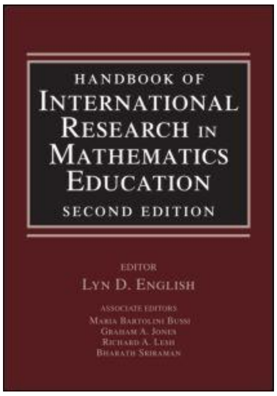


# Handbook of International Research in Mathematics Education

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## Perspectives on representation in mathematical learning and problem solving

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# 9 Perspectives on representation in mathematical learning and problem solving

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> Representation is a crucial element for a theory of mathematics teaching and learning, not only because the use of symbolic systems is so important in mathematics, the syntax and semantic of which are rich, varied, and universal, but also for two strong epistemological reasons: (1) Mathematics plays an essential part in conceptualizing the real world; (2) Mathematics makes a wide use of homomorphisms in which the reduction of structures to one another is essential.
>
> Vergnaud, 1987, p. 227

This chapter is an update of my contribution to the first edition of the *Handbook of International Research in Mathematics Education* (Goldin, 2002a). It incorporates some additional aspects of representation and related themes that have advanced during the past five to seven years. These include "models and modeling" perspectives on mathematics education (Lesh & Doerr, 2003; Lesh, Hamilton, & Kaput, 2007), and an expanded discussion of the affective domain from the standpoint of representation and systems of representation (Gómez-Chacón, 2000a, 2000b; Malmivuori, 2001; DeBellis & Goldin, 2006). The earlier chapter has also been reorganized, placing the discussion of epistemological "paradigms" after that of representation, and incorporating a number of additional remarks.

The first section, adapted from the previous edition, mentions briefly some motivating issues related to the democratization of access to mathematical understanding and skills. It poses the questions of how to characterize the understandings and skills that we should seek to achieve, for whom we should seek to achieve these goals, and with what methods and tools. The section immediately following is devoted to basic concepts in the theory of representation, which are regarded as forming part of the basis for essential methods and tools.

The third section highlights some aspects of models and modeling perspectives in relation to representation, and refers the reader to a few recent sources. This is followed by the previously-published section on abstraction and contextualization in the learning of mathematics, making reference to issues of democratization of access raised earlier. The fifth section addresses affect and representation in mathematical learning and problem solving more extensively, pointing to some new directions.

The chapter ends with the discussion, slightly modified from the previous edition, of current, often-conflicting research "paradigms" and epistemologies, and the adverse consequences that dismissive epistemologies have had for mathematics education. For reasons that become evident, the study of representation does not fit neatly into any of these paradigms, but contributes to an eclectic, potentially unifying research framework.

## DEMOCRATIZATION OF ACCESS



Because mathematics addresses the systematic description and study of pattern, it is not surprising that the world of mathematics opens onto so many other worlds. Most people know generally that mathematics incorporates logical and precise reasoning, and appreciate at least some of its concrete, everyday uses. Many see that it offers a cross-cultural language and set of tools vital for the natural sciences, engineering and technology, economics, business and finance, and that its methods also find application in psychology, the social sciences, law, medicine, and a host of other professions and activities. It is common wisdom that the level of mathematics achieved during school years has enduring consequences, facilitating or impeding lifelong ways of understanding, learning, and communicating. Thus the idea that broad, democratic access to mathematical knowledge serves a valuable social purpose—that of opening the doors of experience and economic opportunity—is not very controversial.

However the *nature* of mathematical power, and the extent to which it is widely achievable, are not so generally agreed on. We lack consensus on the *processes* through which mathematical understanding develops, and on the characteristics of the methods and tools that can most effectively foster that development. As an educator and researcher, my goal is to render accessible the abstract ideas, language, and algebraic and geometric reasoning methods of mathematics, *as well as* everyday computational skills and their applications, to the large majority of learners. But some would see this goal as fundamentally impossible, citing wide ability differences among individuals or populations. In that view, differences in mathematical ability pose an insuperable barrier—what most students are capable of learning is computational arithmetic, consumer mathematics, and possibly elementary algebraic manipulations, when these are simply taught and well drilled; with only a minority able to attain real understanding of algebra and geometry. Some would further characterize the objective of broad accessibility as diametrically opposed to the aim of enabling talented, high-achieving students to accomplish the maximum possible for them. Others would maintain that computational fluency is a *prerequisite* to more abstract understandings, so that the best path toward democratization of access is one that focuses from the beginning on "basic skills" learned in context, especially with less-capable students.

Many educators whose values lead them to embrace the ideal of near-universal access remain at a loss as to how to achieve it. At the least it is an ambitious and difficult undertaking, one that requires sound models at mathematical, psychological, sociocultural, and political levels, and a set of working tools based on them. If the quest is not futile, clear expressions of vision, theory, and method are needed.

Since the 1990s, the issue of democratization has been joined with others in an increasingly vigorous, sometimes rancorous, conflict surrounding public policy in mathematics education in the United States and some other countries.[1] Perhaps because it has an egalitarian, anti-elitist ring to it, the statement of a "universal access" goal is easily subsumed in belief systems and rhetorical and political frameworks that employ popular catchwords, but obscure the nature and central importance of the mathematical concepts, methods, and reasoning capabilities that constitute the very substance of the goal. The language that is used to describe mathematical learning and teaching itself entails assumptions that are increasingly treated as *ideological* rather than scientific. Then we confront the political debate that continues to surround "reform," without the objective, sufficiently-broad and persuasive research base that might resolve it.[2] And some of the influential "paradigms" (see below) still dismiss the very notion of "objectivity" in qualitative research (e.g., Guba & Lincoln, 2005).

In the year 2000, the National Council of Teachers of Mathematics (NCTM) recommended curricular goals for all students that placed emphasis on both content and process (NCTM, 2000), seeking to bridge the growing divide between "traditionalists" and "reformists." In fact, "Representation" is one of the five broad "Process Standards" included and elaborated in this document, and the NCTM devoted its 2001 Yearbook to this subject (Cuoco & Curcio, 2001).

In 2006, the NCTM published *Curriculum Focal Points*, a valuable source identifying those major content areas at each grade level (K–8) that are deserving of central emphasis (NCTM, 2006). But contrary to the stated intent of the NCTM, both the *New York Times* (a respected, politically liberal national newspaper) and the *Wall Street Journal* (a respected, politically conservative national newspaper), as well as other news media, represented the latter document as a major reversal in position by the organization (Fennell, 2006a,b). The press interpreted the NCTM's identification of focal content areas as a victory for the "traditionalists." Evidently, the basis for public understanding in the United States of mathematics curricular goals and of suggested methods for achieving them remains, at best, shaky—a situation for which the mathematics education research community must bear some responsibility.

The need is as pressing as ever for a shared, scientific, non-ideological framework for empirical and theoretical research in mathematical learning and problem solving, one that allows for clarity in expressing curricular goals and methods. The present chapter reflects my view that the constructs of *representation, systems of representation,* and the *development of representational structures* during mathematical learning and problem solving are essential components of such a framework. The fact that their understanding is greatly impeded by prevailing dismissive epistemologies adds to the many reasons for abandoning the latter.

To discuss representation, we must be able to consider at a minimum configurations of symbols or objects *external* to the individual learner or problem solver, configurations *internal* to the individual, *relations* between them, and *structures* within and across them. These basic notions are essential to characterizing the nature of the patterns that mathematics is about. They are likewise essential to a psychologically adequate formulation of what mathematical understanding consists of, and how individuals acquire it. Research on representation thus involves some external and/or behavioral variables, accessible straightforwardly to observation, together with other, internal constructs that require careful, often context-dependent inference. It can and should draw on both quantitative and qualitative research methods and assessment instruments, according to the desired purpose.

The study of representation in mathematical learning allows us—at least potentially—to describe in some detail students' mathematical development in interaction with school environments. It also allows us—at least potentially—to create teaching methods capable of developing mathematical power in the majority of students (e.g., Kaput, 1991; Kaput & Hegedus, 2004; Kaput, Lesh, & Hegedus, 2007). It provides a basis for the effective use of digital technology. It is thus an important tool in achieving wide access to mathematics through public education.

