In his presentation of Dietrich Mahnke and Oskar Becker’s correspondence, Paulo Mancosu expressed his disappointment by saying that “the relationship between phenomenology and exact sciences in the 1920s is still an unexplored area.” The purpose of this paper is to offer an overview of phenomenological reflections on the foundations of formal mathematics during the 1920s, and of the philosophical discussions these reflections triggered. Among the main protagonists of these discussions, some were mathematicians that were interested in phenomenology such as Hermann Weyl. Some were philosophers, former students of Husserl, either in Göttingen, such as Dietrich Mahnke, or in Freiburg, such as Oskar Becker. Finally, others were philosophers, who, without having been, strictly speaking, ‘students’ of Husserl in the academic sense of the word, did nevertheless play the part of intermediaries between the “phenomenological movement” and other philosophical schools, such as Felix Kaufman with the Vienna Circle at the beginning of the 1930s. It is these philosophers interested in mathematics that will be the object of this paper, as well as what Husserl at the time still called their “συμφιλοσοφεῖν”, although the time of disillusionment and of the “great disappointment” was getting near.

24.1 Husserl: formal mathematics and material mathematics (1927)

This overview of the structure of the mathematical field as it was conceived by Husserl will focus on the Natur und Geist course during the 1927 summer semester, which allows a comprehensive view of what mathematics is in its unity and its difference, based on a reflection on the classification of sciences, in particular on their classification according to formal criteria resulting from the characterization of the very idea of science; because mathematics is a “science in the pervading sense of the word” [Wissenschaft im prägnanten Sinne], a science that is itself made of a number of mathematical disciplines that are all interconnected to a certain extent. For instance, plane geometry and 3-D geometry are interconnected, as are also, on the margins of the mathematical field, optics and electric theory. But geometry and pure Analysis can also be brought together from a further unifying perspective.

Husserl’s method here is very different from the one used at the same time by his student Oskar Becker (see the fourth part of this chapter), because Husserl considers that there is no point in thinking about mathematics exclusively as a “historical tradition” [historische
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Überlieferung], in remaining bound by the contingency of facticity, and in doing no more than bringing together what already exists:

A philosophical classification should be a classification of actual and conceivable sciences in general, which implies that it has to follow the pure Idea of science—and to overlook its facticity—or in other words to follow the pure and unconditionally universal sense of a science as such in general, and it must be characterized by an apodictic necessity based on the latter.4

Becker’s method is very different. His ontological reflections on the being-sense of the mathematical objects are not eidetic, and do not aim to be:

For we will try to elucidate—while also interpreting it—the meaning of some mathematical phenomena (that have happened in the history of mathematics). However, we will not ask ourselves if, in addition to of the factual development [neben dem faktischen Verlauf] of the history of ideas, other “pure possibilities” [reine Möglicherkeiten] might have happened otherwise.5

This chapter will not analyze the consequences of this rejection of eidetics on the classification of mathematical sciences, and on the configuration of the field itself.6

Mathematics, as Husserl conceives it, is an a priori science (or a science of essences), not an empirical science; for mathematical propositions are not about either facts or factual beings. They do not even teach us anything about facticities [Faktizitäten], about temporal existence. Mathematical propositions are about pure possibilities [reine Möglichkeiten], or about ideal objects around which pure possibilities are organized. Mathematics is thus an a priori science, because it constitutes an a priori system of a priori propositions or judgments (geometrical judgments on ideal figures, arithmetical judgments on numbers as ideal objects) which, given their inner rational, have to be true judgments, have to be truths—and namely mathematical truths, and thus have to be a priori.

But a priori sciences can be formal (analytical), if their judgments and concepts leave the concrete specificity of their objects in total indetermination, or material (synthetic), if these judgments and concepts include in their conceptual content this concrete specificity. Concepts such as human beings, stones, pictures, etc. have a material content, but if we are concerned about object, propriety, state of affairs, relation, totality, similarity, identity, difference, etc. we are dealing with formal and empty concepts. Judgments that only include empty, formal concepts—and contain no concepts that have any concrete specificity—are called “logico-formal judgments” by Husserl: these judgments constitute a universal a priori science that is related, in an empty generality, to objectivity in general and to its derived forms. Hence the distinction, within a priori sciences, between mathematical and purely logical disciplines that are united under the umbrella of a general logic, or a mathesis universalis, and that a priori sciences that have a material content, such as geometry, the pure doctrine of time, a priori mechanics etc.—a distinction that Husserl sees as being “of the utmost importance.”7

Sciences are furthermore to be divided into abstract sciences and concrete sciences, depending on whether they focus upon a group of concrete individuals, or even a single determined individual taken in its full concreteness, or whether they limit themselves to specific, non-autonomous moments of entire objects and of their concrete essentiality. Zoology and botany are concrete sciences, and so is natural science as far as it is concerned with concrete nature in its totality. Rational mechanics is an abstract science that is relatively autonomous within natural sciences, because
it considers bodies as mere substrates for movements and for driving forces. However, universal physics could itself be seen as abstract, if it turned out that material nature is nothing more than a non-autonomous structure of the fully concrete world. The opposition between abstract and concrete reveals itself here as relative, without this implying anything ambiguous at all regarding the meaning of these concepts. Taken at a general enough level, geometry as the science of space and of spatial figures is abstract, relative to the concrete field of the objects from which its concepts were abstracted, but it is concrete with regards to the theory of conic sections.

