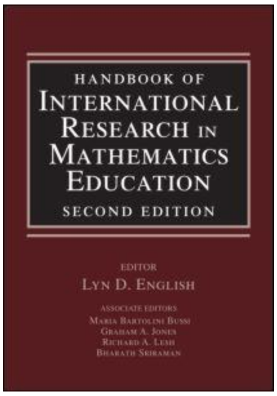

Handbook of International Research in Mathematics Education

Lyn D. English, Maria Bartolini Bussi, Graham A. Jones, Richard A. Lesh, Bharath Sriraman, Dina Tirosh

Elementary students' access to powerful mathematical ideas

Publication details
https://www.routledgehandbooks.com/doi/10.4324/9780203930236.ch6
Cynthia W. Langrall, Edward S. Mooney, Steven Nisbet, Graham A. Jones
Published online on: 19 Jun 2008

6 Elementary students' access to powerful mathematical ideas

Cynthia W. Langrall and Edward S. Mooney
Illinois State University

Steven Nisbet
Griffith University, Mt. Gravatt Campus

Graham A. Jones
Griffith University, Gold Coast Campus

The elementary school is the educational environment where all children are expected to begin the process of accessing powerful mathematical ideas. Although the expectation for elementary students to learn powerful mathematical ideas has been universally accepted, there has been on-going debate as to what constitutes powerful mathematical ideas for the elementary school.

For a substantial part of the 20th century, the prevailing view was that computational skills constituted the important mathematics that was needed for effective citizenry and continuing mathematical growth beyond the elementary school. Whether computational skills constituted powerful mathematics is debatable, especially given the conception of powerful mathematical ideas as it is envisaged today. Even during the 20th century, the heavy emphasis on computation was questioned through periods of reform incorporating an emphasis on meaningful mathematical learning (Brownell, 1935) or a focus on problem solving (Lester, 1980, Krutetskii, 1976, Schoenfeld, 1992). During such periods, there were strong efforts to produce a balance between skill and process, between instrumental and relational understanding (Skemp, 1971, p. 166), and between procedural and conceptual knowledge (Hiebert & Lefevre, 1986, pp. 3–8), However, notwithstanding these periods of reform and enlightenment in elementary mathematics, albeit, all too rare, it is fair to say that curriculum and instruction in elementary mathematics has not provided for *all* children a pervasive access to those important ideas that will prepare them for their life-long journey or even for their continued mathematical development beyond the elementary school.

The emphasis on all students learning powerful mathematical ideas in elementary school is complex and did not come into sharp relief until the latter part of the 20th century. Even so, rhetoric on equitable access has been stronger than fact. For example, in the United States there is a plethora of research that documents the lack of achievement by disproportionate numbers of racial and ethnic groups, speakers of English as a second language, females, and students from lower socioeconomic groups (e.g., Gonzales et al., 2004; Secada, 1992; U. S. Department of Education, National Center for Education Statistics [NCES], 2005). There is evidence, however, that low levels of participation and performance in mathematics by these special groups is not due primarily to lack of ability, but rather to educational practices that deny access to high quality learning experiences (Silver, Smith, & Nelson, 1995; Tate & Rousseau, 2002). This unfortunate reality has prevailed despite efforts to provide a strong mathematics curriculum for all students. In 1990 the National Council of Teachers of Mathematics asserted that the "comprehensive mathematics education of every child is its most compelling goal" (p. 3). This same premise continues to predicate national curriculum statements in many countries (e.g., Australian Education Council [AEC], 1990; Department for Education and Employment & Qualifications and Curriculum Authority, 1999; Department of Education and Science and the Welsh Office [DES], 1991; Weber, 1990). Providing students access to powerful mathematical ideas should be the basis of any comprehensive mathematics education program.

The issue of what constitutes powerful mathematical ideas raises questions that fall within the realm of historical and philosophical research. It is a discussion that will always be inextricably tied to cultural and political forces both within mathematics education and outside of it. In this chapter, we refer to cultural and political forces mainly within the United States, although we have tried to achieve a broader perspective in our review of the research literature. For other international perspectives, we refer readers to the first edition of this *Handboo*k (English, 2002), and specifically to chapters by Moreno-Armella and Block (2002) addressing access to powerful mathematical ideas in a developing country, Amit and Fried (2002) discussing mathematics education reform, and Skovsmose and Valero (2002) on the issue of democratic access to powerful mathematical ideas.

In this chapter, we examine powerful mathematical ideas in retrospect and also in prospect as we try to unfold research directions for the future. Moreover, given the increasing technological sophistication of elementary schools, we devote special attention to the role of technology in making powerful mathematical ideas accessible to elementary children. Our work is organized into four parts. In the first part, we present a retrospective view of the 20th century with regard to elementary school mathematics. In part two, we examine the notion of *cognitive accessibility* and what research indicates about the powerful ideas that cut across the domains of mathematics at the elementary school level. Whereas part two pertains to the powerful mathematical ideas that are cognitively accessible to elementary school children, part three addresses children's opportunity to learn. That is, their *access to learning experiences* through which powerful mathematical ideas can be developed. We focus specifically on issues related to curriculum, instruction, and the role of technology. Finally, we close the chapter with a discussion of some of the challenges for the 21st century and related implications for future research.

POWERFUL MATHEMATICAL IDEAS: IN RETROSPECT

What can we learn about the identification of powerful mathematical ideas for the elementary school from our endeavors in the 20th century? For most of the first half of the century, debate on what constituted powerful mathematical ideas for the elementary school was largely a non-issue. Guided by strong utilitarian and pragmatic needs, and fueled by waning support for mental discipline (Howson, 1982; Jones & Coxford, 1970), elementary mathematics was dominated by the need to train children to perform computational procedures. Even for those students who would progress beyond elementary school, a steady regimen of arithmetic skills was seen as the ideal diet for further manipulation of algebraic symbols in the secondary school.

The debate in the first half of the century was not on powerful mathematical ideas but rather on how arithmetical computation should be taught. Research was designed to compare and contrast computational approaches such as drill and practice, incidental learning, and meaningful learning (Brownell, 1935; Thorndike, 1924). It did not question the importance of, or power attributed to, standard algorithms for whole numbers and fractions. This emphasis on computation was complete and certainly understandable given the lack of computing technology and the needs of society during that first 50 years.

The period of the *new math* was another story. Mathematicians played a key role in arguing for revolutionary changes in mathematics per se (e.g., Howson, 1982; Jones & Coxford, 1970;

Wooton, 1965). Their intent was to generate an elementary mathematics that encapsulated the *structure* of mathematics (Jones & Coxford, 1970, p. 68–86; Page, 1959) and also better reflected the state of mathematics of the day. Although these changes were also accompanied by research on teaching and learning (Biggs & MacLean, 1969; Bruner, 1960; Dienes, 1965) this was a period of genuine change in the content of elementary mathematics.

The introduction of sets as a unifying idea for building concepts of number and space was a pervasive change in the quest for giving students access to powerful mathematical ideas. Through the use of sets, the reform groups of that time generated important representations for operations with whole numbers and fractions—even though the term *representations* appeared later in the century. For example, the addition of whole numbers was represented as the union of disjoint sets and the intent was that connections between sets and numbers would provide a scaffolding for students' learning of operations. Representations like this were expected to not only support the learning of arithmetic but also to facilitate the transition from arithmetic to algebra. Computational algorithms were still a critical part of elementary mathematics in the new math, but underlying place-value representations and structural properties of the relevant operations were made more explicit to increase children's understanding.

The power of sets also extended to the study of geometry and measurement. Sets were used to represent concepts like points, segments, and angles and also to provide meaning for relationships like intersection and parallelism. As it did in the case of number, the notion of correspondence was also implicit in capturing the fundamentals of measurement. Measurement was seen as a function that assigns a number to an object or, more specifically, to an attribute of the object such as length, area, or volume. Accordingly, function as a unifying idea played a subtle but key role in number and geometry, largely as a precursor to its more extensive role in algebra and calculus (De Vault & Weaver, 1970).

Much has been written on the outcomes of the new math and the differences between the intents of its architects and the realities of classroom implementation. It is not appropriate to reanalyze the outcomes of new math except to say that what happened in practice has been called "formalistic game-like plays in and with structures defined in terms of sets and logic; often devoid of sense-making relations to matters outside the structures themselves" (Niss, 1996, p. 31). Our interest is focused on what we might learn from the kind of inquiry approaches and arguments that were used to identify powerful mathematical ideas.

The sources for most of the theoretical and philosophical arguments that generated the new mathematical content were mathematicians. They were in a unique position to make compelling arguments about the need for new content and for a new structural emphasis starting in the elementary grades. Although there were notable examples of collaboration among mathematicians, mathematics educators, and teachers (e.g., Wooton, 1965) in relation to the development of curriculum programs and experimental textbooks (e.g., School Mathematics Study Group [SMSG], School Mathematics Project [SMP]), it was the mathematicians' arguments that determined what powerful ideas were to be included in the school curriculum. Mathematicians were also in a strong position to win external funding for school mathematics projects (Jones & Coxford, 1970) as this was an era of active political support for space exploration and scientific research.

Despite the development of large-scale and heavily funded curriculum projects in mathematics across the world, these projects did not produce the kind of systematic research methodologies that would have on-going significance for the identification of powerful and accessible mathematical ideas. There were two reasons for this. First, it was early days in the paradigmatic shift from scientific-reductionist research in mathematics education to interpretivist research. While there was some evidence of case-study approaches (De Vault & Weaver, 1970; Wooton, 1965), research at that time was more concerned with providing descriptions of the historical process than with analyzing and interpreting the argumentation used to identify key mathematical ideas for the curriculum. Had such research been undertaken it may have revealed the "risks of following specialized mathematics too closely" and consequently selecting "subject matter and elements of mathematical language that do not make much sense outside of specialized mathematics" (Wittmann, 1998, p. 91). In fact, the research of the day was still focused to a great extent on the effect of modern mathematics on student performance (SMSG, 1972) and, as such, it largely followed statistical design models. Second, even if qualitative research had been carried out during this period, it is probable that the new math movement was simply too unique and too spectacular to provide a useful case-study for the future.