## SOME CONCEPTS IN THE THEORY OF REPRESENTATION

This section summarizes briefly some of the key ideas related to representation in the psychology of mathematics education (Janvier, 1987; Goldin, 1987, 1998, 2003a; Goldin & Kaput, 1996; Goldin & Janvier, 1998a,b; Vergnaud, 1998).

### Representational systems

In the most general sense, a *representation* is a configuration that can *represent* something else in some manner. For example, a word can represent a real-life object, a numeral can represent the cardinality of a set, or the same numeral can represent a position on a number line. The nature of the *representing relation* between the one configuration and the other must eventually be made explicit. Kaput (1998) terms this sort of definition (Palmer, 1978;

Kaput, 1985) an "abstract correspondence approach," in that we have (for now) left open the types of configurations we are discussing, the nature of the representing relation, and how it develops.

The representing configuration might, for instance, act in place of, be interpreted as, connect to, correspond to, denote, depict, embody, encode, evoke, label, link with, mean, produce, refer to, resemble, serve as a metaphor for, signify, stand for, substitute for, suggest, or symbolize the represented one. It might do one (or more) of these things by means of a physical linkage or a biochemical, mechanical, or electrical production process, or in the thinking of an individual teacher or student, or by virtue of the explicitly-agreed conventions or the tacitly-agreed practices of a social group or culture, or according to a model developed by an observer.

Rather than distinguish in some fixed and final way the world of representing configurations from that of represented configurations, the relation may frequently be seen as bidirectional. That is, when one configuration represents another, the latter can often be regarded equally usefully as representing the former. In mathematics, for instance, we may take a Cartesian graph as representing an algebraic equation (by depicting its solution set), or we may take the equation as representing the graph (by encoding a relation satisfied by the coordinates of its points).

Written words, numerals, graphs, or algebraic equations are examples of external representations. To be more precise, let us distinguish specific inscriptions of these that are found in books, or produced by individuals doing mathematics—that is, that can be observed and pointed to—from idealized representational configurations that describe socially agreed-upon norms. The latter may be thought of as equivalence classes of inscriptions.

What is the nature of the idealized configurations and the representing relations here? The configurations (e.g., algebraic equations) and relations (e.g., the relation between Cartesian graphs and algebraic equations) became established over a period of time, initially through individual inventions and eventually through shared conventions. These conventions became normative among those doing mathematics, and are today encoded (in ways that neuroscience is just beginning to address) in the brains of millions of people who have studied mathematics. This enables us to interact coherently with each other. To trace this in detail, it is important to have methodological tools for moving beyond external representations, to describe what individual students, teachers, or mathematicians are doing internally.

The examples mentioned (words, numerals, graphs, or algebraic equations) illustrate the idea that individual representational configurations rarely can be understood in isolation. Whether we are speaking of mathematical or nonmathematical representations, we find they belong naturally to wider systems, with internal structure. Numerals, for instance, belong to a system of base ten Hindu-Arabic notation, and Cartesian graphs to a system of conventions for associating pairs of numbers with points in the plane by means of orthogonal coordinate axes. Thus it is essential to define the notion of a representational system to which individual representations belong—indeed, to begin with the idea of the system.

## Primitive components

The building blocks, or primitive components, of a representational system form a class of *characters*. I use this terms when the intent is not yet to ascribe to them any further interpretation or representing relation. These may belong to a *well-defined* set, such as the characters in a system of symbolic logic, the letters in the Roman alphabet, or the bases in a molecule of DNA. We may also work with *partially-defined* or *ambiguously-defined* entities, such as real-life objects and their attributes, or spoken words in the English language. In the domain of mathematics, we may consider concrete characters such as numerals and arithmetic symbols, or abstract entities such as vectors; in physics, we have constructs such as velocities or forces, that (later) vectors may be taken to represent.

### Configurations

A representational system further includes ways of combining the elementary signs into *permitted configurations*. These may be specified by well-defined rules, such as those for creating well-formed-formulas (wffs) in a symbolic logic, or reasonably well-defined lists, such as written words in standard English dictionaries; or they may again be ambiguously-defined, such as arrangements of real-life objects, or grammatical sentences formed from English words. Single-digit numerals may be used to write multi-digit numerals; numerals and operation signs may form mathematical commands or mathematical equations; and so forth. But we still have not discussed the *interpretation* of such configurations.

### Structures within representational systems

Typically, representational systems have higher, more complex structures—such as networks, configurations of configurations, partial or total orderings on the class of configurations, mathematical operations, logical or natural language rules, production systems, and so forth. Rules for moving from one configuration to another, or one set of configurations to another, may create a directed graph structure. Rules of grammar and syntax permit words, designated as parts of speech, to be combined into sentences. Again, we have the possibility of ambiguously-defined structures. In formal logic, inferencing rules permit us to obtain theorems from previously established wffs. Symbol-manipulation rules in algebra or calculus allow us to obtain new formulas from previous ones, or to transform and solve equations.

One sense in which we may speak of the *meaning* of a representational system's characters and configurations is with reference only to structures within the system. This is illustrated by an example familiar from elementary logic, in which the characters read as "and," "or," and "not" are taken as undefined, acquiring meaning exclusively through the axioms and inferencing rules that combine them in certain ways. This is a *syntactic* or *structural* notion of meaning. It complements and contrasts with the *semantic* notion, where the meaning of a representational system's characters and configurations inheres in the things they signify *outside* the system.

### Conventional vs. objective characteristics of representational systems

External representational systems for mathematics, from logical systems described by axioms and theorems to notational systems for arithmetic, algebra, calculus, and so forth, begin with shared assumptions and conventions (such as the axioms defining an Abelian group or a vector space, or the conventions for constructing graphs in Cartesian coordinates). Such systems are structured by their underlying conventions, and when we consider these to be used "correctly," we are referring to conformity with conventional norms. For instance, at the elementary school level, there is nothing "objectively true" about the fact that an expression such as $3 + 4 \times 5$ is evaluated by performing the multiplication before the addition, and not by performing the addition first. It is a matter of commonly agreed notation, open to inventive modification.

On the other hand, once a mathematical system with its rules has been established, the patterns in it are no longer arbitrary. There is a very important sense in which they are now *present to be discovered* in the system. Having assumed the conventional properties of natural numbers, our base ten notational system, the conventional definitions of addition and multiplication, and the conventional definition of a prime number, it is *true* that 23 is a prime while 35 is not. We invoke here no metaphysical or Platonic notions of "absolute truth." Rather we highlight the important and elementary mathematical distinction between that which is conventional, and that which is (objectively) no longer so, once the context of mathematical assumptions is established.

Although the mathematical representations we know have originated with human beings, there is no a priori persuasive argument eliminating the possibility of other intelligent life in the universe developing recognizably similar mathematics in representation of similar external, real-world patterns.

Furthermore, representational systems are here defined quite generally, so that they need not be systems where human beings have invented the configurations or established the representing relations. For hundreds of millions of years, the sequences of base pairs in DNA have encoded in a complex way the amino acid sequences that form protein molecules. Not only protein structures, but the phenotypes of organisms, are represented in DNA through subsequent productions. Human scientists have *discovered* the patterns, and are breaking the code, but this should not obscure the important sense in which the biosystem evolved representational capabilities *apart* from subsequent human knowledge and description of it.

**External and internal representation**

To this point our examples have mostly been systems of representation (including idealized, socially-constructed systems) external to individual learners or problem solvers. Now we want to consider the internal, psychological representational systems of individuals. Such internal systems include their natural language, personal symbolization constructs, visual and spatial imagery, problem solving heuristics, affect, and so forth. Let us consider how these may be understood in relation to that which is external.

Evidently, I cannot under normal circumstances observe the internal representations of anyone else directly. Even the extent to which introspection permits me to describe my own internal representations is questionable. The latter is best regarded as an empirical issue, to be investigated through research. Rather, the idea that individuals have internal systems of representation is an *explanatory theory* framed at a certain level of description. We are to infer such representation from what individuals do, or are able to do, under varying conditions—i.e., from their observable behavior, which may include interactions with observable external representations in their environments.

For example, observation of grammatically consistent spoken English conversation leads us to infer some internally encoded, structured *competencies* forming a (difficult-to-describe) internal system of language representation. The individual may be able to articulate some aspects of this system through conscious introspection (e.g., she may explain how certain words are used, or why they are used in certain ways). Other aspects, though quite stable, are likely to be inaccessible to such introspection (e.g., the native speaker may not be consciously aware of the grammatical rules she uses, or able to express them). The term *internal representation* as I use it is thus not at all synonymous with an individual's "world of experience" or "experiential reality," as radical constructivists employ these terms.

