. But, you will receive $100 from the other person for a profit of $50. The lesson, then, is that the sum of the parts can create an object with distinctly different properties. When stated in terms of our concerns about mathematics education, this lesson becomes a warning to worry, for instance, about the educational consequences resulting from how the parts of a curriculum are assembled. It also identifies opportunities, such as where to improve the educational product.

The main point is that the usual tacit assumptions commonly made about the sum of the parts can be false. Just as mixing the parts of tin and copper could create a mess, or something valuable such as bronze, any hope to describe adequately which “whole” is constructed from the parts requires understanding how the parts are mixed and connected. Indeed, we must expect different ways to combine the parts to define different wholes. To describe what can happen, I borrow an example from my book Saari (2001b) that relates to education, then offer comments about how this parts–whole story speaks to issues of chaos and mathematics education.

In 1999, the state of California instituted an Academic Performance Index (API), “carrot-and-stick” program. The incentives, the “carrots,” involve considerable monetary rewards for teachers in those schools where the students in each of the different ethnic groups show substantial improvement. The “stick” is a threatened intervention for those schools that do not meet certain growth targets. The incentives were sufficiently attractive that teachers in one Orange County junior high school celebrated the improvement over the previous year of each of the two main ethnic groups: Whites and Latinos. But, rather than enjoying substantial financial rewards, the school was threatened with intervention because, overall, it did a poorer job. How could this happen?

The manufactured numbers in Table 25.1 demonstrate this paradoxical situation. Suppose that only two groups of students, the blues and the greens, showed improvement from last year to this year.

With both groups showing improvement, it is reasonable to expect that this hypothetical school did better. But, the data in Table 25.2 show why the school’s overall performance could decline.

This example is hypothetical, but the phenomenon is real and so robust that it must

<table>
<thead>
<tr>
<th>Group</th>
<th>Meeting standards in</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1999 (%)</td>
</tr>
<tr>
<td>Blue</td>
<td>33.33</td>
</tr>
<tr>
<td>Green</td>
<td>45.83</td>
</tr>
</tbody>
</table>

Table 25.1 Improved Performance
be anticipated. Indeed, this effect, where the parts suggest improvement while the whole carries the contrary message of deteriorating standards, affected about 70 schools in California during the first (year 2000) API evaluation cycle! (See Saari, 2001b.) Again, my point is that the sum of the parts and the whole can differ significantly, even qualitatively.  

The parts–whole tree diagram in Figure 25.3 provides a quick visual way to list all the consequences of the parts where “D” and “I” indicate a decline and an improvement over last year. The eight branches identify the eight potential scenarios; for example, the extreme left path of D, D, D indicates that the greens, the blues, and the whole school declined in performance. The intentional similarity of Figures 25.2 and 25.3 underlines my point that the parts–whole analysis is related closely to study of the totality of what can occur. The goal is to understand which branches are admissible; that is, which paths can occur. With Figure 25.3, the answer is that all of them can; for example, the performance of the full school could agree with, or differ from that of the individual parts. Let me provide a challenge that is related to the discussion of the last section: How would you establish that the D, D, I branch can occur—a situation where even though each part declined in performance, the school as a whole did better—and what are the implications?

To explain what happens, imagine the shudders of any elementary school mathematics teacher watching a student trying to add fractions in the following fashion:

\[
\frac{20}{60} + \frac{110}{240} = \frac{(20 + 110)}{(60 + 240)} = \frac{130}{300} \tag{1}
\]

But, this is the correct computational approach when computing the fraction of success for the whole school. After all, the two numerators of 20 and 110 are the number of successes for the two groups, whereas the denominators of 60 and 240 indicate the number of students in each group represented in the first column depicting the 1999 data in Table 25.2 Consequently, and more generally, we must anticipate that whenever parts are combined into a whole in unusual ways, the connecting factor can generate a whole that differs radically from the parts.