However, the distinction between what is abstract and what is concrete, and the use that can be made of this distinction as a tool of universal classification (all sciences are either concrete or abstract) poses problems for mathematics, as the concepts of the abstract and the concrete are not unequivocal with regards to a priori material sciences and a priori formal sciences. In order to retain the universal value of this distinction, a terminological clarification is required—one that will allow us to understand why some mathematical disciplines that are neither abstract nor concrete can be said to be “abstract” in an improper sense of the word. To be sure, every concrete object possesses its own complete essentaility, and as a component of the concreteness, it possesses its own parts and its own moments (length, extension, form, situation, sensory qualities, physical properties, etc.) with their different levels of generality. With this division, we remain in the sphere of material content, which is actually in conformity with the distinction between abstract and concrete. Nevertheless, every object in general, because it is something, also possesses its empty form: it is, for instance, the totality of its parts. The empty form of the totality, with all the variety of relations that relate to it, is reached through the process of “abstraction” of all its material content, in other words through formalization. It is thus only in an improper way, and in order to follow common usage, as Husserl had already highlighted in Ideas 1, that it can be said about the disciplines of the mathesis universalis that they are abstract:

Similarly, universal analytics (mathesis universalis) is categorized amongst abstract sciences, while strictly speaking, as a science of what is formal in every object (and also in every proposition, in every truth, in every science) it is above all sciences, and should not find itself caught in the opposition of the abstract and the concrete.8

Phenomenology, with regards to its methodology, is undoubtedly closer to the intuitive approach of geometry as it operates with ideal figures, than to the formal deductions of mathematicians who proceed axiomatically, because phenomenology belongs to the family of material eidetic sciences9 hence the significance of Becker’s contribution toward the phenomenological foundation of geometry and its physical applications, beyond even what it brings from the strict point of view of a theory of geometrical knowledge.10 Dietrich Mahnke believed, however, that, in contrast to numerous modern philosophies of life and of intuition that rightfully arise suspicions of lack of scientificity amongst mathematicians and scientists used to logical rigor, it is essential for phenomenology to recognize the fundamental value of formal mathematics as “the ideal storeroom of the forms of theories of all exact sciences.”11 Hence the decisive importance of a phenomenological elucidation of formal mathematics, whose historical source lies in Leibniz, and that Hilbert had just brought to its fullest and most complete expression in the developments of his proof theory at the beginning of the 1920s.

24.2. Hilbert and the axiomatic method (1922)

We will now consider Hilbert’s axiomatic method as it was presented at the Copenhagen and Hamburg 1921 conferences that were published in 1922 under the title Neubegründung der
for, as it was noted by Dietrich Mahnke, it is in this presentation that Hilbert is the closest to Husserl’s account of the phenomenological foundation of formal mathematics, which constitutes a good omen about the accuracy of results reached by means of very different considerations.13

Each area of formal mathematics—whether it is arithmetic or a non-Euclidean geometry, the logical structure of Euclidean geometry or an aspect of theoretical physics—is defined as a multiplicity of “things” [Dinge], of which we can have no intuition, of which we know nothing besides the fact they have a certain number of purely conceptual relations, described by a finite number of axioms: axioms of connection, of order, of continuity, etc. Such a multiplicity “exists” (in the mathematical rather than the physical sense of the word) when its axioms are mutually compatible, in other words when it is not possible to derive from some axioms a proposition that would contradict another axiom of the same system, or one of its consequences, via a finite series of inferences. Hence, it follows that one of the main tasks of the axiomatic method is to give out “proofs of existence” by establishing that a given axiomatic system is non-contradictory.

As far as Euclidean geometries, non-Euclidean geometries and physics are concerned, these proofs are achieved by reducing them to areas of mathematical Analysis that are “formally equivalent” or “logically isomorphic” within each system. In other words, the constitutive elements of each follow the same axioms when they are considered in their purely logical and conceptual form, without taking into account their intuitive content. Thus, for example, in the *Grundlagen der Geometrie*,14 Hilbert had solved the problem of the non-contradiction of axioms in plane Euclidean geometry, indirectly and relatively, by reducing it to the problem of the non-contradiction of the axioms that define the set of real numbers as a maximal Archimedean ordered field. This reduction was made possible by the Cartesian co-ordinates method, through which each point on a plane equipped with a Cartesian coordinate system can be associated objectively with a pair of real numbers. The same can be said about proofs of consistency of axioms in thermodynamics, in the theory of radiation or in other physical disciplines,15 which can be reduced to the question of consistency of axioms in Analysis.