Some sources use the expression "mental representation" in a way that seems more-or-less in agreement with what is meant here by "internal representation." But to avoid misunderstanding, I want to stress that I am not suggesting—even tacitly—any sort of "mind-body dualism" (cf. Kaput, 1998, p. 267). My expectation is that internal representations are encoded physically. However, the more reductionist description at the level of neurons and their interaction in the brain is not yet known in detail, nor is it clear how such a level of description would be helpful to mathematics educators.

The creation of shared, conventional (external) representational systems is an important thread in the history of mathematics. Most mathematics teaching involves students learning to interpret such systems and using them to solve problems. Some are mainly notational and symbolic, while others display relationships visually or spatially. Though external mathematical configurations have traditionally been (mostly) static, calculator and computer technology now links them and allows them to change dynamically (Kaput, 1994). But the formal symbolic notations of mathematics, the visual/spatial number lines, complex planes, graphs, and Venn diagrams, the perceived computer-based micro-worlds, and so forth, are also represented and processed internally. It is the internal level that largely determines the usefulness of such external representational systems, according to how the individual understands and interacts with them.

Thus effective teachers *continuously make inferences* about students' internal representations, their mathematical conceptions and misconceptions, based on their interaction with or production of external representations. Sometimes one considers the external to represent the internal (e.g., when a student expresses a relationship he has is mind by drawing a graph). At other times, or even simultaneously, one can take the internal to represent the external (e.g., when a student visualizes what is described by a graph or by an algebraic formula). This again exemplifies the bidirectional perspective mentioned above—and, of course, we must be as specific as possible about the direction and nature of the intended representing relation.

An extremely important aspect is that internal configurations of different kinds can *represent each other* in many different ways (Goldin & Kaput, 1996). An internal visual/spatial image may, for instance, evoke an internal formula-configuration, some kinesthetically-encoded action sequences, a problem-solving strategy, verbal phrases, feelings of comfortable familiarity or anxiety, and so forth. One way to explore what is involved in a student's understanding of a mathematical concept is to consider the variety of distinct, appropriate (or inappropriate) internal representations she has formed, and try to describe and analyze the representing relations she has developed.

## Interacting internal representational systems

To characterize the complex cognitions and affect of individuals, one needs a model or framework that permits the description of internal signs, internal configurations, and higher-level internal structures of different kinds. Often it is a matter of convenience whether we choose to regard some such system as a single, fairly complex representational system (i.e., having much internal structure), or to see it as comprised of two or more simpler systems with representing relations among them.

## Types of internal systems

Elsewhere I have described in more detail a model based on five types of mature systems of internal representation (Goldin, 1987, 1992, 1998). This framework was developed as a way to characterize problem-solving competency in mathematics, and has also proven useful in the study of learning and conceptual development. It connects in obvious ways to the work of others who have focused in depth on just one or two types of representation, or who have focused on learning and problem solving in particular mathematical domains.

My viewpoint is that all five need to be taken as psychologically fundamental, extending earlier "dual code" and "triple code" models (Paivio, 1983; Rogers, 1983; Zajonc, 1980). We have: (1) *verbal/syntactic* systems, that include natural language capabilities—lexicographic competencies, verbal association, as well as grammar and syntax; (2) *imagistic* systems, including visual/spatial, tactile/kinesthetic, and auditory/rhythmic encoding; (3) *formal notational* systems, including the internal configurations corresponding to learned, conventional symbol-systems of mathematics (numeration, algebraic notation, etc.) and how to manipulate them; (4) a system of *planning, monitoring, and executive control* that guides problem solving, including strategic thinking, heuristics, and much of what are often referred to as *metacognitive* capabilities; and (5) an *affective* system that includes not only the "global" affect associated with relatively stable beliefs and attitudes, but also the changing states of feeling as these occur during mathematical learning and problem solving. The characterization of *affective structures* is emerging as an important way to help understand students' mathematical engagement and motivation (see below).

Relations of meaning and symbolization among internal configurations of different kinds relate these systems to each other in complex ways. That is, the various systems are to be regarded not as separate and isolated, but as continually interacting. These internal relations, together with denotative and interpretative relations between internal and external representations, encode the mathematical meanings of the individual's cognitive and affective activity.

Until relatively recently, the most neglected of these systems by researchers in mathematics education were the imagistic and the affective; see English (1997), Presmeg (1998), Goldin (2000, 2002b, 2007), Gómez-Chacón (2000a, 2000b), Malmivuori (2001), DeBellis & Goldin (2006), and references therein.

## Stages of development

Representational systems are not transcribed from outside into human brains like programs being loaded into computers. Over time, they *develop* in learners, structured by the presence of prior systems. It is here that processes of *construction* of knowledge become especially important.

The broad model I bring to such development incorporates three main stages, applicable to each system (and often, to subsystems): (1) an *inventive/semiotic* stage, in which new internal configurations are constructed and first assigned meaning (Piaget, 1969) with reference to previously established representations; (2) a period of *structural development,* driven by the meanings first assigned, during which the higher structure of the new system is largely built with the earlier system serving as a template; and (3) an *autonomous* stage, in which the new representational system detaches partly or even entirely from its previously essential relation to the earlier system(s), and functions flexibly and often powerfully with new or more general meanings in new contexts.

## Representation, pattern, and communication

The word *pattern,* describing the fundamental object(s) of study in mathematics, is already strongly suggestive of some sort of representation. There is a sense in which patterns may be said to "exist," apart from particular individuals who may detect them or know them (or who, alternatively, may not notice them). We are then speaking of representational structures that are external to the individuals. I still use the word *representational* here because the pattern has the *capability* of evoking, and standing subsequently in a certain sort of meaningful relation to, corresponding internal configurations. This contingency is present when a pattern exists, even if it does not always happen:

> … we may say, 'Mathematics is the classification and study of all possible patterns'. Pattern is here used in a way that not everybody may agree with. It is to be understood in a very wide sense, to cover almost *any kind of regularity that can be recognized by the mind.* [emphasis in original] … A bird recognizes the black and yellow bands of a wasp; man recognizes that the growth of a plant follows the sowing of seed. In each case, a mind is aware of pattern. (Sawyer, 1955, p. 12)

There is another important sense, though, in which it is the human individuals or communities of individuals (or "minds") that *invent* the patterns, or construe them in or impose them on their ongoing experiences of the world. Then we are speaking of representational structures internal to individuals, in meaningful relation to the external.

How does such meaningful relation come to be powerful? This is the fundamental question we face as mathematics educators. Seminal work in our field has been based on the idea that children's mathematical ability can be developed *through appropriate interactions with well-designed, carefully-structured task representations embodying the desired patterns* (Bruner, 1960, 1964; Davis, 1966, 1984; Dienes, 1963; Montessori, 1972). Then the mathematical development of the individual takes place through the construction of internal representational systems of the types described above, together with multiply-encoded cognitive/affective conceptual schemata across the different systems.

Mathematical power consists not only in being able to detect, construct, invent, understand, or manipulate patterns, but also in being able to communicate them to others. Thus we can understand mathematics as language, and look at the development of the various types of internal representational systems expressive of mathematics as language learning—that is, occurring through participation in communication, and having structural (syntactic) aspects and representational (semantic) aspects.

Each of the five types of internal systems of representation mentioned above permits the individual to produce a vast array of complex and subtle external configurations that other people interpret meaningfully: (1) spoken and written language; (2) iconic gesture, drawing, pictorial representation, musical and rhythmic productions; (3) mathematical formulas and equations; (4) expressions of goals, intent, planning, decision structures; (5) eye contact, facial expressions, body language, physical contact, tears and laughter, and exclamations that convey emotion. The richness of the resulting communication is what makes the complexity of human social interaction possible.

Thus we have, at least potentially, consilience of the psychological level of description with the sociocultural level, as well as with the neurobiological level.

### Ambiguity and representation

With certain exceptions, *ambiguity* may be a necessary feature in the characterization of a representational system, or its relation to another system. When ambiguity is present in spoken language or in mathematical communication, contextual information is frequently needed to resolve it. Often this requires that one go *outside* the original system—in practice, we interpret uncertain mathematical expressions, diagrams, problem statements, and so forth when we have information about the objects and context to which they refer. Furthermore, ambiguity in the relation between two representational systems is sometimes resolved with reference to yet a third system.