In the wake of the new math era, there was a brief period of return to the traditional roots of elementary mathematics—that is, "back to the basics" of arithmetic (Schoenfeld, 1992). However, growth in technology and dissatisfaction with student mathematical performance especially in processes like problem solving (e.g., Dossey, Mullis, Lindquist, & Chambers, 1988) soon led to a broadening of goals that were intended to "encompass the essential aspects of numeracy and 'mathematical literacy' in society" (Niss, 1996, p. 32). For the elementary school this resulted in greater emphasis on mathematical ideas associated with newer domains like algebraic thinking, data exploration, and probability (e.g., AEC, 1990; DES, 1991; NCTM, 1989). Even in extant areas like number, there was a new focus that emphasized number sense, mental computation, and efficient use of technology in computation (Hembree & Dessart, 1992; Sowder & Schappelle, 1989). Perhaps the most important shift was in emphasizing the powerful ideas associated with mathematical processes. The NCTM *Standards* (1989) encapsulated this worldwide trend by giving preeminence to four process standards: problem solving, communication, reasoning, and connections. Social and utilitarian needs were still important but mathematics was viewed as dynamic rather than static, and constructive rather than prescriptive (Schoenfeld, 1992; von Glasersfeld, 1984). In essence, elementary children were expected to engage in mathematical problem solving, to collaborate with other students, and to build on their own conceptual thinking rather than relying totally on someone else's standard procedures.

By the close of the 20th century, there existed a fairly robust research base in the extant domains of elementary school mathematics: number (whole numbers, fractions, and decimals), geometry, and measurement. For example, in the domain of whole numbers, researchers had classified semantic representations of addition and subtraction problems and identified the kinds of hierarchical strategies that students develop in the early years of schooling (e.g., Carpenter & Moser, 1984; De Corte & Verschaffel, 1987). Similar representations and strategies were generated for multiplication and division of whole numbers (e.g., Mulligan & Mitchelmore, 1997) as well as for multidigit concepts and operations (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). Researchers had also developed conceptual models of learning in the domains of common and decimals fractions, ratio, and proportion (e.g., Hiebert & Wearne, 1986; Lamon, 1993, Mack, 1990, 1995; Moss & Case, 1999; Streefland, 1991), and geometry and measurement (e.g., Chiu, 1996; Lehrer & Chazan, 1998; van Hiele, 1986).

The emphasis on problem solving during the latter part of the 20th century led to a genuine interest in mathematical modeling in the elementary school. Some researchers (e.g., Verschaffel & De Corte, 1997; Verschaffel, De Corte, & Vierstraete, 1999) focused on modeling tasks related to the use of operations with whole numbers, fractions, and decimals. These tasks were intended to not only provide rich experiences in mathematical modeling, but to also reveal different aspects of number and operations and to enhance students' number sense. Other researchers (Lehrer & Romberg, 1996; Lesh, Amit, & Schorr, 1997; Masingila & Doerr, 1998) introduced model-eliciting problems that require a variety of mathematical processes and also draw on conceptual knowledge from mathematical domains less established in the elementary curriculum such as algebraic reasoning, data exploration, probability, and discrete mathematics.

We should note that inclusion of these underrepresented mathematical domains in school mathematics was advocated worldwide, and the general consensus was that they must begin in a significant way in the elementary school (Borovcnik & Peard, 1996; Ralston, 1989). The mathematics education community not only recognized the importance of these topics in the elementary school curriculum, but following research methodologies used in established domains, like numbers and operations, began to develop models and frameworks for algebraic thinking (Bellisio & Maher, 1998); data exploration (e.g., Curcio, 1987; Jones et al., 2000; Watson, Collis, Callingham, & Moritz, 1995); probability (e.g., Fischbein & Schnarch, 1997; Jones, Langrall, Thornton, & Mogill, 1997); and combinatorics (e.g., English, 1991). Thus, unlike the situation that prevailed in new math, we entered the 21st century with conceptually viable mathematical representations in a variety of domains that were accessible to children; in fact, in many cases the representations were the constructed and validated models of children rather than of mathematicians.

As the 20th century drew to a close, technology was seen as a powerful instrument in the development of children's mathematical understanding (e.g., NCTM, 2000). Yet, when examining the emerging role of technology in generating powerful mathematical ideas and potential areas for research, it would be naïve of us not to recognize that there has been reluctance, even resistance, to using technology in elementary school mathematics (Becker & Selter, 1996). Balacheff and Kaput (1996) noted that even the more conceptually oriented software has been seen by some educators "to slow down the curriculum and the student—adding a flexibility and depth of understanding that does not seem to be valued as much as computational facility" (p. 473). And in the realm of calculators, which have become widely available and affordable, Ruthven (1996) identified a number of factors inhibiting their use in schools: public concerns about the effect of calculators on computational learning, testing policies that prohibit the use of calculators, and the treatment of calculators in some of the official curricula and textbooks of some countries. Thus, it appears that as we enter the 21st century, resistance remains in spite of evidence that children's number fact learning and their mental and written computational skills are not diminished by regular use of calculators and other technologies (Ellington, 2003; Kilpatrick, Swafford, & Findell, 2001).

Our examination of elementary school mathematics in the 20th century has revealed that the mathematics curriculum reflected a number of recurring goals: pragmatic and social needs of individuals and society, and general formative goals that related to mathematics and applications outside mathematics. During different time periods, these goals were interpreted by educators of the day as providing a mandate for computational skills in arithmetic and measurement. But as Ralston stated in 1989, "Mathematics education must focus on the development of mathematical power not mathematical skills" (p. 35). By the close of the century, the elementary school mathematics curriculum reflected a broader more balanced representation of mathematical domains.

The enduring pragmatic goals of the 20th century will have core value for the current time period. However, there is evidence that they will be embedded in broader goals leading to powerful mathematical ideas that are different from those emphasized for most of the 20th century. For example, the generally accepted shift to perceiving mathematics as a dynamic process as opposed to a static system has awakened interest in Freudenthal's (1973) theories about teaching and learning mathematics. His views of mathematical activity or *mathematizing* emphasize powerful mathematical processes that include generalizing (generality), justifying and proving (certainty), modeling and defining (exactness), and symbolizing and standardizing procedures (brevity) (Gravemeijer, 1994). These same processes have been highlighted in the NCTM Standards (2000) and are evident in many curriculum documents worldwide (Mullis, Martin, Gonzales, & Chrotowski, 2004). Also, goals for the future will need to take cognizance of the increasing role that technology will play in revealing the powerful ideas of elementary mathematics and in giving children access to them. It is against this backdrop that we consider students' accessibility to powerful mathematical ideas for the 21st century.

POWERFUL MATHEMATICAL IDEAS FOR A NEW CENTURY: COGNITIVE ACCESSIBILITY

The construct *powerful mathematical idea* is multifaceted (Skovsmose & Valero, 2002) and one could debate what makes a particular mathematical idea powerful. In examining powerful mathematical ideas at the elementary school level, we are motivated by Steen's (1990) admonition that "to prepare effective mathematics curricula for the future, we must look to patterns in the mathematics of today to project, as best we can, just what is and what is not truly fundamental" (p. 3). In his book *On the Shoulders of Giants: New Approaches to Numeracy* (1990), Steen describes mathematics as a tree composed of growing branches and supported by a root system of deep ideas that nurture the branches. We find his metaphor useful and have adopted it in considering powerful mathematical ideas as the root system sustaining the various branches of mathematics. In adopting Steen's tree metaphor, we also note that he cautioned that there are more fundamental components to the root system of mathematics than the traditional topics of arithmetic, measurement, algebra, and geometry. Rather, the mathematical root system is a complex structure that also includes attributes, actions, abstractions, attitudes, behaviors, and dichotomies (pp. 3–4).

Although the elementary school mathematics curriculum typically includes the various content domains—number, geometry, measurement, algebra, data, and probability—at each grade level, these strands are not always integrated and the complexity of the mathematical root system is often neglected. Teachers and students alike may not recognize or appreciate the continuities within and across mathematical domains and consequently powerful ideas may go unnoticed. In an effort to explicate some of the powerful ideas at the elementary level, we reviewed the research literature to identify fundamental aspects of mathematics that are cognitively accessible to children.

Some thoughts on powerful mathematical ideas

An important focus of recent mathematics education research has been on understanding and characterizing students' reasoning. As discussed in the previous section, cognitive models describing students' reasoning have been constructed for the various content strands in the elementary school mathematics curriculum. These models, which have taken a variety of forms ranging from frameworks and taxonomies to detailed narrative descriptions and learning trajectories, incorporate key elements of a content domain and the processes by which students grow in their understanding of that content. We examined this rich body of research with an eye toward the complex root system described by Steen (1990); we sought mathematical ideas that recurred in the literature, within as well as across domains. However, we make no claim that our search was exhaustive and recognize that our findings are limited in scope. Nevertheless, we have identified a number of powerful mathematical ideas that are cognitively accessible to elementary school children. They include the following: *unit*, *iteration* and *composition/decomposition*; *generalization* and *formalization*; *variation* and *expectation*; *equality*; and *representation*.

Unit, iteration, composition, and decomposition

To illustrate the essential nature of these ideas in developing mathematical understanding, let us take a closer look at *unit*, *iteration*, *composition*, and *decomposition*. We found these mathematical ideas to be pervasive in the research literature we reviewed. For example, at a fundamental level, number is conceived as a composite of units. Children develop a concept of number through counting and the construction of number sequences that give rise to the development of increasingly more abstract unit types, such as iterable units, composite units, and iterable composite units (Steffe, von Glasersfeld, Richards, & Cobb, 1983). Across the content strands, much of mathematics involves the construction and coordination of abstract units.

In the domain of whole number concepts and operations, detailed models have been constructed that outline a developmental progression from unitary to multiunit conceptual structures (e.g., Fuson, 1992; Jones et al., 1996; Steffe et al., 1983). These models provide insights into children's understanding of single- and multi-digit numbers and how these understandings support the development of operations. Many computational strategies and informal algorithms developed by children rely on a concept of 10 (as a unit and also as a collection of 10 *unities*) and the ability to decompose multidigit numbers. Moreover, McLellan and Dewey (1908) observed that "it is only when the unit is treated not as one thing, but as a standard of measuring numerical values, that addition and multiplication, division and fractions, are rationally correlated" (p. 100).