The direct proof of consistency of the axiomatic system of Analysis, or even of arithmetic or of set theory represented, at the beginning of the 1920s, an unsolved problem that some saw as unsolvable. For instance, Alessandro Padoa had declared about Hilbert’s second problem that it was no more than a “conversation that should be done away with” [une causerie qui se pouvait supprimer].16 Hilbert should refrain from all attempts to use methods used in irrational number theory on the problem of direct proof of the non-contradiction of axioms in arithmetic because of a constitutive dissymmetry. While contradictions or dependences between propositions can only be proved by “deductive arguments” (or in other words “syntactically”), the non-contradictions or independences between propositions can only be proved by “observations” (or in other words “semantically”): “It can be observed,” wrote Padoa, “that each chosen interpretation can or cannot verify the specified.”17

Hilbert had long been convinced that a “simultaneous construction” [simultaner Aufbau] of arithmetic and formal logic was a necessity, since paradoxes relating to the use of arithmetical notions (concept of number, of set …), when explaining logical laws that were expected to serve as a foundation to arithmetic itself, needed to be avoided.18 This vicious circle was not only to be seen in logical principles as they were customarily presented, according to Poincaré; it could also be found in Russell’s logic, which referred to the notion of sets (although he used the word “classes” and considered this notion as a logical one).19 But in this second stage of the theory, the requirement to construct arithmetic and formal logic simultaneously means more. It means that “a strict formalization of the entire mathematical theory, inclusive of its proofs, so that—following the example of logical calculus—the mathematical inferences and conceptualizations
[Begriffsbildungen] become a formal part of the edifice of mathematics should be carried out. Or, in other words, that mathematics, when rigorously formalized, turns into a “stock of formulae” that can be proved by using, on top of the usual mathematical signs, a number of logical signs (the implication that is written →, the universal quantifier, written () with a variable inside these brackets, and later the negation, written ¬ which Hilbert thought at the time he could do without).

As far as the consistency of elementary arithmetic is concerned, Hilbert shows that it can be developed on the intuitive basis of concrete signs. He starts by introducing a number of signs (numbers) that constitute the very objects of mathematics, but that do not mean anything: 1, 1 + 1, 1 + 1 + 1, etc.: “These number-signs, which are numbers and completely make up the numbers, are themselves the object of our consideration, but otherwise they have no meaning of any sort.” Besides these number-signs, Hilbert introduces a number of signs that have a meaning and that we use to communicate. The signs 2 and 3 are the abridged written forms of 1 + 1 and 1 + 1 + 1 (they are signs of signs, signs that refer to other signs); the signs = and > are used to communicate statements. The series of signs 3 > 2 is not an arithmetic formula, but it is only used to communicate the fact that the number 3, the abbreviation of 1 + 1 + 1, is greater than the number 2, the abbreviation of 1 + 1, or in other words that the latter is part of the former.

Similarly, the series of signs \( a + b = b + a \) is not a formula (in the strict meaning of the word according to formal mathematics), but only the communication of the fact that the sign \( a + b \) is the same as the sign \( b + a \). It is possible, says Hilbert to “discern [einssehen] the correctness of the semantic content [das inhaltliche Zutreffen] of this communication.” It can indeed be supposed that \( b > a \), or in other words that the number–sign \( b \) is greater than the number–sign \( a \); and be written that \( b = a + c \), where \( c \) is a sign used to communicate a number–sign. What remains to prove is thus that \( a + a + c = a + c + a \), or in other words that the two number-signs on each side of the equals sign are the same, which is the case if \( a + c = c + a \). Yet this expression is reached from the preceding one by having at least one 1 disappear through a procedure of dissociation of \( a \), and this procedure of dissociation can be continued until the summands agree with each other; because each number \( a \) is by definition composed by assembling signs 1 and +. It can similarly be decomposed by a procedure of dissociation of the signs that compose it.

Arithmetic, when it is thus practiced, does not include axioms and cannot lead to a contradiction. It is made of concrete signs on which operations are performed and about which contentual statements are formulated. However, the whole of Analysis cannot be founded this way, because this intuitive contentual procedure does not allow the formulation of statements about an infinity of numbers or of functions: for when an infinity of numbers is concerned, neither all the relevant number–signs can be written down nor all the required abbreviations can be introduced. In the end we face the incoherencies that Frege highlighted in his attack on the established definitions of irrational numbers.