In mathematics we are used to improving the power of our reasoning by reducing, or eliminating as far as possible, ambiguities in our formal representations. Thus we typically strive for great precision—careful definitions and statements of assumptions, unambiguous notations, and rigorous and detailed proofs. Paradoxically, among the representational systems we are discussing are some whose very power and flexibility seem to depend essentially on the *presence* of ambiguity. Words in natural language that are highly ambiguous out of context, convey meanings flexibly and powerfully in a variety of different contexts. Heuristic processes, problem solving strategies, or "critical thinking" techniques, highly structured and powerful in the individual, may require considerable contextual input before they "make sense" in any given situation. Even greater ambiguity, and greater power, may be associated with individuals' internal emotional states (see below).

## MODELS, MODELING, AND REPRESENTATION

A *model* is a specific structure of some kind that embodies features of an object, a situation, or a class of situations or phenomena—that which the model *represents*. The term *modeling* refers to the construction of models—of meaningful structures, within one or more representational systems (possibly mathematical, possibly physical or iconic, possibly digitally-encoded and dynamic). When appropriately interpreted, the model describes some but not other aspects of the relevant situations or phenomena; hence the central importance of the *meaningfulness* of the representations within which the model has been constructed.

The book edited by Lesh and Doerr (2003) develops a number of important aspects of models and modeling theory in relation to mathematics education. One of these is the view of a model as being both internal and external, reflecting many of the ideas in the preceding discussion of representation and representational systems. Models are seen as generally *distributed* across different representational media. The model is created through *model development sequences* that can be characterized and studied. When problem solvers engage in modeling activity, there typically occur multiple *modeling cycles* that can be characterized and observed, during which *local conceptual development* takes place.

These ideas suggest an explicit focus in mathematics education on *model-eliciting activities,* as distinct from traditional classroom mathematics problems.

The theory is advanced considerably in the recent volume edited by Lesh, Hamilton, and Kaput (2007). For instance Thompson and Yoon propose a taxonomy of situations leading to the need for a model, and examine *modeling ability* and how it can be developed (Thompson & Yoon, 2007; Yoon & Thompson, 2007). Kaput, Hegedus, and Lesh (2007) address the developing technological infrastructure and its interaction with mathematics education, while Hamilton, Lesh, Lester, and Yoon (2007) consider the development of personal problem-solving models in individuals.

A related direction is the study of how individual learners *develop* and *express* particular mathematical ideas through representation. Monk (2003), for example, considers in detail how graphical representation "makes meaning" for students. Goldin and Shteingold (2001) consider young children's developing cardinal and ordinal representations of negative numbers. The related literature is vast, and the notions of representation, representational systems, external and internal representation, structures within representational systems, model-development sequences and modeling cycles, and so forth, all serve as ingredients of a unifying conceptual framework.

## ABSTRACTION, CONTEXTUALIZATION, REPRESENTATION, AND COGNITIVE OBSTACLES

With the above ideas in mind let us consider an alternate way to frame just one of the issues in the continuing debate over mathematics reform, the question of *formal* or *abstract* mathematics (valued for its power in the "traditional" view) vs. mathematics *in context* (valued for its meaningfulness and relevance in the "reform" view).

### Contextualized understanding

Let us try to understand the characteristics of in-context mathematics, or more precisely of *contextualized understanding* of mathematics, from a representational perspective. Familiar contexts are encoded internally as representational configurations in common words, images, formal notations, strategies and operations, and (ideally) comfortable affect. The familiar, or "common sense" nature of the internal structures—expectations, contingencies, beliefs, as well as competencies—associated with such a context (Goldin, 1996, 2003a) means they are likely to be: (1) widely shared, (2) based on everyday experiences that are easily referred to, (3) multiply coded, in highly redundant ways, (4) developmentally prior to the mathematics being learned in the given context, and (5) culturally encouraged or reinforced. Then these internal structures serve as the "templates" for the construction of in-context mathematical representations, which may reasonably be said to encode *contextualized* understandings.

**Example: the "unknown" in algebra.** For students beginning the study of algebra, the notion of a collection of objects is familiar from experience. It is straightforward to develop the idea that one might have such a collection, for instance a bag of peanuts, and not know how many objects are in it, perhaps because the bag is closed and opaque, and the peanuts haven't been counted. The construct "an unknown number of peanuts" can thus be visualized, and the action sequence of opening the bag and counting the peanuts imagined. There are many wider contexts in which such a situation might be set. We now have the possibility of introducing the letter "x" to *stand for* this *specified, but unknown, number*. The students engage in the semiotic act of taking the prior, contextual representation (of the result of the imagined action sequence of counting the peanuts) to be the *meaning* of the character "x" in the representational system of formal algebra.

Evidently, with this representing relation established in the concrete context, quite a few algebraic expressions involving arithmetic operations can be written. Their interpretation makes sense with respect to the contextual template. Thus x + 5 means the result of counting the peanuts and adding 5 more, while 6x refers to the number of peanuts in 6 identical bags, and so forth. Another letter, "y", can stand for a different unknown number of not-yet-counted objects, such as raisins in a box.

The verbal descriptions provide another encoding, increasing the redundancy. Concrete objects, such as might be used with younger children, would serve as an external representational system for connection with these constructs.

Since all this is occurring during the inventive/semiotic stage, where meanings are initially assigned, it is likely that students taught thus will come to understand "the value of an unknown number," encoded in multiple ways, as *the real meaning* of "x" in algebra, or *the one meaning that is easy to understand,* or even *the only meaning that is possible,* at least for a period of time. That is, "x" and "y" always *stand for* numerical values (their actual values); they *must* do so; we just don't know what these values are. The algebraic understanding to this point is entirely in context.

### *A cognitive obstacle*

Eventually, it will be important to *abstract* from the initial meanings. A small, straightforward abstraction is to see "x" and "y" as symbols that could also stand for other specific, unknown values, in other concrete contexts (not just a whole number of peanuts in a bag or raisins in a box). We anticipate no important difficulty in this step. But in developing a powerful algebraic representational system, the students at some point need to interpret the letter symbols as *variables*. That is, "x" no longer will stand for a specific unknown number, but will be able to flexibly assume *any* of the values in some numerical domain. The contextualized understanding is likely to make this cognitive representation quite problematic, since the actual value of "x" (which, multiply encoded, served as its semantic interpretation) has disappeared entirely. The context now can result in a *cognitive obstacle* to the more abstract mathematical understanding. It is constraining the desired representation, and a dramatic breakthrough is needed.

This pattern, where the contextualized representations *first assist and then constrain* the subsequent cognitive development, is quite common in mathematics.

## Decontextualized representation

One "reform" trend associated with radical constructivist methods has been toward teaching most or all mathematics by fostering students' in-context reinvention of every mathematical concept. The contextualized mathematics is romanticized, and the abstract devalued. This is, in my view, an unfortunate but understandable reaction against the widespread tendency toward teaching mathematics as *decontextualized representation*.

I suggest the latter term to describe formal mathematical notations and rules of procedure introduced as syntax without semantics, or rules and methods without context—a practice seen often in traditional teaching. The *good intention* behind such decontextualized representation is to avoid the contextual constraints, to *embody* that which is abstract in mathematics. But at best, the result is likely to be the construction of an internal, formal system lacking important semantic connections.

The student may, for instance, learn to "move the 'x' to the other side of the equation and give it a minus sign," without understanding what such a step means, why it is valid, or what it accomplishes. The procedure is formal. The period of structural development for the algebraic notational system with accompanying operations, based on a meaningful representational relation with a prior system, has been bypassed. The student may or may not learn to do some algebra in the form of school exercises (i.e., in the original decontextualized format in which the algebra was practiced). But the system might never come to function flexibly and autonomously, as a bona fide abstraction.

### Abstraction and contextualization processes

Decontextualized representation is not the same as abstraction. To emphasize the limitations of decontextualized mathematics is *not* to insist that all mathematics be "in context," especially when the contexts are those that will pose natural obstacles to later abstraction.