Table 25.2 Overall Decline in Performance

<table>
<thead>
<tr>
<th>Group</th>
<th>Meeting standards in 1999</th>
<th>Meeting standards in 2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue</td>
<td>20 out of 60</td>
<td>90 out of 240</td>
</tr>
<tr>
<td>Green</td>
<td>110 out of 300</td>
<td>120 out of 300</td>
</tr>
<tr>
<td>School</td>
<td>130 out of 300</td>
<td>120 out of 300</td>
</tr>
</tbody>
</table>

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Figure 25.3 A “Part–Whole” Tree.
But, if an example as simple as this one can create surprise and confusion—as it did throughout the state of California—we must anticipate even greater surprises with the far more complex events of mathematical education. This includes parts–whole problems of understanding how our students assimilate and connect the information of the parts from our curricula. These comments suggest the importance of placing as much, or perhaps even more, attention on how the parts are combined and the consequences. As only one illustration, although we design the course material carefully so that the parts coincide with our vision of the whole, we often make the implicit (and probably incorrect) assumption that the parts can and will be assembled in only one manner. Just as the often-told story where a student learning that
\[ \lim_{x \to 8} \frac{1}{(x - 8)^2} = \infty \] (2)
imics the pattern carefully by answering the question \( \lim_{x \to 5} (x - 5)^2 \) with a 5 lying on its side, there are many unanticipated ways to connect information. The interface, the role of the connections, must be understood better.

Finally, let me use Figure 25.3 and the above to illustrate my earlier comments about the chaos tree diagram in Figure 25.2. A natural way to try to establish that the D, D, I path can occur is to construct an example. Unfortunately, a particular example fails to identify the underlying structure of the parts–whole construction, and it can be difficult to find through the usual trial-and-error approach. Instead, let me demonstrate the pool hall approach by indicating how to find all the examples that support a D, D, I path. To do so, let \( G_{I,99} \), \( B_{I,99} \) denote, respectively, the numbers of greens and blues that met the standards in 1999; \( G_{D,99}, B_{D,99} \) describe the numbers that failed to meet the standards. Similarly, \( G_{I,00}, B_{I,00}, \) and \( G_{D,00}, B_{D,00} \) describe the year 2000 data.

The first step is to establish all the ways in which the first two steps, D, D, of the D, D, I path can occur. This means that the two inequalities:
\[
\frac{G_{I,00}}{G_{I,00} + G_{D,00}} < \frac{G_{I,99}}{G_{I,99} + G_{D,99}}, \quad \frac{B_{I,00}}{B_{I,00} + B_{D,00}} < \frac{B_{I,99}}{B_{I,99} + B_{D,99}} \] (3)
must be satisfied. Each inequality imposes no constraint of any kind on the other. Hence, this independence of the variables make this possible.

Now suppose that the full school does better even though each group declined. This performance of the whole requires satisfying the inequality:
\[
\frac{G_{I,99} + B_{I,99}}{(G_{I,99} + G_{D,99}) + (B_{I,99} + B_{D,99})} < \frac{G_{I,00} + B_{I,00}}{(G_{I,00} + G_{D,00}) + (B_{I,00} + B_{D,00})} \] (4)

These three inequalities are much easier to solve and analyze than trying to invent an illustrating example. Even more can be accomplished; we can determine all eight branches of Figure 25.3 by keeping, or reversing, the various inequalities. Consequently, a full analysis is achieved merely by replacing each inequality with an equality and determining how these three equations in eight unknowns split up the space. In this manner, we can consider the totality of the tree (along with the missing branches that indicate ties) in a far more complete manner. So, trying to create an illustrating example makes the issue difficult, but analyzing the problem in this more general and ambitious framework, which reduces to three equations in eight unknowns, makes it...
fairly apparent that anything can happen. Although other mathematical tools often are needed, a similar effect arises with voting and with other places where this approach has been used.

What does this mean about the modeling of mathematics education? It is my sense that moving to the general analysis of part–whole connections will result in a richer theory along with a way to identify the valued and basic assumptions. Just as the above analysis identifies what kinds of interactions cause all possible different consequences, it would be exciting if something similar could be developed for, say, the understanding of mathematical principles.