More recently, researchers (Biddlecomb, 2002; Olive, 1999; Saenz-Ludlow, 2003; Steffe, 2003, 2004) have examined the development of students' understanding of fractions as an extension of their numerical schemes for whole numbers. According to Steffe (2004), "the major operations of the partitive fractional scheme—partitioning, iterating, and disembedding—have their genesis in the child's numerical concepts" (p. 155). Similarly, Saenz-Ludlow attributed students' transition from understanding fractions in relation to continuous wholes to an emergent understanding of fractions in discrete wholes to their ability to express *number as manifold of units*—that is, being able to simultaneously express the same number as a unit of units in multiple ways.

Working within the context of a computer microworld, Olive (1999) facilitated students' construction of fractions as iterable units and as operations. They constructed units (sticks or bars) by iterating a given part of the unit, as well as by partitioning units into equal parts and reconstructing the unit from any one of those parts. A goal was for students to construct a scheme that included fractions as measurement units that would allow "the construction of any fraction from any other by finding a comeasurement unit for the two fractions" (p. 294). As Olive described, "With fractions as measurement units, division of fractions becomes meaningful. For example, the questions 'How much of 3/4 is 1/8?' or 'How many 8ths in 3/4?' require finding the measure of 3/4 in terms of 1/8 as a measurement unit" (p. 294). These ideas resonate with Lamon's (1993, 1996) work in the area of ratio and proportion, which highlighted the importance of developing students' perception of ratios as units that can be used as referents for interpreting other relationships. Lamon described this as *unitizing* (i.e., constructing a reference unit and interpreting situations in terms of that unit) and also described a related process of *norming* that involves reinterpreting a situation in terms of a composite unit. Both of these processes characterize students' construction of increasingly complex quantity structures.

Unit, iteration, and composition/decomposition are also relevant beyond the domain of number and operation. Given the close connection between concepts of number and measure, the notion of an iterated unit is a central, unifying idea for all of quantity measurement (Outhred & Mitchelmore, 2000). Research by Clements, Battista, Sarama, Swaminathan, and McMillen (1997) pointed to three levels of development for length concepts. Initially, students do not segment lengths or associate the number of a measure with the actual length of a segment. However, experiences iterating segments and partitioning unsegmented lengths support the development of students' understanding of measure in terms of equal-interval partitions and eventually, the conception of an abstract unit of length as a measure.

By way of extension, the research of Reynolds and Wheatley (1996) and Outhred and Mitchelmore (2000) underscores the importance of the construction and coordination of units and iterable units in understanding the concept of area and area measurements. For example, Outhred and Mitchelmore reported that the construct of an *iterable row* is necessary for understanding the structure of an array as a basis for area measure. Studies examining students' experiences with 3-dimensional figures leading to concepts of surface area and volume have shown similar needs for the coordination of units, for example in terms of layers or faces (e.g., Battista & Clements, 1996).

Saenz-Ludlow (2003) described *iteration* and *decomposition* as inverse operations. This perspective is consistent with Clements, Wilson, and Sarama (2004) who claimed that "the ability to describe, use, and visualize the effects of composing and decomposing geometric shapes is a major conceptual field and a set of competencies in the domain of geometry" (p. 164). The learning trajectory they constructed to describe children's development of shape composition identified levels of competency that included combining shapes to make pictures and designs; creating and operating on a shape as a unit with measurable attributes; conceptualizing two or more shapes as a composite unit viewed as a separate entity. This learning progression is significant in that it "goes beyond existing van Hielian thought in adding the composition and decomposition processes as essential elements of geometric knowledge" (p. 166).

As these research studies indicate, *unit*, *iteration*, and *composition/decomposition* undergird the development of mathematics concepts within various content strands. Moreover, the research findings presented above show that these fundamental ideas are not only cognitively accessible to students in the elementary grades, but that they are used intuitively and as building blocks for further conceptual development. The constraints of this chapter preclude us from presenting such detailed evidence of the pervasiveness of the other fundamental mathematical ideas we identified in our review of the literature. However, we provide brief descriptions of each to allow the reader to assess the merit of these ideas as reflecting powerful mathematics that is accessible to elementary school children.

Generalization and formalization

One develops an understanding of the nature of mathematics, its structure and concepts through the processes of generalizing and formalizing. "Generalization and formalization involve the articulation and representation of unifying ideas that make explicit important mathematical relationships" (Carpenter & Levi, 2000, p. 2). These unifying ideas span the domains of the elementary mathematics curriculum. More specifically, making generalizations and expressing them in increasingly more formal and conventional ways is the foundation of geometric and algebraic reasoning (Carraher, Schliemann, Brizuela, & Earnest, 2006; Kaput & Blanton, 2000). There is a growing body of research indicating that elementary school children are capable of developing these processes through instruction that emphasizes the underlying properties and structure of number and operation (e.g., Carpenter, Franke, & Levi, 2003; Kaput & Blanton, 2005; Verschaffel, Greer, De Corte, 2007; Warren, 2003a).

Variation and expectation

The constructs of variation and expectation lie at the core of data and chance, and also extend beyond mathematics into other content areas and everyday events. "Variation is ubiquitous, and being able to reason about its qualities comprises a form of literacy with very broad scope" (Lehrer & Schauble, 2004, p. 672). A substantial body of research literature pertains to children's understanding of variation and expectation (e.g., Jones, Langrall, & Mooney, 2007; Petrosino, Lehrer, Schauble, 2003; Shaughnessy, 2007; Watson, Kelly, Callingham, & Shaughnessy, 2003).

Equality

The connection between equality (i.e., quantitative sameness) and students' interpretation of the equal sign is often tenuous. Many students, even in the secondary grades, persist in interpreting the equal sign as an indicator of the answer or as an operator that signals a calculation

(Kieran, 1981; Saenz-Ludlow & Walgamuth, 1998). However, there is ample evidence in the research literature that elementary school children can develop an understanding of the equal sign as expressing a relation when they experience arithmetical equations in a variety of formats and contexts (e.g., Carpenter, Franke, & Levi, 2003; Dickinson & Eade, 2004; Freiman & Lee, 2004; Jones & Pratt, 2005; Verschaffel et al., 2007; Warren, 2003b).

Representation

The literature is replete with studies that have examined the role of representation in the development of children's mathematical understanding (e.g., Brizuela & Schliemann, 2004; Carpenter & Levi, 2000; Gray, Pitta, & Tall, 2000; Heirdsfield & Cooper, 2002; Kato, Kamii, Ozaki, & Nagahiro, 2002; Klein & Beishuizen, 1998; Lehrer & Schauble, 2004; Nisbet, Jones, Thornton, Langrall, & Mooney, 2003; Thomas, Mulligan, & Goldin, 2002). In *all* areas of the mathematics curriculum, elementary school children use forms of representation (e.g., mental models, symbolic notation, tables, graphs, number lines, drawings) to organize and advance their thinking as well as to convey what they know. Indeed, Lesh (2000) claimed that "representational fluency often is at the heart of what it means to understand many of the most important underlying mathematical constructs" (p. 180).

Why are they powerful mathematical ideas?

The constructs *unit, iteration* and *composition/decomposition*; *generalization* and *formalization*; *variation* and *expectation*; *equality*; and *representation* transcend domain-specific topics. They are fundamental components of mathematics corresponding to the attributes, actions, abstractions, attitudes, behaviors, and dichotomies of Steen's (1990) root system. They provide a conceptual base that supports engagement in more complex or abstract mathematics, and they increase in mathematical power as they become formalized in students' thoughts. In describing the mathematics that students will need for success in the 21st century, Lesh (2000) noted that

> to provide powerful foundations … the kind of understandings and abilities that appear to be most needed are not so much about new topics as they are about broader, deeper, and higher-order treatments of traditional topics such as rational numbers, proportions, and elementary functions—topics that have been part of the traditional elementary mathematics curriculum, but that have been treated in ways that are far too narrow and shallow. (p. 183)

The constructs we have identified here reflect broader, deeper treatment of traditional topics. In many respects, they resonate with Freudenthal's view of mathematical activity in terms of generality, certainty, exactness, and brevity (Gravemeijer, 1994). Thus, we consider them to be *powerful* mathematical ideas, and more specifically, ones that are cognitively accessible to students at the elementary school level.

In summarizing this part of the chapter on cognitive access to powerful mathematical ideas, we note that research (e.g., studies cited in the previous section) supports our position that children's knowledge structures, both informal and formal, accommodate powerful conceptual ideas that constitute "the deep ideas that nourish the growing branches of mathematics" (Steen, 1990, p. 3). When children are given opportunities to build upon their informal knowledge structures to make sense of problem situations, they are capable of understanding significant mathematics that was once reserved for older students or an elite minority (Romberg & Kaput, 1999). Thus, when understanding is perceived as emerging over time, we are able to broaden the range of powerful mathematical ideas considered accessible to children. Certainly the ideas we identified in our review of the literature are only a beginning. In the

21st century, mathematics education researchers will need to continue to investigate what is truly fundamental and cognitively accessible to elementary school children. Furthermore, research will need to reveal that curriculum and instruction can be informed by these powerful ideas.

ACCESS TO POWERFUL MATHEMATICAL IDEAS: OPPORTUNITY TO LEARN

As shown in the previous section, there is strong evidence that powerful mathematical ideas are identifiable for, and cognitively accessible to, elementary school students. We direct our attention now to a different form of accessibility—opportunity to learn. Although many factors are germane to this issue, curriculum, instruction, and the role of technology are essential elements in providing students access to powerful mathematical ideas. First, the mathematics curriculum needs to include content goals addressing fundamental mathematical ideas and pedagogical goals (e.g., problem solving and conducting investigations) that support students in developing those powerful mathematical ideas. Second, classroom practice needs to be based on instructional models that accommodate and promote the development of powerful mathematics. Teachers need to be supported in their quest to improve their practice through opportunities to appreciate and promote new pedagogies. Third, the provision of and training in the use of technology to engage students in exploring powerful mathematical ideas must be recognized for its role in providing students opportunity to learn.

Curriculum reform

At critical junctures during the 20th century, mathematics education leaders throughout the world called for reform in the school mathematics curriculum, in classroom implementation of that curriculum, and in related assessments (e.g., AEC, 1990; College Entrance Examinations Board [CEEB], 1959; Commission on Post-War Plans, 1944; Council for Cultural Cooperation, 1988; Cockroft, 1982; NCTM, 1989, 2000; Report of the Mathematical Association: The Teaching of Mathematics in Public and Secondary Schools, 1919 cited in Howson, 1982). The second half of the 20th century saw the new mathematics movement of the 1950s and 1960s, the back-to-basics movement of the 1970s, the promotion of problem solving in the 1980s, and the push for greater numeracy skills and standards-based reform which started in the 1990s and has continued into the 21st century. Each of these reform endeavors sought to improve students' access to the mathematical ideas considered important at the time. As noted earlier in this chapter, calls for restructuring the elementary mathematics curriculum have reflected on-going societal needs, growth in the discipline of mathematics, changes in our understanding of students' mathematical learning, stronger commitment to "mathematics for all," and increased availability and use of technology.