The *process* of abstraction is one that involves reaching the autonomous stage in the functioning of a representational system. This can occur *after* relations with prior systems (involving some context or contexts) have been established through semiotic acts, and *after* some structural development of the new system. As starting points, we should use those representational contexts that permit maximum ease of structural development, and limit our reliance on those that impose the most difficult constraints. Since most initial contexts eventually create some cognitive obstacles, the process requires the progressive detachment of representations from their initial contexts as structure is built.

*New* semiotic acts then permit the "same" familiar representational configurations to acquire new meanings in new semantic domains. This is the process I would like to call *contextualization*. It is a kind of complement to the abstraction process, and in my view equally important to powerful mathematics. Through contextualization, students learn to construct special cases, to "see the particular in the general," to move toward the concrete in a new representational situation, and to take these steps spontaneously and flexibly. Through abstraction, they learn to generalize, to "see the general in the particular," to move away from inessential details of the concrete, representational situation, and to do these things also spontaneously and flexibly.

In short, the representational perspective permits us to relinquish the idea that "mathematics in context" is somehow the opposite of "formal, abstract mathematics." Instead, we identify abstraction and contextualization as complementary representational processes. Both are essential to depth of understanding in mathematics, and developing both in students should be our goal as mathematics educators.

Now let us refer back to our discussion of democratization of access. The goal of developing mathematical proficiency in the large majority of students needs to be formulated in a way that does not focus on decontextualized skills, but on the power of abstract mathematics—i.e., within each mathematical topic, we seek to develop representational structures that promote fluency in the processes of mathematical abstraction and contextualization. Then, the power of mathematics as an abstract discipline becomes accessible, together with the psychological empowerment that comes from working in context with insight and understanding.

## AFFECT AND REPRESENTATION

Both mathematics education researchers and cognitive scientists are highlighting the importance of the affective domain (Evans, 2000; Malmivuori, 2001; Leder, Pehkonen, & Törner, 2002; Dai & Sternberg, 2004; Hannula, 2002, 2004; Lesh, Hamilton, & Kaput, 2007). While much of the research has focused on individual expressions of affect during mathematical problem solving (e.g., Zan, Brown, Evans, & Hannula, 2006), the most recent efforts in which I have been involved focus on the affect of children engaged in conceptually challenging mathematical activity in the context of urban classrooms (Schorr & Goldin, in press; Goldin, Epstein, & Schorr, 2007).

The view I have taken, that affect should be regarded as an internal *representational* system, is not such a common one. Usually emotion is seen as a concomitant of cognitive processes. It is of course recognized that the individual's emotional state can enhance cognition (e.g., through mathematical curiosity) or impede it (e.g., through "math anxiety"). But to see affect as representational is to hypothesize (with considerable empirical support) that emotions encode information—detailed, context-dependent, rapidly-changing information essential to the doing of mathematics (as well as other human activities, of course). Speaking colloquially, feelings have meanings—sometimes fleeting, transient meanings, and sometimes deeper, more enduring ones.

Expanding on McLeod's pioneering work (McLeod, 1989, 1992, 1994), DeBellis and I consider four components of the affective domain—emotional feelings, attitudes, beliefs, and values. We have pursued the perspective that affective states involve complex structures, and that these include *meta-affect* (i.e., affect about affect, or affect about cognition about affect, or the monitoring of affect). We focus considerable discussion on the constructs of *mathematical intimacy* and *mathematical integrity*, in connection with students' engagement in problem solving (DeBellis & Goldin, 1997, 1999, 2006). Many researchers have studied *beliefs*, *mathematical (self-)identity* and *mathematical self-efficacy* (e.g., Nosek, Banaji, & Greenwald, 2002; Leder, Pehkonen, & Törner, 2002; Stevens, Olivarez Jr., Lan, & Tallent-Runnels, 2004; Anderson & Gold, 2006), which may likewise be regarded as long-term, internal affective/cognitive structures. Meta-affect is crucial to understanding these structures from a representational perspective. It can transform the experience of emotional feelings, so that (for example) feelings such as impasse, frustration, and disappointment contribute to mathematical engagement and satisfaction in achievement. It can stabilize (or destabilize) belief systems pertaining to mathematics (Goldin, 2002b, 2007; DeBellis & Goldin, 2006).

Affect may thus encode one's expectations of the nature of the subjective consequences of approaching a mathematical task. It may carry evaluative information regarding the success or failure of a strategic approach to a problem, up to a certain point in time. It may reflect one's tacit appraisals of the emotional states (actual, or potential) of other people, with whom one has meaningful relationships connected to mathematics (a teacher, a parent, or a friend). From a models and modeling perspective, one may consider an individual's relevant affective structures to be functioning as a kind of *self-model in relation to mathematics*. The structures may function as *descriptive* representations of the person's perceptions of his or her interest, ability, or characteristic feelings and behaviors in relation to mathematics. They may also function as *prescriptive* representations—modeling ideals against which the student measures himself or herself in these respects. Then affect also signals, for example, whether or not one is meeting the self-expectations flowing from one's sense of identity in relation to mathematics. As with other sorts of models, one can then seek to understand how affective structures—desirable or undesirable—develop through a sequence of modeling cycles, with local affective development occurring during each cycle. Here is the opportunity for valuable interventions by teachers or mentors.

Early results from a study of the affect of urban students engaged in conceptually challenging mathematics in their classrooms (Alston et al., 2007; Epstein et al., 2007; Schorr et al., in press; Goldin, Epstein, & Schorr, 2007) have led to a description of several *archetypal affective structures*—idealized, recurring patterns in individuals that include sequences of emotional feelings, characteristic behaviors and social interactions, self-talk, meta-affect, and other interacting components. Here the social dimension is stressed. Examples include: "Don't Disrespect Me": the student's experience of a perceived challenge to his or her well-being, status, dignity, or safety, where the need to maintain face supersedes the initial, mathematical engagement; and "Check This Out": the student's realization that learning the mathematics or solving the problem can pay off tangibly or emotionally or both, leading to heightened intrinsic extrinsic or intrinsic interest, and possibly drawing others into the task.

## IDEOLOGICAL DEBATE, PARADIGMS, AND DISMISSIVE EPISTEMOLOGIES

I have alluded to the fact that the concept of representation, despite its demonstrated value to the theory and practice of mathematics education, does not fit well within any of the fashionable, conflicting paradigms for educational research. This section, adapted from my earlier chapter, outlines some of the problems exacerbated by the absence of a suitable, shared theoretical framework in mathematics education research. Behind these problems lie tacit or explicit belief systems, based on epistemologies I have termed "dismissive" (Goldin, 2003b, provides a more extensive discussion). We shall see why these systems have downplayed, skewed, or disallowed the notion of representation.

My main purpose here is to advocate for a unifying theoretical foundation for mathematics education, one that can accommodate the most helpful and applicable constructs from a variety of approaches, including the paradigms discussed—but without the dismissive aspects. Then it becomes feasible to approach currently debated issues in mathematics education as empirical questions, not ideological ones. For this I think that a framework based on the study of representations and representational systems can be of great assistance.

### Mathematics education issues and ideologies

As this is written, traditionalists continue to challenge reform in mathematics education. The following idealized descriptions of the two sets of ideas, written about 6 years ago, does not seem to require significant revision today. Since the descriptions are idealized, I have not sought to attribute them to particular individuals.

#### *Traditional views*

Traditionalists, including some leading mathematicians, advocate curriculum standards that stress specific, clearly-identified mathematical skills at each grade level. Ideally, these are to be developed step-by-step, and then abstracted and/or generalized in higher level mathematics. This recognizes that much of mathematics is structured hierarchically, with more advanced techniques presupposing mastery fluency in more basic ones. Arithmetic operations with whole numbers, fractions, and decimals are fundamental at the elementary level, forming the basis of most of the mathematics that follows. Abstract or formal mathematical methods are valued for their power.

Principal attention should be given at all levels to the strength of the curricular content, the correctness of students' responses, and the mathematical validity of their methods. Standards should be measurable, and standardized achievement tests based on explicit goals should provide the main objective measures of standards attainment. Expository teaching methods are valued, including considerable individual drill and practice to ensure not only the correct use of efficient mathematical rules and algorithms, but also students' ability to interpret and apply them appropriately. Based on this mastery, more complex mathematical ideas can be successfully introduced.