Social Norms

How are social norms developed? As an illustration, in the United States, we drive on the right-hand side of the road, rather than the left. Why? The obvious response is that this is the way we do things. In other words, in some manner, we adopt particular social norms.

This social norm situation, where the accepted this-is-the-way-we-do-things customs are sufficiently strong to overrule proposed changes, has powerful positive consequences of ensuring orderliness and efficiency. On the other hand and of particular concern to education, is the reality that norms can create a serious drag that can destroy valued reform efforts. Sandy Sharma, one of my Ph.D. students, is studying how social norms from her country, which include bribery and the dowry, inhibit progress. She is trying to understand the nature of the policy changes needed to change these norms.

Can social norms be changed? I observed an example in 1983 when I was in Recife, Brazil, to deliver a series of lectures. Upon arrival, I noticed the peculiarity where, without hesitation, my host would drive through all the red lights. Although I tightened my seat belt, I said nothing until we approached a green light—where he immediately slowed down! In response to my obvious inquiry, he stated, “I had to stop because someone might be driving through the red light in the other direction.” It turns out that there was an excellent rationale for this reversed social norm. This region was suffering an unfortunate crime wave where criminals would dash toward cars waiting at a red light to accost the occupants. For fitness’ sake, drivers needed to drive through red lights.

In recent years, an actively studied question is to understand how social norms are created and modified. The current tools of preference involve dynamical systems. In keeping with the spirit of this chapter, let me introduce basic notions in an intuitive manner and leave technicalities to references.

Start with what is known as the ultimatum game. Think of this as where you have an opportunity to earn a portion of a certain sum of money, say, $100,000. The rules are as follows: you propose a split of this money to another person. If that person agrees with your division, that is the agreement. But if the person disagrees, neither of you get anything. What division should you propose?

In the United States, the standard response is the “fair,” 50–50 outcome; this sense is supported by experimental evidence where large numbers of people reject offers involving sizable amounts of money if the proposed division is viewed as not being fair. A surprise, however, is that this division is not universal. Conducting experiments in a manner to ensure the anonymity of the two players, Heinrich (2000) discovered that the stable division can differ in different societies even to the extreme where the proposer could demand consistently 80 percent of the reward and get it. Why? What is going on?
To understand by using the kind of example considered by Skyrms (1996), suppose that there are two types of individuals. The first are those who demand two-thirds of the proceeds in any setting, and the others demand only one-third. Assume that these individuals meet at random where the conclusion of their encounter determines the “fitness” of each species. If an individual receives something, this type is reinforced in that there will be more individuals of this type. On the other hand, if the encounter ends in nothing, then the discouragement leads to fewer individuals of this type. When assuming large numbers of people, Skyrms found that the dynamic reached an equilibrium where half of the individuals were of one type and the other half of the other type. Another illustration of the fitness dynamic is the driver in Recife going through a red light; when comparing the consequences of an unlikely traffic ticket to a robbery, fitness supported those who drove through the red light. Here, unanimity in ignoring the red lights characterized the social norm.

Now turn to my student Ms. Sharma’s “toy example” that she developed at the start of her studies for the purpose of developing her intuition. It starts with a group of people trying to create a public good; as an illustration, think of this as providing security—say, hiring a doorman or guard—for their private community. In her model, only those who cooperate enjoy the rewards. The problem is that those who cooperate incur a personal expense, and any real payoff of social benefit requires the participation of enough people. After all, if only one or two people participate, they could not afford a guard, and they would abandon their efforts quickly. A sufficiently large group, however, could afford this security easily. Then, beyond the cooperators, there are others who are not interested in contributing to this community benefit if only because of the added personal cost. As a slight but important digression: notice how this description resembles educational reform where the incurred expense involves learning and adapting to the new methods, where only those who sign on to the reform procedure can benefit from their involvement, and where the full community benefits only after enough people are involved. Read on; the similarity is intentional.