Mathematics educators today, enlightened by the experiences of the 20th century and informed by the findings of local as well as large-scale assessments (e.g., TIMSS, PISA) recognize clear discrepancies among the *desired curriculum*—as it exists in a national goal statement or a ministry of education syllabus, the *implemented curriculum*—as it plays out in classrooms, and the *achieved curriculum*—in terms of what children learn. Ultimately, while these inconsistencies remain, we cannot guarantee that all elementary students will have access to powerful mathematical ideas.

Mathematics curriculum documents and research reports around the world (e.g., Mullis et al., 2004; NCTM, 2000; Oliveira, Segurado, da Ponte, & Cunha, 1997) have emphasized the notion of mathematical investigation as a means of teaching and learning mathematics, and have identified thinking, reasoning, and working mathematically as a means to engaging with powerful mathematical ideas. These kinds of recommendations have been the catalyst for curricular change or reform in mathematics education in many parts of the world.

Despite the impetus for reform in mathematics education that highlights the need for students to develop more sophisticated ways of mathematical reasoning, educators in some countries (e.g., Australia, New Zealand, United Kingdom) have recently experienced pressure, exerted by politicians, to emphasize basic numeracy skills that often focus narrowly on recall of number facts, mental arithmetic, written computation skills. Concerns expressed over national standards of numeracy have led to great debates and battles of policy and implementation between an alliance of progressive educators and industry-based technological pragmatists on one side and industrial trainers on the other (Ball, 1990, cited in Brown, Millett, Bibby, & Johnson, 2000). The debates exposed tensions between pragmatism and ideology, traditional and modern knowledge bases, as well as equity and equality (stemming from different interpretations of equity—namely equality of opportunity, equality of treatment, and equality of outcome).

The United Kingdom National Numeracy Project (1996–98) and National Numeracy Strategy (NNS, starting 1999) have been characterized by an emphasis on calculation, a three-part template for daily mathematics lessons, detailed planning using a week-by-week schedule of objectives, and a standardized national training program for teachers. Despite this attempt at tight prescription and control, varying perceptions of the need for change on the part of teachers and curriculum leaders, along with multiple interpretations of the features of the National Numeracy Strategy across schools and regions have led to large variations in teacher practice, and the impact and outcomes of the NNS (Brown, Millett, Bibby & Johnson, 2000).

In Australia, a National Literacy and Numeracy Plan was initiated federally in 1997, and adopted in all states and territories with the overall goal of improving students' performance in literacy and numeracy. The main purposes of the plan were (1) to identify students at risk, (2) to conduct intervention programs, (3) to assess all students against national benchmarks, and (4) to introduce a national numeracy reporting system (Department of Education, Training & Youth Affairs, 2000). Consequently, annual compulsory state-wide testing was introduced for students in Years 3, 5 and 7 in 1998. Subsequently in 2001, the Australian government implemented the Numeracy Research and Development Initiative (Numeracy Initiative) to support numeracy improvement. The Numeracy Initiative comprised national and state projects, the purpose of which was to investigate a broad range of teaching and learning strategies that contribute to enhanced numeracy outcomes (Department of Education, Science and Training [DEST], 2005).

The findings of the Numeracy Initiative projects point to seven key aspects which contribute to enhanced numeracy outcomes—namely (1) school, home and community philosophies which support mathematics/numeracy learning, (2) whole-school approaches to the improvement of numeracy, (3) teachers who have strong mathematics content knowledge appropriate to the levels they are teaching, (4) teachers who have a high level of mathematics pedagogical knowledge, (5) teachers who recognize numeracy situations in the world around us, (6) respect for the learning needs of all students, (7) early intervention programs for children identified at risk, and (8) effective use of resources, including technology (DEST, 2005).

The New Zealand Numeracy Projects (NZNP) were implemented as part of the New Zealand Ministry of Education's Literacy and Numeracy Strategy to reform mathematics teaching and learning, and increase student achievement by improving the professional capability of teachers (Young-Loverege, 2005). Key aspects of NZNP included a research-based framework to describe children's progression in mathematics learning, individual task-based interviews and ongoing reflective professional development for teachers. Evidence shows that students have benefited from the projects regardless of ethnicity, socio-economic status, gender or age (Thomas, Tagg, & Wood, 2002).

Calls for reform usually achieve varied responses in the general community and the educational community. Zevenbergen (2003) claimed that there are three types of teachers classified according to their responses to calls for reform in mathematics education—the Conservatives who prefer the status quo, the Pragmatists who are concerned about the practical issues related to the implementation of reforms, and the Contemporaries who see the value and need for reform. Little wonder that debates rage and stakeholders and interested parties engage in so-called "math wars" over issues related to mathematics education reform.

Brown, Millett, Bibby, and Johnson (2000) see the problem of curriculum reform and teacher change as one of interpretation. For example, with the implementation of the National Numeracy Strategy in the U.K., there was a triple layer of interpretation from the director of the project via numeracy consultants and key school personnel, to teachers. As Fullan and Hargreaves (1991) noted, the consistency of interpretation, vis-à-vis major educational innovation, depends on the clarity with which it is perceived by those who have to implement it; for example, problems arise because the adopted change may not highlight what teachers are expected to do differently. The clear implication for reform in mathematics curriculum is that much time and effort needs to be spent on the issue of clarification of purpose and the thrust of the reform at all levels and between all levels. This is essential in ensuring that change initiatives do in fact inform teachers about new practices and hence provide students with access to powerful mathematical ideas.

Instructional models

Whereas recommendations for reform provide a vision of the curriculum that is desired, the implemented and achieved curriculum is a result of the classroom learning environment and instruction provided by the teacher. Sfard (2000) stated that the principal aim of reform in mathematics education is the improvement of the teaching of mathematics, and that despite decades of intensive research in mathematics education, many questions about students' learning, quite vital to any pedagogical decision, still wait to be answered. For instance, why are certain mathematical concepts inadmissible and certain mathematical arguments unconvincing to many students? From Sfard's perspective, part of the answer lies in acknowledging that mathematical learning can be conceptualized as gaining access to a certain discourse, rather than acquisition of knowledge. A key question is from where should the meta-rules that make a classroom discourse come? Although the classroom discourse should be close to the discourse engaged in by mathematicians, a mathematician's argument would fail to convince a child, and vice versa. The discourse that develops in reform classrooms may turn out to be different from the professional mathematician's discourse, however, for elementary school children the concept of access to the discourse is pre-eminent. The establishment of greater rigor and discipline in school mathematical discourse is a necessary but time-consuming process, and the rules of mathematical discourse can be learned by students in participation with not only their peers but also experts in the discourse, namely, their teachers. As students learn the rules of mathematical discourse, they are able to participate more fully in the classroom discourse. Such participation by all students in the classroom should be the goal of all teachers as they strive for greater equity for their students.

In her writing on equity, Lubienski (2002) has called for mathematics educators to take stock of their knowledge and commitment to equity. Strides have been made in terms of achieving equity, but much work remains to be done in this era of pedagogical and curricular reform. Making further strides towards equity requires moving beyond the dichotomy between traditional and reformed teaching. Lubienski claimed that socio-cultural studies of mathematics classrooms hold promise for informing our efforts to empower all students. The goal is to learn more about the complexities of successfully implementing meaningful instructional methods equitably with students who differ in terms of social class, ethnicity, and gender.

We have begun the 21st century with a substantial body of research describing instructional models that can accommodate and promote students' development of important mathematical concepts. There is strong evidence of the effectiveness of instruction that promotes inquiry and fosters understanding and making sense of mathematics (Hiebert et al., 1997; Kilpatrick et al., 2001; Senk & Thompson, 2003). We elaborate on three related theoretical perspectives that we believe will continue to inform research on curriculum and instruction aimed at providing students opportunity for accessing powerful mathematical ideas.

Realistic mathematics education

At the forefront of this research is the instructional theory of *Realistic Mathematics Education* (RME). Since the 1960s, the research of Dutch mathematics educators (e.g., Freudenthal, 1968; Gravemeijer, 1994; Streefland, 1991; Treffers, 1987) has provided the theoretical basis for their "realistic approach" to the teaching and learning of mathematics. The original Wiskobas Project, which served as the catalyst for the reform of elementary school mathematics in the Netherlands, set in train the shift from a mechanistic orientation to teaching and learning to an approach that emphasized learning through reconstructive activity grounded in reality and sociocultural contexts. This involved the development of a new curriculum, textbooks, and tests; design of preservice and inservice teacher education programs related to that curriculum; preparation of counselors and instructors; and on-going research to monitor this activity.

The work that began with the Wiskobas Project has continued to this day. From the RME perspective, school mathematics should be embedded in rich problem contexts that allow instruction to proceed from the reality of students' informal strategies. Teaching in this kind of learning environment involves globally guiding students to be reflective and to develop increasingly abstract levels of mathematical reasoning that eventually lead to formal mathematization (Gravemeijer, 1991; Streefland, 1991). The theoretical base of RME has been constructed through a distinctive process of *developmental research* (Gravemeijer, 1994, 1998), also known as design research, which combines curriculum development and educational research in such a way that the development of instructional activities is used as a means of elaborating and testing instructional theory. Rather than research providing a formative evaluation of curriculum development, developmental research lays the foundations for the work of professional curriculum developers; it is an iterative process in the sense that the development of instructional theory is gradual and cumulative over a large set of individual research projects (Gravemeijer, 1998, pp. 277–279).

By design, Realistic Mathematics Education (RME)accommodates students' development of powerful mathematical ideas, and there is evidence that students engaged in RME are especially successful in higher-level problem solving and reasoning when compared to students who receive more traditional instruction (Anghileri, 2004; Streefland, 1991; Treffers, 1987). Moreover, the Dutch have implemented large-scale evaluation measures to validate the ongoing cycle of developmental research, curriculum development, and teacher enhancement. In essence, RME provides a model that might serve to narrow the gap between the intended curriculum, the implemented curriculum, and the achieved curriculum.