In this spectrum of opinion, calculator-based work should be deemphasized until computational skills are well established. Children are recognized as differing greatly in mathematical ability, so that some significant numbers of them may not have the capacity to succeed in higher mathematics; for these children, achieving the basics is especially important. Class groupings should tend to be homogeneous by ability, at least after a certain grade level, to permit advanced work with high-ability students and attention to the basics with slower learners.

### *Reform views*

Reformers, including many leading mathematics educators, advocate curriculum standards in which high-level mathematical reasoning processes are central and universally expected. They value students' finding patterns, making connections, communicating mathematically, and engaging in real-life, contextualized, and open-ended problem solving from the earliest grades, with correspondingly reduced emphasis on routine arithmetic computation. Such learnings are best assessed through open-ended, authentic, or alternative assessment methods, and assessed least well through short-answer, standardized skills tests.

Hands on, guided discovery teaching methods are encouraged that involve exploration and modeling with concrete materials. In this spectrum of opinion, teachers should have children solve problems cooperatively in groups as well as individually, encouraging them to invent, compare, and discuss mathematical techniques as they construct their own, viable mathematical meanings. Contextualized mathematics is valued for its meaningfulness and relevance.

Extensive, early use of calculators and computer technology is seen as desirable, with the goal of pursuing more advanced mathematical explorations and projects unhindered by the limitations of pencil-and-paper computation. It is recognized that children have different learning styles; for example, those who seem to learn routine arithmetic or algebra operations slowly or imperfectly sometimes show surprisingly strong visual, spatial, or logical reasoning ability in less routine mathematics. Thus low (or preferably, high) expectations may be self-fulfilling. Most often it is thought that children should be grouped heterogeneously, to allow interaction among those with different learning styles and characteristics, and to achieve greater equity.

### *Discussion*

Which school of thought is right? Without the distorting lenses of ideology, it is evident that most of the stated ideas are not contradictory at all, but complementary. In particular, skills and reasoning are not opposites; each involves the other. I see much of value in both sets of views—and of course would introduce a few important qualifications. Either set alone is, in my judgment, *wholly insufficient*.

Some of the statements are open to empirical study. The methods advocated (such as expository or guided discovery teaching, individual or group work, homogeneous or heterogeneous grouping) are likely to be *appropriate under the right conditions,* and optimized in a reasonable balance that takes into account many variable factors—characteristics of the teacher, the students, the community, the mathematical knowledge to be developed, the problem solving tasks, school organizational constraints, available resources, etc. Good research makes the effects and interplay of such factors explicit, provides useful empirical information, improves on our theoretical constructs, and leads ultimately to generalizable results.

Unfortunately, as so often happens in the political and social arena, the most pure or most radical exponents of a belief system receive on balance the most attention. Rational advocacy of complex solutions to complex problems is drowned out by the noise of sound bites. Thus, to a certain extent, extreme positions emerge powerfully. Each camp accommodates itself to the willful disregard of contravening evidence, and tacitly adopts negative, value-laden terminology for characterizing the views of others. We then move from thoughtful, research-based consideration of difficult problems, with possibly complex ideas for solving them, to a state of ideological and political conflict. Consider, for a moment, some of the extremes.

### *Ideological poles*

Some traditionalists, at least tacitly, *define* mathematical knowledge to be that which is measured by the standardized tests they favor, and mathematical ability to consist exclusively in students' accuracy of response under timed test conditions. With these definitions, the only acceptable interpretation of meaningful understanding, real achievement, talent, or educational merit, is to be found in high test scores—speed and accuracy become the outcome observables. Quantitative measures are admissible, while qualitative ones are not—a view that has influenced educational policy at the federal level in the United States.

The main way for less talented students to achieve speed and accuracy on traditional tests is through the discipline of systematic drill in the skills to be tested. The admissible empirical evidence demonstrates, then, that drill focused on testable skills raises scores. Inclusion of the long division algorithm in the core of any proposed curriculum becomes a quick litmus test for its mathematical soundness, while calculators are to be banned entirely from the lower grades.

Opponents are stereotyped, in their ideas as well as personally. Open-ended exploration of any but the most directed kind is called fuzzy mathematics. Those who favor guided discovery learning are accused of valuing all children's responses equally, and of devaluing or negatively valuing correct answers. Those who use calculators are said to want to "dumb down" the curriculum. Heterogeneous grouping of any kind is regarded as denial of ability differences, and equity concerns are dismissed as "political correctness." The term *constructivist* is generalized to label, without distinction—but with considerable stigma attached—all who might think any of these things. Advocates of reform or educational equity are characterized as mathematically unqualified, ignorant people, holding positions in schools of education that do not value mathematical knowledge or objective research, and caring more about equality of outcome by race and gender than about mathematical achievement.

On the other hand, some reformists *define* the teaching of mathematical rules and algorithms, with accompanying student drill and practice, as exemplifying—by its very nature—meaningless or rote learning. The placing of value on correct responses, or on the objective validity of mathematical reasoning, is labeled rigid, absolutist, or destructive of children's natural creativity or inventiveness. Indeed, the very terms *correct*, *objective*, or *valid* are taken as highly objectionable. Quantitative measures are negatively valued, and qualitative ones esteemed. The problem of basic skills prerequisites for higher mathematical learning is denied and circumvented, rather than addressed. Calculators and computers are sometimes seen as having rendered computational skills obsolete.

And opponents are, as above, stereotyped. Exponents of expository teaching methods are characterized as advocates of authoritarianism. Those who seek the highest levels of achievement by the most capable students must be elitists, spokespersons for class, race, or gender privilege. Those who question calculator use at the expense of learning fundamental mathematical operations are considered Luddites, who place oppressive classroom rituals ahead of modern technology. Abstract mathematics, standardized testing of any kind, formal logical reasoning, or homogeneous class grouping, are deemed *per se* racist, sexist, or both. Professional mathematicians are stereotyped as an arrogant group of men claiming special access to truth, ignorant of schools and their problems, and expressing the narrow values of a White, Western, masculinist culture—one that values abstract rules and theorems at the expense of human beings.

As with most stereotypes, there are (unfortunately) individuals whose ideas seem to fit the caricatures drawn by each camp—while the majority of those who care about educational issues do not.

### *A quick historical look*

The current math wars are, of course, not a wholly new phenomenon. I was educated in the traditional mathematics of the 1940s and 1950s in the United States, with a great deal of memorization, rule learning, and training in routine problem solving. The pendulum swung. In the late 1950s and 1960s, my younger siblings began to study the new mathematics, the product of a mathematician-led movement funded by the U. S. National Science Foundation, to teach concepts and structures rather than procedures (cf. Sharp, 1964). Topics such as operations with sets, systems of numeration other than base ten, structural properties of number systems, probability, and transformational geometry supplanted the flash cards, the tables of arithmetic facts, and the memorization of rules and algorithms. Pattern-finding and mathematical discovery became valued over rule learning.

This was called a revolution—and there followed, inevitably, the counter-revolution (NCTM, 1968, 1970; Kline, 1973). The "back to basics" movement of the 1970s, intensely critical of the leadership by academic mathematicians, refocused attention on computational skills and rule learning with emphasis on measurable, behavioral outcomes (Mager, 1962; Sund & Picard, 1972). Most of the earlier innovations were discarded, or at most reserved for select student populations, and the mathematics community seemed to withdraw, licking its wounds, from its former leadership involvement in public education.

But the pendulum swung again. Non-routine problem solving came into fashion in the 1980s, and by the 1990s in the United States there had developed—at least at the level of rhetoric, though not as frequently in practice—a renewed emphasis on mathematical exploration and discovery, group activities, open-ended questions, alternate solution methods, contextualized understandings, and uses of technology (NCTM, 1989). There was a corresponding de-emphasis on computational algorithms, and on uniform curriculum standards based on them.

In the 1990s and 2000s, the restoring force of a second back to basics movement overtook the trend. In one of those ironic twists of history, academic mathematicians—whose community's advocacy of the new math was challenged by the back to basics movement of the 1970s—became leaders in the current traditionalist movement.

## The role of dismissive paradigms and epistemologies

Opposing forces over the years have found supporting intellectual bases in the academic research arena. Extreme educational ideologies often draw, tacitly or overtly, on radical theoretical or epistemological "paradigms" whose exponents have achieved prominence in part by dismissing—often on a priori grounds—the most important constructs of other frameworks.