This is why Hilbert moves to a higher level, where axioms, formulae, and proofs of mathematical theory are themselves taken as the objects of a contentual investigation. To achieve this, the contentual arguments that are often used in mathematical theories first have to be replaced by formulae and rules; in other words, they have to be reproduced by formal structures. Hence the “strict formalization” that we mentioned earlier, which gives all its meaning to the requirement of simultaneous construction of arithmetic and formal logic. The axioms \( a = a \), \( 1 + (1 + 1) = (1 + a) + 1 \), etc., formulae and proofs that thus constitute the formal structure of mathematics are exactly what the number–signs were empty of any sense in the former construction of elementary arithmetic; and they now become, just as was the case for the arithmetical signs, the theme of contentual arguments or, in other words, of proper thoughts. Hence the strict distinction between the level of formal arguments and formulae and that of
contentual arguments. The ultimate foundation of mathematics is thus reached, according to Hilbert, through a critique of proof, analogous to the critique of reason: “Just as the physicist examines his apparatus, the astronomer his position, just as the philosopher engages in critique of reason, so the mathematician needs his proof theory, to secure each mathematical theorem by proof critique.”

Husserl’s analysis had taught us that mathematics is not a science of facts, and that it even tells us nothing about facts: how is it, then, that the signs that Hilbert introduces into elementary arithmetic to produce contentual arguments regarding simple numerals do evoke facts? How can we understand for instance that they are used to communicate “the fact that 2 + 3 and 3 + 2 are the same number-sign” [die Tatsache dass 2 + 3 und 3 + 2 dasselbe Zahlzeichen sind], or “the fact that the sign 3 extends beyond the sign 2” [die Tatsache dass das Zeichen 3 über das Zeichen 2 hinausgeht]? It is worth mentioning here that the challenge is not only a verbal one, as the “facts” that are communicated through arithmetical arguments are not, according to Husserl, facts, but a state of affairs. In other words, they are objects of higher order that bear the mark of predication, and the “fact” that the sign $b$ is greater than the sign $a$, because the later contains at least one extra 1 than the former in the way they are written, is not a simple “fact” that (once $a$ and $b$ are high enough) can be apprehended via a sensory perception, just as we cannot perceive that the earth is larger than the moon. Such issues lead, however, to a debate about the status of “things” and of signs. In Dietrich Mahnke’s words, they challenge the philosopher to further the work.

### 24.3. Dietrich Mahnke and the phenomenological elucidation of the axiomatic method (1923)

In his article “From Husserl to Hilbert” (1923), whose purpose is to introduce readers with a science background to phenomenology and to its method, starting with the example of the phenomenological analysis of axiomatized arithmetic, Mahnke endeavors to elucidate, from the point of view of the theory of knowledge, the meaning of the axiomatic method and of its objects. What, for instance, are these “things” that Hilbert refers to? Naturally, they are not physical realities, nor are they ideal concepts as those of intuitive geometry. They are rather “mere conceptual skeletons without the covering of sensory material” they are forms of things comparable to variable functions with empty positions $f(*)$, forms that are introduced by Hilbert along with individual signs to constitute the formal structure of mathematics. These simple supports for relations can be compared to the “purely grammatical categories” such as unity and plurality, the whole and the part, the subject and the predicate, that Husserl refers to in his *Logical Investigations*. For, just as in the formula “$S$ is $p$,” $S$ and $p$ can be replaced by any terms, with the first term as a subject and the second as a predicate; similarly, very different things in terms of their content can be introduced within the propositions of formal arithmetic, as long as in their mutual relations, they validate the laws of commutativity, of associativity, etc., such as segments, times, energy, frequencies, etc.

The system of formal arithmetic is thus not a specific scientific discipline, but merely the logical form of theories that is used as a “common deductive scaffolding” in all “logically isomorphic” or “formally equivalent” disciplines. The whole of formal mathematics collaborates, according to Husserl, with pure logic as a theory of all nomological theories, insofar as it works out the form of the essential types of possible theories or fields of theory, and investigates their legal relations with one another. All actual theories are then specializations or singularizations of corresponding forms of theory, just as all theoretically worked-over fields of knowledge are individual manifolds.
The constitution of a theory of the forms of possible theories is not only of the highest importance on a theoretical level, but also on a practical one, as the possibility to classify a theory within its formal class can be very useful on a methodological level. It then thus becomes possible to solve some problems that arise within the framework of a theoretical discipline, or of one of its theories, by a recourse to the categorial type, or to the form of the theory, or even by going over to a more comprehensive form or class of forms, and to its laws. Examples of this could be the theory of $n$-dimensional manifolds, which arises from generalization of geometric theory, Lie’s theory of transformation-groups, etc.

This elucidation of the nature of “things” in axiomatic mathematics shows that it is a philosophical error to state that intuitive numerical signs are the true objects of arithmetic; however, Mahnke asserts that this erroneous statement contains a “deep truth” that phenomenology can help to elucidate. What leads Hilbert to this statement is that he considers that simple arithmetic formulae are not enough to prove the consistency of arithmetic, and that intuitive operations such as composition and decomposition of signs, and contentual inferences, are necessary. From that perspective, mathematics, just like metaphysics, does not accept “purely conceptual proofs of existence,” even if the concept of existence is not univocal with regards to mathematics and metaphysics.