Learning trajectories

An important feature of the RME model is the teacher's role in mapping out a learning route for instructional tasks. This is accomplished by anticipating paths of development in students' understandings and skills by drawing on knowledge of how conceptual structures develop within a particular content domain and insights into children's informal knowledge structures. Compatible with the RME model, Simon (1995, 1997) constructed a framework, called the Mathematics Teaching Cycle, that describes "the relationships among teacher's knowledge, goals for students, anticipation of student learning, planning, and interaction with students" (1997, p. 76). A key component of this teaching cycle is the *hypothetical learning trajectory* or "the teacher's prediction of the path by which learning might proceed" (p. 77). Simon's hypothetical learning trajectory is essentially the same as the learning route in RME. It includes the teacher's goal for student learning, plan for learning activities, and hypothesis of the student learning process. Teacher knowledge and interactions with students reflexively inform the generation and modification of hypothetical learning trajectories. More specifically, teachers draw on their knowledge of how children learn in general, and of how particular mathematical understandings are developed as they design sequences of instructional tasks that support developmental progressions of children's thinking and learning.

More recently, researchers have begun to adopt and elaborate on the hypothetical learning trajectory construct. Models of children's developmental progressions have been constructed within different mathematical domains (see *Mathematical Thinking and Learning*, *4*[2], 2004 special issue on learning trajectories). These models tend to highlight powerful mathematical ideas as they describe learning paths that lead to increased levels of abstraction and formalization in children's thinking; they also take into account the mathematical processes in which children engage and their dispositions towards mathematical activity.

Modeling

Although mathematical models and the process of modeling are themselves powerful mathematical ideas, a modeling approach to instruction establishes a learning environment in which students are engaged in both using and developing important mathematical concepts. From one perspective, modeling is "a process whereby a situation has to be problematized and understood, translated into mathematics, worked out mathematically, translated back into the original (real-world) situation, evaluated and communicated" (Greer, Verschaffel, & Mukhopadhyay, 2007, p. 2). Another view, known as emergent modeling (Gravemeijer, 2004), involves learners in model-eliciting tasks aimed at developing mathematical concepts, in addition to applying concepts and tools already within their repertoire. As students engage in solving realistic problems set in real-world contexts, the models that emerge from their informal activities gradually develop into models for more formal mathematical reasoning.

Greer et al. (2007) have identified a number of studies with elementary school children that point to the effectiveness of a modeling approach with respect to students' performance, motivation, and affective aspects of learning. They also reported the potential of modeling activities in engaging students in critical analyses of contemporary social issues. The promise of a modeling approach for providing children the opportunity to engage in powerful mathematics is especially evident in the longitudinal research of English and Watters (2004). They concluded that children's engagement with model-eliciting problems enabled them to develop mathematical concepts and processes that are not typical of instruction in the early grades. For example, Grade 3 students in their study examined quantitative relationships, analyzed change and varying rates of change, aggregated and averaged data, and explored aspects of probability. They accessed these ideas at varying levels of sophistication, and also developed their abilities at mathematical description, justification, explanation, and argumentation.

Role of technology

Technology can impact the instructional methods that teachers use and the mathematical ideas accessible to students. Research over the last 30 years has revealed the potential of technology to provide students access to powerful mathematical ideas. In this section, we consider the role of technology in generating powerful mathematical ideas and then discuss the issue of students' access to technology.

Access through technology

Technology can introduce children to mathematical ideas that are limited by or not possible with the use of paper and pencil (Arnon, Nesher, & Nirenburg, 2001). It can provide immediate feedback (Ainley, Nardi, & Pratt, 2000; Reimer & Moyer, 2005) and has the potential to allow children to "say and do things with suitably-designed systems that they may be unable to say or do without them" (Noss, 2002, p. 2). Moreover, children often work in an unconventional manner using technology. Research has shown that technology can aid teachers in providing students access to powerful mathematical ideas.

CALCULATORS

Research reveals that elementary children who use calculators identify new insights into modes of calculation, build earlier conceptions of large numbers, and develop different perspectives on checking arithmetical calculations (Groves, 1994; Ruthven, 1996; Shuard, Walsh, Goodwin, & Worcester, 1991). For example, in comparing the problem-solving processes of students who were expected to use calculators with students who had no access to calculators (Wheatley, 1980), the calculator group exhibited more exploratory behaviors and spent more time attacking problems and less time computing. They also used different predominant processes for solving problems and checking their solutions. With regard to checking, the calculator group used worthwhile processes significantly more often than the non-calculator group. These processes included the following: checking that the conditions of the problem had been met, retracing their steps, and checking the reasonableness of their answers.

Groves (1994) noted that primary children who had taken part in projects emphasizing the development of mental methods of calculation in conjunction with the use of the calculator did not necessarily use calculators more often, rather they made more appropriate choices of methods of calculation. There is even evidence that children use calculators in unanticipated yet important ways. Stacey (1994) gave illustrations of children learning to write numerals like 2 and 5 correctly by looking at the appropriate keys on their calculator. This finding appears to be consistent with Ruthven's (1996) more general claim that pupils with less confidence in, or enjoyment of number, seem to experience through the calculator a means of matching the demands of schoolwork to their mathematical capabilities.

Although research has shown that the calculator can be a powerful tool in students' development and use of mental mathematics and problem solving skills, there is little research, especially at the elementary level, on other effective uses of the calculator (Kirkpatrick et al., 2001). With the development of calculators with graphical features, technology has the potential to help students develop powerful mathematical ideas beyond those associated with number and number operations. Graham and Smith (2004) found that students in their study, ages 7 to 11, had no difficulty or discomfort in using graphic calculators. Yet, the focus of the study was aimed at helping students' recall multiplication facts. Clearly, more research is needed to examine how the calculator can be used most effectively to provide students access to powerful mathematical ideas.

SPREADSHEET PROGRAMS

Computers often come with pre-loaded programs that can be used to teach mathematics. A spreadsheet program is an example of such software, that when used appropriately, can "promote open-ended exploration of mathematical concepts, take advantage of [its] capabilities that allow the learner to extend beyond or significantly enhance what could be done using paper-and-pencil, and give teachers and students an opportunity to discover mathematical concepts in a laboratory-like setting" (Drier, 2001, p. 178). Filloy and Sutherland (1996), Rojano (2002), and Sutherland and Rojano (1993) have used spreadsheets to focus on number concepts from the perspective of patterns and relationships. They suggested that this approach supports students' thinking in making the key transition from arithmetic to algebra. Moreover, their research reveals that children can use spreadsheet language to build conceptions of functions and their different representations: rule (both written and symbolic forms), graph, and table. Rojano claimed that spreadsheets provide access to the power of algebraic language thus removing one of the key obstacles associated with the development of algebraic thinking. She also maintained that the use of the computer frees children from the arithmetical activity of evaluating expressions, thus enabling them to focus on the structural aspects of algebra.

Ainley, Nardi, and Pratt (2000) used spreadsheets to aid children in interpreting trends in data. Traditionally in mathematics classes, emphasis has been on designing graphs by hand with little time spent on interpretation of graphs. The benefit of using a spreadsheet environment is the ease in which children can produce a variety of graphs. The dynamic nature of the spreadsheet enables students to produce graphs immediately after data are entered or altered, change the size and shape by dragging the corners of the graph window, and modify a graph's appearance in terms of scale, orientation, and labeling. The researchers reported that these features give children the opportunity to examine trends and interpret data without being encumbered by the construction of the data display.

VIRTUAL MANIPULATIVES

Virtual manipulatives are "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer, Bolyard, & Spikell, 2002, p. 373). Typically, virtual manipulatives are equivalent to physical manipulatives such as base-ten blocks, pattern blocks, or geoboards. They can be moved like their physical counterparts, linked to other representations, and can facilitate students' exploration of a variety of mathematical ideas by providing access to unlimited quantities of materials (Moyer, Niezgoda, & Stanley, 2005; Reimer & Moyer, 2005). Virtual manipulatives can help students build early ideas of patterns and relationships as well as the composition and decomposition of number. Moyer, Niezgoda, and Stanley (2005) found that kindergarten students using virtual pattern blocks created more patterns and more complex patterns that when drawing or using wooden pattern blocks. Second-grade students in their study were able to translate their understanding of regrouping using virtual base-ten blocks to written addition problems.

MICROWORLDS

The computer microworld can present a problem-rich environment in which children solve challenging problems by programming the computer to exhibit arithmetic, geometric, and algebraic relationships or by exploring ideas in open virtual "worlds." Seymour Papert (1980), through the development Logo and its accompanying philosophy of learning, pioneered the development of these kinds of computer environments. Papert created intellectual environments that fostered learning through interactions involved with programming the computer. These *turtle* microworlds, as Papert named them, were "constructed realities" (p. 204) structured to allow children to connect their intuitive understandings with formal mathematical knowledge. Papert's microworlds were "sufficiently bounded and transparent for constructive exploration and yet sufficiently rich for significant discovery" (p. 208). In this way, he believed that the computer added "new degrees of freedom" (p. 209) to what children learned and how they learned it.

Noss and Hoyles (1996) claimed that when children work with the computer turtle in the Logo environment, they tend to build ideas of ratio and proportion naturally and as a consequence begin to think multiplicatively. More specifically, they claimed that once procedures for drawing figures were established, students often posed for themselves the issue of enlarging and shrinking. While growing and shrinking do not necessarily involve proportionality, Hoyles and Sutherland (1989) documented how students have used Logo input as a scale factor to change the sizes of drawings and build procedures that reflected the internal relationship between figures.

More recently, the goals of microworlds have shifted from having children program computers to having children devise their own tasks and subtasks for constructing and reconstructing mathematical objects and relationships (Noss & Hoyles, 1996). For example, the *Tools for Interactive Mathematical Activities* (*TIMA*) microworld was designed to give students opportunities to perform such mathematical operations as unitizing, disembedding, fragmenting, partitioning, iterating, and measuring (Steffe & Olive, 2002). In *TIMA: Sticks*, a child can draw a stick of any length, make multiple copies of the stick, join sticks together, partition sticks, and designate any stick of any length as a unit for measuring. In this way, children use the objects and operations of the *TIMA: Sticks* microworld to generate and abstract mathematical objects and relationships; for example, to build conceptions of unit and nonunit fractions as invariant relations (Tzur, 1999).