In her modeling, technical conditions are imposed to capture how people decide whether to cooperate or not. These conditions are based on how individuals interact randomly with others and determine whether this cooperation is, or is not, to their personal advantage. What Ms. Sharma discovered with her dynamical model is the common finding for this area: a threshold effect of the kind indicated in Figure 25.4.

To describe this figure, the extreme ends represent where “nobody” and “everyone” is involved in the cooperative venture. The key is the bullet that indicates a threshold effect. To explain, suppose that there is only a small initial involvement that locates the starting position (the dagger) of the movement to the left of the threshold. According to the figure, the natural dynamic of interaction, involvement, and discouragement will cause a decline in cooperation, leading to its eventual demise. On the other hand, with a sufficiently massive start that places the initial position to the right of the bullet, the dynamic of encouragement indicates a gradual increase in the level of community cooperation, leading to a situation where cooperation takes over the full society. A key factor in Ms. Sharma’s analysis is the location of the threshold; she finds that its location is a combination of the entry costs and the extent of the benefit for the community. For instance, a community benefit that offers more rewards for a smaller cost has a

Figure 25.4 Dynamics of Social Norms.
smaller threshold; it is one that is easier to overcome. But, a higher expense of entry requirements, such as needing to learn a new approach, combined with uncertain benefits, will define a discouraging location for the threshold. Here, it would be unlikely to sustain the change, even with a massive starting bloc.

Will this model explain the problems and adoption of reform procedures? Because Ms. Sharma’s model was intended (at this early stage) to provide intuition and guidance, where several accompanying relevant features are ignored, it is too simple. On the other hand, her model does provide insight. (For a more general perspective, see my notes in Saari, 2002). Let me offer a suggestive example before providing a warning. Ever since the introduction long ago of the Apple IIc computer, the potential value of computers in the classroom had been appreciated and discussed. But, even with the pioneering efforts of several advocates, it took a surprisingly long time for this approach to take hold in our schools. The explanation is easy to understand in terms of Figure 25.4. In the early years, the location of the threshold point reflected the sizable commitment. Relative to today, desktop computers were expensive and learning how to use them effectively was difficult and hindered by poorly written manuals. In other words, the entry expense was relatively high. Moreover, software for the classroom was primitive and scarce, not many families had computers at home that would allow for homework, and the internet with its vast sources of readily available information was yet to be developed. Stated in Ms. Sharma’s terms, the personal costs were high and the community benefits were limited, so the threshold (the bullet) was located far to the right. This location indicates the need for a sizable movement to achieve wider acceptance. Thus, it is not surprising that computers were relatively scarce in classrooms and restricted to more affluent communities where both the expense and the community benefit (e.g., the availability of computers at home) were met. What we had, then, was computer usage restricted to certain communities, as indicated by the dagger in Figure 25.4, but this was only until improved conditions moved the location of the threshold. With corporations donating computers and software, with the cost of computers coming down, with programs becoming more user-friendly, the entry costs and continuing expense came down. Once more software, the internet, and larger numbers of homes with computers became available, the benefits became greater. Accordingly, the social threshold point (the bullet) moved to the left. The model predicts that once the threshold (the bullet) moves to the left of the current practicing standard (the dagger), we should expect a rapid increase in adoption, and that is precisely what we observed. On the other hand, suppose that there were no pockets of acceptance; suppose that the current level of the use of computers still remained to the left of a less imposing threshold; we still would expect a tendency toward extinction.

The message and warnings suggested by this simple model are clear. For instance, expect difficulty in getting a reform approach adopted. Even with an initial success, unless and until some societal threshold is passed, the approach will tend to be ignored. This, of course, has been the fate of several reforms. During the meeting on the modeling of mathematics education in Santa Fe in December 2001, several private conversations debated the wisdom of introducing reform in a small dosage or with a larger approach. Figure 25.4 supports the larger approach, but with the constraint that we have no notion about the location of the threshold point. Consequently, Ms. Sharma’s example suggests that a more sophisticated and successful approach toward reform must include understanding how to reduce the cost of getting involved and how to maximize the benefits. Stated in words, we must find ways to reduce entry costs and share the benefits; we need to change the threshold location.
Social Movement