On this point, Hilbert agrees with Husserl’s fundamental idea that ultimately all scientific statements can only be justified by being referred to an adequate intuition, in which the object of knowledge is given just as it is reached for. Husserl, however, draws a distinction between the objects that are given in a sensory intuition and those that can only be given through evidence in a rational act of a different kind, in a categorial intuition:

The “a” and the “the”, the “and” and the “or”, the “if” and the “then”, the “all” and the “none”, the “something” and the “nothing”, the forms of quantity and the determinations of number, etc.—all these are meaningful propositional elements, but we should look in vain for their objective correlates in the sphere of real objects, which is in fact no other than the sphere of objects of possible sense-perception.

But on the basis of this sensory perceptions, higher rational acts are built: colligation, counting for instance; and it is precisely in these logical experiences, which are nevertheless founded on sensuous experiences, that collectiva and numbers are given. In other words, what we think when we use the words “and” or “or” is filled by the corresponding categorial intuition. Hilbert is thus right to say that the foundation of mathematics requires “extra-logical discrete objects” to be given beforehand.

Hilbert also agrees with Leibniz when he chooses to use as extra-logical discrete objects not natural things, but artificially created numerical signs; like all good mathematical symbols or characters, these signs “express” the purely arithmetical relations that exist between countable things, such as order and connection, without having to bring into play material determinations foreign to the question: “There is in the characters,” wrote Leibniz in his *Dialogus de connexione inter res et verba*, “some relation, some arrangement that is the same that the one that exists between the things, especially if these characters are well imagined.” Similarly, in geometry, characters have to be substituted to geometric shapes, as stated in the fragments of the 1679 *Geometric Characteristic* to which Mahnke refers here, because shapes are far too often intertwined and confusing for those who look at them. Yet, Descartes’ analytical geometry cannot totally be freed of shapes, because it only shows relations of magnitude in its calculations, thus assuming that the relations of situation are known from the observation of shapes; hence, the project of a geometric characteristic that would rely exclusively on the use of characters...
that would also convey the situation, or in other words the relation the geometric shapes have with each other. Following up from these comments, Mahnke will write in his thesis on Leibniz:

Leibniz’ point of view is in total agreement with Hilbert’s current perspective as in his latest texts he undertakes (against former attempts of a purely logico-formal foundation of mathematics, but also against Brouwer’s and Weyl’s “intuitionistic” attempts to only tolerate mathematical concepts that can be constructed intuitively in a determinate number of stages) a new grounding of the whole of mathematics up until to his works, which holds on to the formalizing and axiomatizing method but which, in order to produce a proof of existence, also and essentially mobilizes intuitive operations with extra-logical and intuitively given signs of numbers and of formulae.33

But Hilbert is wrong to deduce from this that “these signs are themselves the objects of arithmetic,” and that they have no other meaning, because the sign $1 + 1$ has a meaning, just as much as the abbreviated sign 2, which according to Hilbert has a meaning: “It is the sensible representation (Versinnlichung) of a purely logical relation: the collective combination.”34 The true objects of arithmetic are precisely these logical relations, and the true value of numerical signs lies in the fact that they are the more appropriate sensory foundations for categorial intuitions of numbers. Hilbert himself recognizes indirectly that the signs of formulae also have a meaning as he accepts that next to the “formal inferences” of mathematics in the narrow sense of the word, and so that he can prove its consistency, there are also “inferences that have a semantic content” belonging to metamathematics. Mahnke, however, notices that the distinction between simple symbols and significations in the strict sense of the world intersects another one: the distinction between ideal object and real acts of mathematical thought. The sign $\rightarrow$ means there exists a relation involving two propositions, but there is necessarily a match between the perceptible sign, as well as the logical consequence it expresses, and the act of the mind that aims at the objective state of affairs, or in a signitive way on the basis of a subjective perception of a sign, or on the basis of a categorial intuition in which the intended object is self-given. Just as it is shown by Husserl in §14 of the Sixth Logical Investigation, a “signitive intention” always has an “intuitive support” [intuitiver Anhalt]—in this case the sensuous side of the expression—but for all that, it still does not have an “intuitive content” [intuitiver Inhalt]. It still needs to be articulated to an intuitive act, while both remain specifically different.35 In this correlation between rational objects and objectifying rational acts, between the noema and the noesis, it is noematics which, according to Mahnke, finds its foundation in noetics.

Is it possible, however, to deduce the objective existence of the set of numbers, as Hilbert suggests, from a study of human, subjective inference procedures? Are things in conformity with our thought process? Mahnke answers positively, because the “things” of formal logic and mathematics are mere “categorial forms of things” that cannot be found in the sensory material, but are only constituted within form-giving, rational acts. In this operation, the sensory contents of things remain unchanged, and the same goes for their figural qualities: “Categorial forms do not glue, tie or put parts together, so that a real, sensuously perceivable whole emerges. They do not form in the sense in which the potter forms.”36 This is why the course of the world can neither contradict nor confirm ideal laws of formal logic and mathematics: “Laws which refer no fact,” writes Husserl in §65 of the Sixth Logical Research, “cannot be confirmed or refuted by a fact.”37 Categorial forms blend into the essence of reason, which gives form.