To be clear, the frameworks I am terming "ideological" or "dismissive" are those where the system is *closed to falsification* either by empirical evidence or by rational inquiry, and/or where the fundamental tenets exclude by fiat consideration of the theoretical or empirical constructs of nonadherents.

In the psychology of mathematics education, such schools of thought have come in and out of fashion like clothing styles, dependent more on the cultural climate and marketing than on their rational coherence or the empirical evidence for them. This process may be explained partly by the simplistic appeal of all-encompassing constructs, especially those that are sufficiently vague or general as to lend themselves to the popular jargon. The sociology of university-based research in the "soft sciences" appears to favor—with fame, or at least with wide attention—"isms" that distinguish themselves by branding as illegitimate the conceptual entities of rival perspectives. Rarely does the new movement build on or acknowledge the value of what went before. In succession, the dismissive theory arises, gains adherents, educates graduate students in its tenets, and after some decades is discarded—not because it is wrong (as a theory in the physical sciences might be abandoned in the face of contravening evidence), but because newer fashions have rendered it no longer in vogue.

This pattern repeats itself, despite the fact that those who study mathematical learning and problem solving from the different perspectives of such theories do want ultimately to understand and explain similar observable phenomena.

### *Behaviorism*

One such fashion, which provided theoretical support for back to basics advocates across roughly four decades, has been the psychological school of behaviorism, and its subsequent elaboration as neo-behaviorism. Founding their movement on the radical empiricist epistemology called positivism (Ayer, 1946), behaviorist psychologists rejected *on first principles* any incorporation into theory of internal mental states, mental representations or cognitive models, thoughts or feelings, understanding, or information gained through introspection. Since none of these are susceptible to direct, empirical observation, behaviorists claimed that according to the verifiability criterion of meaning asserted by the positivists, none could possibly have meanings beyond the observable behaviors from which they might be inferred. Therefore, they were simply ruled out—the words were forbidden.

Reinforcement of observable stimulus-response (S-R) connections through timely reward, an empirically verifiable phenomenon, was adopted as a nearly all-encompassing mechanism to which learning, including mathematical learning, could be and should be reduced (Skinner, 1953, 1974). Neo-behaviorists were somewhat more flexible, accepting constructs built up from "internal responses" to previous responses that could serve as stimuli (allowing chains of S-R bonds, rules, and so forth), and focusing more directly on structures in external environments—while continuing to reject all mentalistic explanations on first principles.

The statistical methods of psychometrics were compatible with the behaviorists' insistence on predefined, observable outcomes. Together these provided an academic rationale in the United States for the "behavioral objectives" approach to mathematics education, combined powerfully with performance-based accountability measures. Legions of mathematics teachers rewrote their schools' curricular objectives during the 1970s to accord with the approved terminology. Qualitative research was devalued to the extent that it became unacceptable in some journals. Today, it is difficult to appreciate how dominant behaviorism became in U.S. mathematics education during this period, and how unacceptable were other points of view.

Though the behaviorists claimed to be scientific, their epistemology was not so. It is true that earlier in the 20th century, positivism had gained credibility from the need to address through *operational* definitions the modified concepts of space and time associated with Einsteinian relativity, and the problems of measurement raised by quantum mechanics. However, successful scientific theories have always relied not only on observable data, but also on constructs that are not themselves directly observable, but help to unify empirical observations and provide explanatory or predictive power. This aspect of physics did not change with the advent of relativity or quantum theory. Furthermore, qualitative and exploratory research have continued to play well-established, essential scientific roles—most apparent in the biological sciences, astronomy, and emerging disciplines of chaos and complexity theory.

The behaviorists' ban on internal, mental states and related ideas as legitimate constructs was, from the standpoint of sound philosophy of science, a wholly arbitrary one—but it energized back to basics advocates greatly. Without the admissibility of internal or mental phenomena, mathematics educators could focus easily on discrete, testable skills, but were forbidden from discussing cognitive structures or conceptual understanding. And without the possibility of complex, explanatory models for students' cognitions, psychometrics—claiming statistical rigor—lent support to the *reification* of some behavioral patterns as aptitudes, abilities, traits, or "general intelligence" and the neglect of other, perhaps more important indicators.

Challenged by Piagetian developmental psychology, unable to resist the appeal of the information processing sciences generally or cognitive science in particular, and never able to account for the complexities of mathematical or language learning, radical behaviorism went into decline. There is no opprobrium today in criticizing it within most mathematics education research circles. Rather it seems trite to do so, as few mathematics education students spend time learning about behaviorism.

But ideologies rarely become influential without some grains of truth. The important and valid reasons that fueled the ascendance of behaviorism were, in its rejection, also largely forgotten. One of these was a prior reliance on inadequate or overly simplified mentalistic constructs as psychological explanations, where the process of inferring these had no scientific reliability or validity. A related reason, perhaps more important for us today in mathematics education, was the tendency of psychology to lose touch with its scientific, empirical foundations, to mistake values for evidence, and to over-generalize from anecdotal reports and clinical interviews.

### *Radical constructivism and social constructivism*

A second fashion, one that fueled the reform movement in mathematics education in the 1980s and 1990s, is radical constructivist epistemology and its offshoot, radical social constructivism (cf. von Glasersfeld, 1987, 1990, 1996; Ernest, 1991; Confrey, 2000). In contrast to the behaviorists, who barred internal or mentalistic constructs, radical constructivists rejected on a priori grounds all that is external to the worlds of experience of human individuals. Excluding the very possibility of knowledge about the real world, they dismissed unknowable, objective reality to focus instead on experiential reality. Mathematical structures, as abstractions apart from individual knowers and problem solvers, were likewise to be rejected. In advocating the (wholly subjective) idea of viability they dismissed its counterpart, the notion of (objective) validity. Thus cognition and learning were seen *exclusively* as adaptive to the individual's experiential world, and *never in principle* as reaching truths about the real world. Those who paid close attention to the processes of constructing knowledge during learning and problem solving, but did not accept the radical constructivists' fundamental denial of the notions of objectivity and truth, were labeled "trivial constructivists" or "weak constructivists."

Radical social constructivists saw mathematical (and scientific) truth itself as merely social consensus, and dismissed the possibility of any objective sense in which reasoning could be correct or incorrect. This perspective was consonant with the fashionable trend toward ultra-relativism. Since each cognizing individual constructs his or her own knowledge, population studies or empirical investigative methods in education based on controlled experimentation were to be effectively replaced by in-depth case studies—research on human beings could never be replicated, as no two individuals or populations could (in principle) ever be shown to be the same.

Radical constructivist epistemology, unlike positivist epistemology, aimed more at challenging the supposed objectivity of science than it did at claiming scientific validity for itself. But it was deeply flawed (Goldin, 1990). It offered no explanation of the extraordinary degree to which science and mathematics *succeed* in permitting accurate prediction, control, and design, while superstitious belief systems do not. If I apply its initial assertions, that cognizing individuals have access only to their worlds of experience and can never have knowledge about the real world, directly to myself, I arrive at a well-known and not-very-useful solipsism. If I apply it to others as well as myself, as radical constructivism intended, I simply bypass the problem of how I (having access only to my own experiential world) can validly infer cognition in others. If I can do that, am I not assuming knowledge about a real world in which other human beings and their experiential worlds exist? If I cannot have such knowledge, how can I consistently make assertions about other cognizing individuals and what they may or may not have access to? The response to such objections, asserting not the validity but the viability of knowledge, created a system impervious to argument or evidence. Each belief system was viable for its adherents—and there one had to stop.

The radical constructivists' ban on objective knowledge begged important questions in the philosophy of science. But in challenging scientific hegemony, it proved a powerful energizing force for reformers intent on overthrowing behaviorist ideology in mathematics education. No longer were there correct answers in mathematics, only more-viable or less-viable constructions; and this appeared to strengthen the legitimacy of researchers' wanting to study and interpret students' spontaneous, non-standard ways of reasoning. Complex, explanatory discussions of cognition, cognitive structures, and conceptual understanding became not just admissible but highly desirable, as long as no objective validity was claimed for them. Mathematics educators could now devalue the objectivity of discrete, testable skills—not based on empirical evidence, but on the a priori basis of a philosophical movement.