Pratt (2000) has reported impressive results in the development of 10- and 11-year-old children's probabilistic thinking when they used the *Chance-Maker* microworld. Using this dynamic and interactive environment the children articulated their meanings for chance through their attempts to "mend" the computer tool so it would function like it was supposed to. Children explored ideas such as expectation and fairness by controlling aspects of probability gadgets including generated outcomes and strength of spin.

Access to technology

The access to powerful mathematical ideas and the opportunities to learn through technology will be of no use if students do not have access to the technology. As stated earlier in the chapter, the acceptance and use of technology in mathematics classrooms is limited. This is particularly true at the primary level. In the 2003 Trends in International Mathematics and Science Study (TIMSS), 44% of countries reported having national curriculums that contained statements about calculator use at the fourth-grade level. At the eighth-grade level, 64% of the same countries reported having national curriculum statements about the use of calculators. Teachers in the TIMSS countries reported that greater percentages of fourth-grade students were not permitted to use calculators than the percentage reported by eighth-grade teachers. Only four countries reported permitting widespread calculator usage, 90% of the students or more, at the fourth-grade level. However, even in those countries, teachers reported that they rarely incorporated calculator activities into more than half their mathematics lessons (Mullis et al., 2004).

The use of the computer is even more limited than the use of calculators in the elementary grades. Thirty-six percent of the TIMSS countries reported having national curriculum statements regarding the use of computers at the fourth-grade as opposed to 48% of national curriculum statements for the same countries at the eighth-grade level. Computer access remains a challenge in many countries. On average, internationally, teachers reported that computers were not available for 58% of the fourth-grade students and 68% of the eighth-grade students. Also, countries with relatively high computer availability reported rarely using computers a majority of the time (Mullis et al., 2004).

There are efforts, however, in several countries to make learning through technology more accessible to students. For example, in Australia and New Zealand, The Learning Federation ([TLF], 2007) have developed online mathematical *learning* objects for students in Years P (Preparatory) through 12. These technology-driven mathematical learning objects have been disseminated to all public, private, and independent schools in the two countries and have paid special attention to the needs of indigenous, ESL (English as a Second Language) and remote students. As such, they provide a rare example of a national curriculum initiative that has the potential to provide opportunities for all students to gain access to mathematical learning via technology.

Technology can provide access to powerful mathematical ideas not previously available to teachers and their students. However, consideration must be made in the design and implementation of these technologies in the classroom. Research has not only focused on designing powerful technology for the classroom; but, also on the design process (e.g., Sarama & Clements, 2002) and the implementation of technology to give students the best access to powerful mathematical ideas (e.g., Ainley, Nardi, & Pratt, 2000). Even so, issues about the acceptance of computer and calculator technologies in elementary schools prevail. Thus, researchers will need to examine the political and pedagogical interventions needed to successfully place these technologies into the hands of all students.

CHALLENGES FOR THE 21ST CENTURY: IMPLICATIONS FOR RESEARCH

The discrepancy among the desired curriculum, the implemented curriculum, and the achieved curriculum is not a new problem in mathematics education, but it is an intractable one. Moreover, it is a problem with consequences regarding students' access to powerful mathematical ideas. When the results of the first international mathematics study were announced, critics of the new math blamed the comparatively poor performance of U.S. students on the new math curricula (which emphasized powerful mathematical ideas). However, the U.S. National Advisory Committee on Mathematical Education (NACOME, 1975) declared that, despite formal changes in school syllabi and curriculum texts of the new math era, the actual mathematical experiences of elementary school students during the 1960s reflected little of the reformers' intended curricula. Consistent with this comment, Cooney (1988) later claimed that criticisms of the new math were inappropriate because "studies that carefully detail what happened in classrooms during the modern mathematics movement are virtually nonexistent" (p. 352). In essence, although studies revealed differences between the desired and the achieved curriculum, there was virtually no research at the time that examined differences between the desired and the implemented curriculum.

As we begin the 21st century, we are better positioned to examine all aspects of the curriculum—desired, implemented, and achieved. Powerful research designs have emerged that support fine-grained analyses of teaching and learning and that recognize the complexities and affordances of conducting research in classrooms and other learning environments (Kelly & Lesh, 2000). These methodologies include developmental or design research (Gravemeijer, 1998; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), classroom teaching experiments (Cobb, 1999; Confrey & Lachance, 2000, Wittmann, 1998), and teacher development experiments and accounts of practice (Simon, 2000; Simon & Tzur, 1999). Models are also emerging for research examining teacher knowledge (e.g., Hill, Rowan, & Ball, 2005) and curriculum development (Clements, 2007).

One challenge for researchers is to use these emerging methodologies and fresh theoretical perspectives to examine how powerful mathematical ideas become instantiated in the school mathematics curriculum and reflected in measures of accountability; targeting the issue of discrepancies between the desired, implemented, and achieved curriculum. There are two vital elements that should be addressed in narrowing these discrepancies. The first is the need to appreciate the central role of the teacher in classroom practices during the process of implementing a new curriculum, in association with the power of the teacher to misinterpret, subvert and even ignore the unfamiliar curriculum and curriculum materials (Remillard, 2005). How do teachers come to recognize and appreciate powerful mathematical ideas as core mathematical experiences for students at the elementary school level? This question points to the second vital element, which is the recognition of the potential of well-designed professional development programs for teachers to move the curriculum change process forward and change teachers' attitudes and beliefs as well as their practices (Fullan, 1982; Guskey, 1985; Clarke & Peter, 1993; Nisbet, 2005). Research pertaining to these issues will have the potential to improve curriculum development and implementation processes for the long-term improvement of the effectiveness of mathematics pedagogy and students' mathematical learning.

The integration of technology into the mathematics curriculum and classroom instructional practice presents an enduring challenge for the 21st century. There is little evidence in the literature of widespread use of technology in the elementary grades. For example, although research has shown that technologies such as computer microworlds hold great potential for student learning, most of the research has been conducted in conjunction with the design and development of these programs. The integration of these technologies into regular classroom instruction appears to be limited. Availability of the materials as well as the infrastructure to support their use are likely culprits; and ones that are related to issues of equity. Moreover, teachers' willingness and readiness to incorporate these powerful tools into mathematics instruction remains an issue to be resolved. Even though calculators are prevalent in most elementary classrooms, there is disagreement among educators about when and how they should be used. Questions about the role of technology in developing both concepts and skills continue to be raised and need to be examined in a systematic, reasoned way.

Perhaps the most important challenge for the 21st century is to address the issue of access and equity in mathematics education in a scholarly and realistic way. The catch cry of "mathematics for all" has multiple interpretations amongst the various communities of stakeholders—government, business, schools, parents, and mathematicians. Perceptions vary from a post-modernist position seemingly supporting low-level numeracy programs, to an economic-rationalist position that bemoans the shortage of mathematically-trained people for industry and teaching, and advocates large-scale testing as a measure to halt the purported lowering of standards. Many who call for "mathematics for all" usually assume the existence of mathematics opportunities for every child and the desire to furnish them with high-quality mathematics education that will give them access to professions and careers of their choice (Malloy, 2004). Research needs to identify the extent to which this assumption can be validated. According to Schoenfeld (2002), the achievement of equity of access requires four systemic conditions to be met, namely, (1) a high quality curriculum; (2) a stable, knowledgeable, and professional teaching community; (3) high quality assessment that is aligned with curricular goals; and (4) stability and mechanisms for the evolution of curricula, assessment and professional development. Each of these four conditions offers agendas and challenges for researchers in the 21st century, in order to determine the various environments that lead to greater access to powerful mathematical ideas.

CLOSING REMARKS

Our analysis in this chapter suggests that the direction for elementary school mathematics in the 21st century will be more reflective of the last two decades of the 20th century than the first 80 years. There is increasing research evidence that elementary school children need to engage in a "cultural initiation" (Chevallard, 1989) that will enable them to mirror the kinds of experiences in which mathematicians engage. This means that process goals that focus on problem solving, mathematical discourse, reasoning, and connections with technology will take precedence over pragmatic goals that have less salience in a society where technology has packaged the computational skills needed for effective citizenry.

Instruction in the elementary grades should enable *all* children to develop powerful mathematical ideas and use them with competence, confidence, and enjoyment. The emphasis on *all* preempts a need for continued research to ensure that equity permeates the teaching and learning of elementary mathematics. Discrepancies among the intended curriculum, the implemented curriculum, and the achieved curriculum have proved a barrier to elementary children's access to powerful mathematical ideas, especially for children from minority groups and poverty areas. This curriculum hiatus will continue to challenge us in the 21st century but there are hopeful directions emerging. Access to powerful mathematical ideas must be the right of every elementary student whatever their cultural background. While it is neither realistic nor desirable to search for a solution to cognitive and curriculum access that is unique to every culture, increased globalization and technology offer unprecedented opportunities for international collaboration on these critical and enduring issues.

REFERENCES

Ainley, J., Nardi, E., & Pratt, D. (2000). The construction of meanings for trend in active graphing. *International Journal of Computers for Mathematical Learning*, *5*, 85–114.

Amit, M., & Fried, M. (2002). Research, reform, and times of change. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 355–381). Mahwah, NJ: Erlbaum.

Anghileri, J. (2004). Disciplined calculators or flexible problem solvers? In M. J. Hoines & A. B. Fuglestrad (Eds.), *Proceedings of the 28th PME International Conference* (Vol. 1, pp. 41–46).

Arnon, I., Nesher, P., & Nirenburg, R. (2001). Where do fractions encounter their equivalents: Can this encounter take place in elementary school? *International Journal of Computers for Mathematical Learning, 6*, 167–214.

Australian Education Council. (1990). *A national statement on mathematics for Australian schools.* Carlton, Victoria: Author.

Balacheff, N., & Kaput, J. J. (1996). Computer-based learning environments in mathematics. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Part 1, pp. 469–503). Dordrecht, The Netherlands: Kluwer.

Ball, S. J. (1990). *Politics and policy-making in education*. London: Routledge.

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education*, *27*, 258–292.

Becker, J. P., & Selter, C. (1996). Elementary school practices. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Part 1, pp. 511–564). Dordrecht, The Netherlands: Kluwer.

Bellisio, C. W., & Maher, C. (1998). What kind of notation do children use to express algebraic thinking? In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the Twentieith Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (Vol. 1, pp. 161–165). Columbus, OH: The ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Biddlecomb, B. D. (2002). Numerical knowledge as enabling and constraining fraction knowledge: An example of the reorganization hypothesis. *Journal of Mathematical Behavior*, *21*, 167–190.