Is it possible, in these conditions, to grant to procedures of logical inference a general logical meaning rather than just a simple individual, psychological meaning? Once more, Mahnke answers positively, because the fact that these founded acts “inform” the sensual object, which is conse-
quently constituted as a modified objectivity, does not mean in any way that these actions falsify the reality into a phenomenal world that would only have a psychological, individual validity, or at the most a typically human one. Categorial forms do not change reality; which does not mean, however, that they only have a personal and limited value, or that they only have a value for human souls for instance. They have a value for any mind that possesses reason in general, or in other words that does not just have moments of sensory experiences, but that is also able to grasp a scientific truth, i.e. to grasp an objectivity that is formed by categorial functions:

The truly “objective” noematic laws of form, those which alone give rise to objectivity, are grounded not in empirical-psychological laws of fact which have subjective validity, or at best anthropological validity, but rather in phenomenological-noetic laws of essence which have objective universal validity.  

The noeses of reason, in general, are not real lived experiences of an individual psyche, but they “reside” in these lived experiences more or less like geometrical circles “reside” in physical ones. The noesis is, relative to the mental experience, what the noema is to the physical “thing” and the idea to reality. Mahnke rediscovers here the idea that had been developed by Natorp, and that he believed he could trace to Heraclites, according to which all individual psyche, all “animate monad” could perceive in its own innermost depth the eternal “Logos” common to all; it is in this context that Natorp quoted Schiller’s line: “Es ist nicht draussen, da sucht es der Thor; Es ist in dir, du bringst es ewig hervor.”

Hence the criticism of what we could call the “Heideggerian point of view” in the philosophy of mathematics, which only recognizes as “existing” those objects that are accessible to a human consciousness. This will be precisely the point of view of Becker, who will try to promote an “anthropological” conception of knowledge against the “absolute” one, in full awareness of Husserl’s former criticisms of logical anthropologism, and hoping nevertheless to give an deeper ontological meaning to Husserl’s doctrine of categorial intuition.

24.4. Oskar Becker and the criticism of Hilbert’s axiomatic formalism (1927)

In the sixth and final section of Mathematische Existenz, Becker offers an “hermeneutical analysis of demonstrative mathematics and deductive mathematics.” To what extent, asks Becker, is the existence of mathematical objects expressed as a particular guise of the care [Sorge] and of significance [Bedeutsamkeit]? How should we understand the opposition between formalism and intuitionism, between mathematical existence understood as a non-contradiction and mathematical existence understood as constructability? Which guise of the care of significance is hidden behind this opposition?

The history of mathematics teaches us that demonstrative (intuitive) mathematics is, in all respects, the most original. Strictly speaking, purely deductive mathematics was only established at the end of the nineteenth century (around 1870), even if it was anticipated in various of its former branches (since Leibniz). Thus, Becker sees it as a “late bloom,” a “degenerated form.”

The non-contradiction requirement has, according to Becker, a very specific meaning. It is the condition that allows the indefinite extension of a purely formal deduction. It articulates “the care to extend the formal deduction itself indefinitely”, Becker 1927, 629. in other words, “the care to preserve the guise of care that exists specifically within academics whose field is formal mathematics.” The deductive mission [Betrieb der Deduktion] has to be guaranteed, whatever, on the other hand, the things or problems in question or that could be in question. Reassigning the sense of mathematical existence to the non-contradiction of the “existing” formation means
ignoring the sense of being of the formation. It implies a deliberate hindrance to any questioning of the mode of being of the formations.

Formal mathematics thus somehow represents the triumph of the “aphaeretic tendency” [aphairetische Tendenz] that Becker believes he can discern in the history of mathematics: “The expression ‘signs that signify nothing’ [Zeichen, die nichts bedeuten] used by Hilbert (for the arch-objects [Urgegenstände] of mathematics) is not correct, even if, in all appearances, it targets, albeit confusedly, something that is right.” For, undoubtedly, mathematical objects are in no way given “in a signitive manner” (in an empty intention), for they are given in a categorial intention. But this also implies that the tension between the empty intended object [Leerintendiertem] and what fills it [Erfüllendem] between the sign [signum] and the meaning [significatum] disappears. Signs that signify nothing are precisely signs that are freed from indicative function; in other words, signs that are not signs at all. Hilbert’s incorrect expression nevertheless contains something correct, because it allows us to get rid of the idea that, because of the noetic–noematic parallelism, mathematical objects (the contents) and the intentionalities that aim at them (the relations) allegedly belong to the same ontological level. Yet this is not the case, for just as the being-a-sign is created in the indicative relations, the “archontic” sign of the relations (or the ontic accent), in the global phenomenon of “the mathematical,” is to be found, so to say, on the relations themselves. The ontic priority belongs to the noesis, the noema being secondary to it, and it is exactly the reverse that happens in the case of the sensory perception, as shown for instance by Kant’s narrative of the “receptivity” of sensibility and the “spontaneity” of the understanding. Synthetic activity thus somehow “produces” mathematical objects of higher order. They are not “encountered” as ideal objects in the world of ideas.