While sharp criticism of radical constructivism still invites powerful disapproval in some academic circles, it is becoming clear that the movement as a whole has become passé. And, as in the case of behaviorism, the most important reasons for its ascendance are also being forgotten—the inadequacy of behavioral measures alone in describing meaningful learning and understanding, the need for complex, cognitive-developmental models to describe and account for mathematical learning and development, the value in complex domains of qualitative as well as quantitative research investigations, the importance of social and cultural variables in understanding learning in classroom contexts, and so forth.

### *Other dismissive paradigms*

These are, of course, not the only examples of dismissive theorizing. For instance, insisting that all thinking must be information processing, some "artificial intelligence" oriented cognitive scientists maintained in effect that theoretical models are impermissibly vague unless they are written as computer code. This lent great legitimacy to descriptions of cognition by readily-programmed constructs such as problem-solving search algorithms, while thought processes more difficult to simulate were downplayed. Some cognitive theorists maintained for a while that all cognitive encoding should be represented propositionally, on *a priori* grounds of parsimony, thus rejecting any kind of internal imagistic representation (Pylyshyn, 1973).

At another extreme, some language theorists claimed that all mathematics is metaphor, attributing the fact that theorems "remain proven" to the stability of metaphor, and dismissing the study of formal foundations as metaphor (Lakoff & Nuñez, 1997, 2000). Thus ultrarelativism with regard to the notion of right or wrong in mathematics became no longer the exclusive province of radical constructivists, postmodernists, and critical theorists. From the perspective of "embodied cognitive science," Núñez (2000, p. 19) suggests, "The so-called 'misconceptions' are not really misconceptions. This term as it is implies that there is a 'wrong' conception, wrong relative to some 'truth.' But Mathematical Idea Analysis shows that there are no wrong conceptions as such, but rather variations of ideas and conceptual systems with different inferential structures …"

Unfortunately, some advocates for qualitative research identified their advocacy very closely with dismissive postmodernist and other highly relativist (and mutually contending) epistemologies such as "critical theory" (Guba & Lincoln, 2005). The rejection on first principles of notions such as understanding recurs in mathematics education research. For instance Lerman, adopting a sociological/postmodernist perspective, writes, "First, it is high time we abandoned words and phrases such as 'understanding', 'misconceptions', and 'acquisition of concepts' in mathematics education. They are useless from a teacher's and a researcher's point of view, since they are in essence totally unobservable, and are effectively tools of regulation, since we take it upon ourselves to be the only ones qualified to identify when understanding has taken place" (Sfard, Nesher, Lerman, & Forman, 1999, pp. 1–85). In this view, the terms are not only epistemological unsound but morally offensive—a dimension that empowers critical theorists in their efforts to characterize those who take a scientific perspective as (hegemonistic) proponents of a positivist or post-positivist paradigm.

**Consilience and unification**

We should learn from the history of progress in the natural sciences that the denial on first principles of the admissibility of one or another kind of construct is rarely fruitful. The need is for a theoretical framework that is not ideological or fashion-driven, but scientific—where complex models are permissible, constructs are subject to validation, claims are open to objective evaluation, and conjectures can be confirmed or falsified through empirical evidence.

The paradigms and epistemologies I have mentioned make their most valuable contributions by focusing attention and study on particular domains of empirical phenomena, or particular sets of theoretical constructs—structures of observable behavioral patterns, and their reinforcement (behaviorism); cognitive-developmental processes and subjective experience in the construction of knowledge (radical constructivism); the role of social and cultural processes in knowledge development (social constructivism); the importance and ubiquity of metaphor, especially bodily metaphor, in human language (embodied cognitive science); the important dimension of power in social interactions (critical theory); and so forth. But a single-minded insistence on excluding other phenomena and other constructs, even to the point of the words that describe them being forbidden, is intellectually insupportable. It leads to built-in, unnecessary limitations.

The idea of the coherence and compatibility of knowledge in different domains, termed "consilience" and discussed interestingly by Edward O. Wilson (1998), is perhaps useful here. At the most reductionist level, we might come to describe human learning, understanding, and problem solving (including mathematics) biologically, particularly at the levels of genetics, evolution, and neuroscience. But cognitive science, the information sciences, linguistics, and developmental and cognitive psychology all provide different and useful ways to describe knowledge structures, and their development, including mathematical knowledge of various kinds, at a more holistic level. The idea of consilience suggests that none of these are fundamentally contradictory. Ultimately, we are likely to discover in detail how higher-level constructs are encoded or represented in the brains of thinking human beings.

While we do not yet know the specifics of representation at the level of networks of actual neurons, or how the human brain as an organ of the body is encoded and evolved genetically, we can still say a lot about mathematical knowledge structures at the psychological level. To do this, we study patterns in verbal and nonverbal mathematical behavior in controlled or partially-controlled task environments, from which we seek to draw increasingly reliable inferences about internal cognitive structures and their development.

The societal level, involving as it does variables descriptive of populations of individuals, culturally normative beliefs and expectations, and so forth, is still more holistic. But descriptions at holistic levels do not preclude or contradict more reductive descriptions (cf. Hofstadter, 1979). Rather, the former may anticipate the latter descriptions, be consilient with them, and eventually be explained in terms of them—as the theory of evolution proved useful before we understood its basis in molecular biology (thus unifying previously disparate areas of study), or the physical field of thermodynamics became well established prior to its reduction to more fundamental principles through statistical mechanics.

Thus, it seems within the realm of possibility to envision a theoretical framework for mathematics education that can unify useful and valid ideas, without the *a priori* dismissals associated with "isms."

**Implications for and from the construct of representation**

As we have seen, the abstract notion of representation involves a relation between two (or more) configurations, with one representing another in a sense to be specified. In the concrete context of the psychology of mathematical learning and problem solving, we must be able to consider internal configurations and structures, external configurations and structures, possible representing relations, socially shared configurations and structures, and so forth.

Evidently, the a priori dismissal by the behaviorists of internal configurations as acceptable constructs renders the very notion of representation in this sense inadmissible. Behaviorists have much less difficulty with relations (such as physical linkage) among different configurations that are external, and therefore observable, as long as the relations themselves involve no questionable internal constructs.

Radical constructivists, on the other hand, are deeply reluctant to acknowledge the admissibility of external representational configurations and structures—the inherent unknowability of the external by the individual forbids their discussion. However, radical constructivists have much less difficulty with relations among different internal configurations (cf. von Glasersfeld, 1987, 1996).

The parallels here with traditional and reform views in mathematics education are not accidental. To the extent that we dismiss or deemphasize the internal, we tend to focus by default on students' easily-observed productions—their mathematical skills performance, their achievement of behavioral objectives—without addressing the nature of their mathematical understanding or its development. This imbalance has tended to characterize the traditionalist approach. And to the extent that we dismiss or deemphasize the external, we focus on students' cognitive processes and qualitative conceptual understandings, possibly unreliably inferred, to the exclusion of measurable skills attainment or the validity of their mathematics. This imbalance has tended to characterize the reform approach.

Whichever dismissal one adopts, the notion of representation as descriptive of interaction between the internal and the external is effectively banned. Conversely, we have the demonstrated value of the analysis of representation as contributing on many levels to mathematics education theory and practice. We should draw the natural conclusion—that it is time to set aside dismissive epistemologies, in order to proceed with concepts that can unify the understandings reached from disparate perspectives.

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## NOTES

1. For instance, "Mathematically Correct" (http://www.mathematicallycorrect.com, 2002–2003 posting) quotes McEwan (1998, p. 119): "Who's to blame for the math crisis? The answer to this question is very simple: The National Council of Teachers of Mathematics (NCTM), to whom teachers, curriculum developers, and administrators have always looked for expert advice, has betrayed us."
2. A pointed and wryly humorous column by Diane Ravitch, Assistant Secretary for Educational Research and Improvement and Counselor to the Secretary, US Department of Education during the Bush administration from 1991–1993, contrasts the research-based medical treatment she received with the state of educational research. Ravitch writes, "Medicine, too, has its quacks and charlatans. But unlike educators, physicians have canons of scientific validity … Why don't we insist with equal vehemence on well-tested, validated education research? Lives are at risk here, too." (Ravitch, 1999). The column formed the basis of a plenary panel in July 2000 at the 24th Conference of the International Group for the Psychology of Mathematics Education, where discussants took a variety of positions on the feasibility and value of achieving validity and reliability in mathematics education research.

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