Remember those Friday night beer parties during our college days when a hat was passed around seeking donations to buy a keg of beer and when each person was to contribute an amount of money compatible with the amount of beer that will be consumed. You remember the problem: some paid little and drank much. Surprisingly, ways to resolve this free-rider problem in collegiate beer parties offer insight about how to convert an educational reform movement into a broader social movement. Insight and help come from mathematical economics where a goal of the incentive literature is to find ways to convert the societal and/or organizational structures and reward system so that it now is in the best interest of an individual to cooperate with a specified objective. With the beer party illustration, the goal is to design an approach so that each person feels it is in his or her best interest to contribute an amount of money commensurate with his or her personal beer consumption. Similarly, when worrying about converting educational reform into a social movement, part of the goal must be to find ways to encourage individuals to make it in their personal interest to sign on. Such encouragement must continue until the societal threshold is passed, where the social dynamic of encouraging benefits takes over.

How is this done? Although the technical material involves gradients and game theory, the intuition is clear. If you want me to do something, the rewards and punishments must be expressed and designed to make it compatible with my personal interests. This most surely is the case when trying to involve in a project educators and others who are overworked and have an agenda that already is too full. The incentives must be high, but remember, for educators some rewards may be much stronger than money. As an illustration, professional recognition and research opportunity are important motivating factors in any community of educators and researchers.

I leave the technicalities of the design of incentives to the reader; they are easy to find by checking several of the current, graduate-level books in microeconomics. But, for intuition, consider the task of attempting to enlist teachers to try out a new approach for the teaching of mathematics. If an invitation involves a certain degree of prestige and recognition from fellow teachers and the administration, it is reasonable to anticipate active involvement and participation. However, when it is time to enlist more teachers, resistance can occur. Why? This is a mystery because some of the personal costs of being involved have been eliminated through the efforts of the first group of teachers. So why is the second group not as enthusiastic? The answer is clear: although personal costs have dropped, so have the personal incentives of recognition of participating in a pioneering group. In other words, the incentives for the next wave of teachers must be thought through carefully; they cannot be “more of the same.”

This intuition captures a technical lesson that comes from the incentive literature. Namely, do not expect to find a one-size-fits-all incentive; the usual case is that the incentive must be designed for the individual. But, with reflection, this important lesson is natural and instinctive. Let me suggest that readers interested in this topic check the incentive literature from economics.

Conclusions

I started this chapter by acknowledging the underlying complexity of the issues in mathematics education. Of course, I have strong views about these topics. But beyond
my personal explorations where I worry about how to stimulate my students to appreciate the power and beauty of mathematics, to explore new ways to convey information efficiently and accurately in a large class, to encourage students to separate concepts from technical details, I have no expertise in this particular area. On the other hand, my research interests include decisions, aggregation, dynamics, and complexity. It is my sense that, if used carefully, general principles from these areas will help address the research concerns of mathematics education.

But just as I want my students in a course on analysis, or dynamical systems, or calculus to separate the concepts and the technicalities, it is worth repeating my warning in the introductory section. Although it is reasonable to seek help from mathematics, be careful. I am skeptical whether anything useful will follow from technical descriptions and conclusions. On the other hand, basic concepts phrased properly often are transferable to other disciplines. Guided by these principles, we can expect to identify better central assumptions for mathematics education and to uncover the sources of complexity.

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Notes

1 Also, I highly recommend checking out R. Devaney’s delightful web page: http://math.bu.edu/DYSYS.
2 This parts-whole conflict is the theme of the book Decisions and elections: explaining the unexpected (Saari, 2001b). Please see this publication for a detailed discussion.
3 The reader might suspect that the change in numbers in each unit for the two years played a role; as explained in Saari (2001b), this is the case. This migration of different ethnic groups occurred in all the schools affected by the paradox of the state of California’s Academic Performance Index.

References

Devaney, R. (1999). The dynamical systems and technology project at Boston University, at http://math.bu.edu/DYSYS.