What is the consequence of this acknowledgment that the ontic weight of mathematical phenomena resides in the sense of the relation [Bezugssinn]? Becker notes that the relation [Bezug] as such is never ontically independent. What provides it with its facticity is always the completion [Vollzug] of mathematical synthesis (syntaxes). This simple observation has an astounding use within the ontological controversy between formalistic and intuitionistic definitions of mathematical existence; for the ontic force of the mathematical has to reside in the completion of syntaxes, which have to be factual [faktisch] in the strict sense of the term, or in other words, which have to be effectively completed. However, “transfinite” syntaxes cannot be so. Hilbert’s transfinite axioms articulate the need for syntheses that are factually unattainable, such as Cantor’s continuum.

Becker concludes from this that “phenomenological analysis as Hermeneutics of the Dasein settles the controversy about the definition of mathematical existence in favor of intuitionism.” For the intuitionistic requirement according to which all existing mathematical objects should be able to be “presented” [dargestellt] in a construction that is both in concreto and de facto attainable (almost in the sense of the chemical “synthesis” [Darstellung] of a pure body) does not imply anything more than this postulate: “All mathematical objects should be able to be obtained via factually ineffectual synthesis”; which furthermore means that “authentic (existing) mathematical phenomena ‘are’ only to be found among factually attainable synthesis.” Mathematics thus acquires an anthropological foundation, but with this quasi-operative meaning of concepts, mathematical phenomenology reveals itself as compatible with a kind of “operationalism” as it was defined by Bridgman in the 1920s; and it is this double heritage, both operationalist and phenomenological, that is characteristic of Felix Kaufmann’s philosophy of mathematics.

24.5. Felix Kaufmann: phenomenology and logical empiricism (1930)

Becker’s criticism finds an interesting development in Felix Kaufmann’s Das Unendliche in der Mathematik und seine Ausschaltung, published in 1930. In its Chapter 2, “Symbolism and
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Axiomatics,” Kaufmann maintains a double thesis which seemingly contradicts Hilbert’s theory of proof, while he nevertheless considers this theory as the most important discovery in the research on the foundation of mathematics.

According to Kaufmann, the concept of a “senseless sign” is a contradiction in terms, because the assertion according to which visual or sound phenomena are “signs” already contains the assertion that through these signs something can be understood, that through these signs the thought of others can be grasped; which corresponds with what Carnap calls the “expression relation” [Aussdrucksbeziehung],47 or, in other words, the relation that finds itself at the foundation of the inference through which something psychic is deduced from a physical process. For instance, a physical movement expresses a state of the psyche. It is, however, not necessary that every single visual or sound phenomenon that is temporally or spatially independent has an independent sense. It is even possible that the sense does not increase until a series of psychic phenomena are somehow connected. It is not totally rigorous, in that case, to use the expression “dependent signs,” because there is no sign, in the proper sense of the word, until this relation happens, and thus to which the “meaning relation” happens. These should rather be described as “incomplete symbols,” an expression that has been accredited by Russell and Whitehead’s Principia Mathematica: “By ‘incomplete’ symbol we mean a symbol which is not supposed to have any meaning in isolation, but is only defined in certain contexts.”48

Thanks to its sense—in other words, thanks to its connection with thought, which is Kaufmann’s second thesis—a sign is indirectly connected to the thing that constitutes the object of thought. Consequently, it is essential that each sign should mean something; in other words, that it should precisely mean what constitutes the object of thought it expresses. This is what corresponds to what Carnap calls the “designation relation” (Zeichenbeziehung), in other words, the relation that “holds between those physical objects which ‘designate’ and that which they designate.”49 For instance, the written sign “Rome” designates the city of Rome. Yet, because any object of conceptual knowledge could, in principle, be designated in any arbitrary way, whatever the category it fits into, it belongs to “the converse domain of the designation relation.”