Biggs, E. E., & MacLean, J. R. (1969). *Freedom to learn: An active learning approach to mathematics.* Don Mills, Ontario: Addison-Wesley.

Borovcnik, M., & Peard, R. (1996). Probability. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Part 1, pp. 239–288). Dordrecht, The Netherlands: Kluwer.

Brizuela, B., & Schliemann, A. (2004). Ten-year-old students solving linear equations. *For the Learning of Mathematics*, *24*, 33–40.

Brown, M., Millett, A., Bibby, T., & Johnson, D. (2000). Turning our attention from the What to the How: the national Numeracy Strategy. *British Educational Research Journal, 26*(4), 457–471.

Brownell, W. A. (1935). Psychological considerations in the learning and teaching of mathematics. In *The teaching of arithmetic* (10th yearbook, pp. 1–31). New York: Teachers College, Columbia University.

Bruner, J. (1960). *The process of education*. Cambridge, MA: Harvard University Press.

Carpenter, T. P., & Levi, L. (2000). *Developing conceptions of algebraic reasoning in the primary grades.* Madison, WI: National Center for Improving Student Learning. (ERIC Document Reproduction Service No. ED470471)

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, *15*, 179–202.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education*, *29*, 3–20.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, *37*, 87–115.

Chevallard, Y. (1989). Implicit mathematics: Their impact on societal needs and demands. In J. Malone, H. Burkhardt, & C. Keitel (Eds.), *The mathematics curriculum: Towards the year 2000* (Collected papers on this theme from the 6th International Congress on Mathematical Education) (pp. 49–58). Perth, Western Australia: Science and Mathematics Education Centre, Curtin University of Technology.

Chiu, M. M. (1996). Exploring the origins, uses, and interactions of students' intuitions: Comparing the lengths of paths, *Journal for Research in Mathematics Education*, *27*, 478–504.

Clarke, D. J., & Peter, A. (1993). Modelling teacher change. In B. Atweh, C. Kanes, M. Carss, & G. Booker (Eds.) *Contexts in Mathematics Education. Proceedings of the Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia (MERGA),* (pp. 167–175). Brisbane: MERGA.

Clements, D. H. (2007). Curriculum research: Toward a framework for "research-based curricula." *Journal for Research in Mathematics Education*, *38*, 35–70.

Clements, D. H., Battista, M. T., Sarama, J., Swaminathan, S., & McMillen, S. (1997). Studnets' development of length concepts in a logo-based unit on geometric paths. *Journal for Research in Mathematics Education*, *28*, 70–95.

Clements, D. H., Wilson, D. C., & Sarama, J. (2004). Young children's composition of geometric figures: A learning trajectory. *Mathematical Thinking and Learning*, *6*, 163–184.

Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning*, *1*(1), 5–43.

Cobb, P., Confrey, J., diSessa, A. A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*, 9–13.

Cockcroft, W. H. (1982). *Mathematics counts.* London: Her Majesty's Stationery Office.

College Entrance Examinations Board, Commission on Mathematics. (1959). *Program for college preparatory mathematics.* New York: Author.

Commission on Post-War Plans. (1944). The first report of the Commission on Post-War Plans. *The Mathematics Teacher*, *37*, 22–232.

Confrey, J., & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven research design. In R. Lesh & E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 231–265). Mahwah, NJ: Erlbaum.

Cooney, T. (1988). The issue of reform: What we have learned from yesteryear. *Mathematics Teacher*, *81*, 352–363.

Council for Cultural Cooperation. (1988). *Innovation in primary education: Final Report_*(DECS/EGT 987 23, Project No. 8). Strasbourg: Council of Europe.

Curcio, F. R. (1987). Comprehension of mathematical relationships expressed in graphs. *Journal for Research in Mathematics Education*, *18*, 382–393.

De Corte, E., & Verschaffel, L. (1987). The effect of semantic structure on first graders' strategies for solving addition and subtraction word problems. *Journal for Research in Mathematics Education*, *18*, 363–381.

De Vault, M. V., & Weaver, J. F. (1970). Designing a contemporary elementary school mathematics program: 1952-present. In P. S. Jones (Ed.), *A history of mathematics education in the United States and Canada* (32nd yearbook, pp. 133–154). Washington, DC: National Council of Teachers of Mathematics.

Department for Education and Employment & Qualifications and Curriculum Authority, (1999). *Mathematics: The National Curriculum for England – Key stages 1 – 4.* Norwich, UK: Authors.

Department of Education and Science and the Welsh Office. (1991). *National Curriculum: Mathematics for Ages 5 to 16.* York, UK: Central Office of Information.

Department of Education, Science and Training (DEST), (2005). *Numeracy research and development initiative 2001–2004: An overview of the numeracy projects.* Canberra: Commonwealth of Australia.

Department of Education, Training and Youth Affairs. (2000). *Numeracy, a priority for all: Challenges for Australian schools.* Canberra: Author.

Dickenson, P., & Eade, F. (2004). Using the number line to investigate the solving of linear equations. *For the Learning of Mathematics*, *24*(2), 41–47.

Dienes, Z. P. (1965). *Building up mathematics.* London: Hutchison Educational.

Dossey, J. A., Mullis, I. V. S., Lindquist, M. M., & Chambers, D. L. (1988). *The mathematics report card: Are we measuring up?* Princeton, NJ: Educational Testing Service.

Drier, H. S. (2001). Teaching and learning mathematics with interactive spreadsheets. *School Science and Mathematics*, *10*, 170–179.

Ellington, A. J. (2003). A meta-analysis of the effects of calculators on students' achievement and attitude levels in precollege mathematics classes. *Journal for Research in Mathematics Education*, *34*, 433 – 463.

English, L. D. (1991). Children's strategies for solving two- and three-dimensional combinatorial problems. *Journal for Research in Mathematics*, *24*, 255–273.

English, L. D. (Ed.). (2002). *Handbook of international research in mathematics education.* Mahwah, NJ: Erlbaum.

English, L. D., & Watters, J. J. (2004). Mathematical modeling in the early school years. *Mathematics Education Research Journal*, *16*(3), 59–80.

Filloy, E., & Sutherland, R. (1996). Designing curricula for teaching and learning algebra. In A.J. Bishop, K. Clements, C.Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Part 1, pp. 139–160). Dordrecht, The Netherlands: Kluwer.

Fischbein, E. F., & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Journal for Research in Mathematics Education*, *28*, 96–105.

Freiman, V., & Lee, L. (2004). Tracking primary students' understanding of the equality sign. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME International Conference*, *2*, 415–422.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht, The Netherlands: Reidel.

Fullan, M. (1982). *The meaning of educational change.* New York: Teachers College Press.

Fullan, M., & Hargreaves, A. (1991). *What's worth fighting for? Working together for your school: Strategies for developing interactive professionalism in your school.* Hawthorne, Victoia: Australian Council for Education Administrations in association with the Ontario Public School Teachers Federation.

Fuson, K. C. (1992). Research on whole number additiion and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). New York: MacMillan.

Gonzales, P., Guzmán, J. C., Partelow, L., Pahlke, E., Jocelyn, L., Kastberg, D., et al. (2004). *Highlights from the Trends in International Mathematics and Science Study (TIMSS) 2003* (NCES 2005–005). U.S. Department of Education, National Center for Education Statistics.

Graham, T., & Smith, P. (2004). An investigation into the use of graphics calculators with pupils in Key Stage 2. *International Journal of Mathematical Education in Science and Technology*, *35*, 277–237.

Gravemeijer, K. (1991). An instruction-theoretic reflection on the use of manipulatives. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 57–76). Utrecht, The Netherlands: CD-b Press.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education, *Journal for Research in Mathematics Education*, *25*, 443–471.

Gravemeijer, K. (1998). Developmental research as a research method. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (Part 1, pp. 277–296). Dordrecht, The Netherlands: Kluwer.

Gravemeijer, K. (2004). Local instructional theories as a means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*, 105–128.

Greer, B., Verschaffel, L., & Mukhopadhyay, S. (2007). Modelling for life: Mathematics and children's experience. In W. Blum, P. L. Galbraith, H. W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education*. New York: Springer.

Gray, E, Pitta, D., & Tall, D. (2000). Objects, actions, and images: A perspective on early number development. *Journal of Mathematical Behavior*, *18*, 401–413.

Groves, S. (1994). *Calculators: A learning environment to promote number sense.* Paper presented at the Annual meeting of the American Educational Research Association, New Orleans, LA.

Guskey, T. (1985). Staff development and teacher change. *Educational Leadership*, *42*(7), 57–60.

Hembree, R., & Dessart, D. (1992) Research on calculators in mathematics education. In J. Fey & C. Hirsch (Eds), *Calculators in mathematics education* (pp. 23–32). Reston, VA: National Council of Teachers of Mathematics.

Heirdsfield, A. M., & Cooper, T. J. (2002). Flexibility and inflexibility in accurate mental addition and subtraction: Two case studies. *Journal of Mathematical Behavior*, *21*, 57–74.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., & Murray, H. (1997). *Making sense: Teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Erlbaum.

Hiebert, J., & Wearne, D. (1986). The acquisition of decimal number knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Erlbaum.

Hill, H.C., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371–406.

Howson, A. G. (1982). *A history of mathematics education in England*. Cambridge, UK: Cambridge University Press.

Hoyles, C., & Sutherland, R. (1989). *Logo mathematics in the classroom*. London: Routledge.

Jones, I., & Pratt, D. (2005). Three utilities for the equals sign. In H. L. Chick & J. L Vincent (Eds.), *Proceedings of the 29th PME International Conference*, *3*, 185–192.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909–955). Greenwich, CT: Information Age Publishing and National Council of Teachers of Mathematics.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1997). A framework for assessing and nurturing young children's thinking in probability. *Educational Studies in Mathematics*, *32*, 101–125.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Nisbet, S. (2002). Elementary students' access to powerful mathematical ideas. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 113–141). Mahwah, NJ: Erlbaum.

Jones, G. A., Langrall, C. W., Thornton, C. A., Mooney, E. S., Wares, A., Jones, M. R., et al. (2001). Using students' statistical thinking to inform instruction. *Journal of Mathematical Behavior, 20*, 109–144.