These two points seem to contradict Hilbert’s theory of proof, whose fundamental idea, as we saw in the second section, was that “all the propositions that constitute mathematics are converted into formulae, so that mathematics proper becomes an inventory of formulae. These differ from the ordinary formulae of mathematics only in that, beside the ordinary signs, the logical signs → (implies), & (and), γ (or), (not), (x) (all), (Ex) (there exists) also occur in them.”50

Hence the “critical correction” [kritische Berechtigung] suggested by Kaufmann, which is not directed against the theory of proof as such, and about which Kaufman asserts, with a slight lack of caution, that its very important mathematical and epistemological scope “will become increasingly clear in years to come.”51 But against Hilbert’s and Bernays’ philosophical interpretation of their own theory, he states: “Whenever concepts are missing, a sign is introduced at the right moment. This is the methodological principle of Hilbert’s theory.”52 Kaufmann sees this observation as typical of the kind of blindness that can plague scientists when they offer spontaneous epistemologies of their own scientific practice. As a matter of fact, according to this interpretation, proofs are intuitively given figures totally lacking in meaning. Yet, the rules that preside on the use of signs in the formulation of “figures of proofs” themselves contain the sense that belongs to logical transformations as such. This appears very clearly, when considering that, in the proof figures, different groups of signs are also used. Individual and variable signs are thus definite, but also the types of individual signs (the signs 1, +, which constitute parts of numbers; the signs =, ≠, >, which are the mathematical signs of the formal structure; the sign →, which is a logical sign) and also the types of variable signs (basic variables a, b, c, etc.; variable functions f(*), g(*); variable formulae A, B, C, etc.).
Hilbert’s theory of proof represents, according to Kaufman, the most radical formalizing project that can be conceived. Thus, if therefore we accept that the signs and formulae of the theory of proof have a sense, we have also to accept that the implicit definitions upon which the axiomatic system of Euclidean geometry is based have a sense too. The axioms of geometry are thus to be understood as judgments about particular arithmetical or logical relations between random objects, or in other words, that they are “a logico-arithmetical (relational) schema that can be variously filled in by intuitive or pseudo-intuitive objects.”

There are, within Hilbert’s axiomatization of geometry, exactly three systems of things between which a specific relation of order etc. is specified; here is what constitutes the semantic concept of the axiomatic system. These questions lead us to the issue of the phenomenological elucidation of material mathematics. However, we will not tackle them in the context of this presentation, whose aim was only to present the phenomenological discussions of the 1920s on the foundation of formal mathematics.

By April 1933, the “συμφιλοσοφεῖν” had been washed away by history. Husserl had been removed from his University position and was more alone than ever, relegated to a “spiritual ghetto” where the racial laws had excluded his works from the spiritual history of Germany, like “a poison from which we need to be protected and that needs to be extirpated.”

Notes

1 Mancosu 2005, 229.
2 On the “συμφιλοσοφεῖν” see Hua-Dok III/3, 457; and his Letter to Felix Kaufmann, 16. XII. 1931 in Hua-Dok III/3, 187. On the “great disappointment,” see Husserl’s letter to Mahnke, 8. I. 1931 (Hua-Dok III/3, 473). Translations of all texts are mine unless otherwise noted.
3 Hua XXXII 30–65.
4 Hua XXXII, 31.
5 Becker 1927, 622.
6 See the draft classification elaborated by Becker 1923, 388–396.
7 Hua XXXII, 35.
8 Hua XXXII, 42. On the distinction between abstract sciences and concrete sciences, Hua III/1 §72.
9 Cf. Hua III/1, 150/161.
10 Becker 1923.
11 Mahnke 1923, 34; Mahnke 1977, 75.
14 Hilbert 1899; Hilbert 1902. On the axiomatization of Euclidean geometry, see Geiger 1924.
15 On the proof of consistency of the axioms of the elementary theory of radiation, and their compatibility with the elementary laws of optics, see Hilbert 1914, 275–298; Hilbert 1935, 252–257.
16 Padoa 1903, 85.
17 Padoa 1903, 90.
18 Hilbert 1905, 176; Hilbert 1967a, 131.
19 Cf. Poincaré 1906, 17.
22 It is not a “formula” in the narrow sense of formalized mathematics; but it could be said, in Paul Bernays’ words, that it is a “formula with meaning” if we give a broad sense to this term.
24 Cf. Frege 1903.
26 Mahnke 1923, 35; Mahnke 1977, 79.
27 Hua XVIII, 251/157.
28 Mahnke 1923, 36; Mahnke 301977, 80.
29 Hua XIX/2, 667/346.
We do not mean by this a philosophy of mathematics that could be reconstructed on the basis of various analyses spread through Heidegger’s works. For a project of that kind, see Souan 2006. On what we call here the “Heideggerian point of view” in philosophy of mathematics, see Gethmann 2002; Gethmann 2003.

See on this point Mahnke’s criticism in his Letter to Becker, 8. IX. 1927; see Peckhaus 2005, 262 and Mancosu 2005, 238–239.


On the distinction between “expression” and “meaning,” see Carnap 1928, 24; Carnap 2003, 33.

Russell and Whitehead 1910, 69.


Kaufmann 1930, 47; Kaufmann 1978, 46.

Bernays 1922, 16; Bernays 1998, 215–222.

Kaufmann 1930, 49; Kaufmann 1978, 47.

Husserl’s Letter to Mahnke, 4/5.V. 1933 (Hua-Dok III/3, 491–492).

References


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