Jones, G. A., Thornton, C. A., Langrall, C. W., Mooney, E. S., Perry, B., & Putt, I. J. (2000). A framework for characterizing children's statistical thinking. *Mathematical Thinking and Learning, 2*(4), 269–307.

Jones, G. A., Thornton, C. A., Putt, I. J., Hill, K. M., Mogill, A. T., Rich, B. S., et al. (1996). Multidigit number sense: A framework for instruction and assessment. *Journal for Research in Mathematics Education*, *22*, 310–336.

Jones, P. S., & Coxford, A. F. Jr. (1970). Reform, "revolution," reaction. In P.S. Jones (Ed.), *A history of mathematics education in the United States and Canada* (32nd yearbook, pp. 67–86). Washington, DC: National Council of Teachers of Mathematics.

Kaput, J. J., & Blanton, M. L. (2005). A teacher-centered approach to algebrafying elementary mathematics. In T. A. Romberg, T. P. Carpenter, & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 99–125). Mahwah, NJ: Erlbaum.

Kaput, J. J., & Blanton, M. L. (2000). *Algebraic reasoning in the context of elementary mathematics: Making it implementable on a massive scale*. Dartmouth, MA: University of Massachusetts–Dartmouth. (ERIC Document Reproduction Service No. ED441663)

Kato, Y., Kamii, C., Ozaki, K., & Nagahiro, M. (2002). Young children's representations of groups of objects: The relationship between abstraction and representation. *Journal for Research in Mathematics Education*, *33*, 30–45.

Kelly, A. E., & Lesh, R. A. (2000). *Handbook of research design in mathematics and science education*. Mahwah, NJ: Erlbaum.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*, 317–326.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up*. Washington, DC: National Academy Press.

Klein, A. S., & Beishuizen, M. (1998). The empty number line in Dutch second grades: *Realistic* versus *gradual* program design. *Journal for Research in Mathematics Education*, *29*, 443–464.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

Lamon, S. J. (1993). Ratio and proportion: Connecting content and children's thinking. *Journal for Research in Mathematics Education*, *24*, 41–61.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education*, *27*, 170–193.

Lehrer, R. & Chazan, D. (1998). *Designing learning environments for developing understanding of geometry and space*. Mahwah, NJ: Erlbaum.

Lehrer, R., & Romberg,T. A. (1996). Exploring children's data modeling. *Cognition and Instruction*, *14*(1), 69–108.

Lehrer, R., & Schauble, L. (2004). Modeling natural variation through distribution. *American Educational Research Journal*, *41*, 635–679.

Lesh, R. (2000). Beyond constructivism: Identifying mathematical abilities that are most needed for success beyond school in an age of information. *Mathematics Education Research Journal*, *12*, 177–195.

Lesh, R., Amit, M., & Schorr, R. Y. (1997). Using "real-life" problems to prompt students to construct conceptual models for statistical reasoning. In I. Gal, & J. B. Garfield (Eds.), *The assessment challenge in statistical education* (pp. 65–83). Amsterdam, The Netherlands: IOS Press.

Lester, F. K. (1980). Research on mathematical problem solving. In R. J. Shumway (Ed.), *Research in mathematics education* (pp. 286–323). Reston, VA: National Council of Teachers of Mathematics.

Lubienski, S. (2002). Research, reform, and equity in U.S. mathematics education. *Mathematical and Learning, 4*(2&3), 103–125.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*, 16–32.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, *26*, 422–441.

Malloy, C. E., (2004). Equity in Mathematics Education Is about Access. In R, Rubenstein & G. Bright (Eds.), *Perspectives on the teaching of mathematics, Sixty-sixth yearbook* (pp. 1–14). Reston, VA: NCTM.

Masingila, J. O., & Doerr, H. M. (1998). Use of function ideas in everyday activities. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the Twentieith Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 562–568). Columbus, OH: The ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

McLellan, J. A., & Dewey, J. (1908). *The psychology of number and its applications to methods of teaching arithmetic.* New York: D. Appleton and Company.

Moreno-Armella, L,, & Block, D. (2002). Democratic access to powerful mathematics in a developing country. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 301–321). Mahwah, NJ: Erlbaum.

Moss, J., & Case, R. (1999). Developing children's understanding of rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, *30*, 122–147.

Moyer, P. S., Bolyard, J. J. & Spikell, M. A. (2002). What are virtual manipulatives? *Teaching Children Mathematics, 8*, 372–377.

Moyer, P. S., Niezgoda, D. & Stanley, J. (2005). Young children's use of virtual manipulatives and other forms of mathematical representations. In W. J. Masalski & P. C. Elliott (Eds.), *Technology-supported mathematics learning environments: Sixty-seventh yearbook* (pp. 17–34). National Council of Teachers of Mathematics: Reston, VA.

Mulligan, J. T., & Mitchelmore, M. C. (1997) Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education*, *28*, 309–331.

Mullis, I. V. S., Martin, M. O., Gonzales, E. J., & Chrotowski, S. J. (2004). *TIMSS 2003 international mathematics report: Findings from IEA's trends in international mathematics and science study at fourth and eighth grades.* Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Lynch School of Education, Boston College.

National Advisory Committee on Mathematics Education. (1975*). Overview and analysis of school mathematics, K-12.* Washington, DC: Conference Board of the Mathematical Sciences._

National Council of Teachers of Mathematics. (1990). *NCTM News Bulletin, 27.* Reston, VA: Author

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). P*rinciples and standards for school mathematics.* Reston, VA: Author.

Nisbet, S. (2005). Regional differences in the professional development needs and preferences of teachers of primary mathematics. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A McDonough, R. Pierce, & A. Roche. *Building connections: Research, theory and practice. Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australasia (MERGA). Vol. 2* (pp. 577–584). Melbourne, Vic.: MERGA.

Nisbet, S., Jones, G. A., Thornton, C. A., Langrall, C. W., & Mooney, E. S. (2003). Children's representation and organisation of data. *Mathematics Education Research Journal, 15*, 42–58.

Niss, M. (1996). Goals of mathematics teaching. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Part 1, pp. 11–48). Dordrecht, The Netherlands: Kluwer.

Noss, R. (2002). Mathematical epistemologies at work. *For the Learning of Mathematics, 22*, 2–13.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers.* Dordrecht, The Netherlands: Kluwer.

Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning, 1*, 279–315.

Oliveira, H., Segurado, I., da Ponte, J., & Cunha, H. (1997). Mathematical investigations in the classroom: A collaborative project. In V. Zach, J. Mousley, & C. Breen (Orgs.) *Developing practice: Teachers' inquiry and educational change* (pp. 135–142). Geelong, Australia: Centre for Studies in Mathematics, Science and Environmental Education.

Outhred, L. N., & Mitchelmore, M. C. (2000). Young children's intuitive understanding of rectangular area measure. *Journal for Research in Mathematics Education*, *31*, 144–167.

Page, D. (1959). The University of Illinois Arithmetic Project. *American Mathematical Monthly*, *66*, 412–422.

Papert, S. (1980). Computer-based microworlds as incubators for powerful ideas. In R. Taylor (Ed.), *The computer in the school: Tutor, tool, tutee* (pp. 203–210). New York: Teachers College Press, Columbia University.

Petrosino, A., Lehrer, R., & Schauble, L. (2003). Structuring error and experimental variation as a distribution in the fourth grade. *Mathematical Thinking and Learning*, *53*, 131–156.

Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education*, *31*, 602–625.

Ralston, A (1989). A framework for the school mathematics curriculum in 2000. In J. Malone, H. Burkhardt, & C. Keitel (Eds.), *The mathematics curriculum: Towards the year 2000* (Collected papers on this theme from the 6th International Congress on Mathematical Education) (pp.33–42). Perth, Western Australia: Science and Mathematics Education Centre, Curtin University of Technology.

Reimer, K., & Moyer, P. (2005). Third-graders learn about fractions using virtual manipulatives: A classroom study. *Journal of Computers in Mathematics and Science Teaching*, *24*, 5–25.

Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211–246.

Reynolds, A., & Wheatley, G. H. (1996). Elementary studetns' construction and coordination of units in an area model. *Journal for Research in Mathematics Education*, *27*, 564–581.

Rojano, T. (2002). Mathematics learning in the junior secondary school: Students' access to significant mathematical ideas. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 143–163). Mahwah, NJ: Erlbaum.

Romberg, T. A., & Kaput, J. J. (1999). Mathematics worth teaching, mathematics worth understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 3–17). Mahwah, NJ: Erlbaum

Ruthven, K. (1996). Calculators in the mathematics curriculum: The scope of personal computational technology. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Part 1, pp. 435–468). Dordrecht, The Netherlands: Kluwer.

Saenz-Ludlow, A. (2003). A collective chain of signification in conceptualizing fractions: A case of a fourth-grade class. *Journal of Mathematical Behavior*, *22*, 181–211.

Saenz-Ludlow, A., & Walgamuth, C. (1998). Third graders interpretations of equality and the equal symbol. *Educational Studies in Mathematics*, *35*, 153–187.

Sarama, J. & Clements, D. (2002). Building blocks for young children's mathematical development. *Journal of Educational Computing Research*, *27,* 93–110.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.) *Handbook for research on mathematics teaching and learning* (pp. 334–370). Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H. (2002). Making mathematics work for all children: Issues of standards, testing, and equity. *Educational Researcher*, *31*(1), 13–25.

School Mathematics Study Group. (1972). *The prediction of mathematics achievement* (NLSMA Reports No. 27). Stanford, CA: Author.

Secada, W. G. (1992). Race, ethnicity, social class, language, and achievement in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 623–660). New York: Macmillan.

Senk, S. L., & Thompson, D. R. (2003). *Standards-based school mathematics curricula.* Mahwah, NJ: Erlbaum.

Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning*, *2*(3), 157–189.

Shaughnessy, J. M. (2007). Research on statistics learning and reasoning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning.* Greenwich, CT: Information Age Publishing and National Council of Teachers of Mathematics.

Shuard, H., Walsh, A., Goodwin, J., & Worcester, V. (1991). *Calculators, children and mathematics.* London: Simon & Schuster.

Silver, E. A., Smith, M. S., & Nelson, B. S. (1995). The QUASAR Project: Equity concerns meets mathematics education reform in the middle school. In W. G. Secada, E. & L. B. Adajian (Eds.), *New directions for equity in mathematics education* (pp. 9–56). Cambridge, MA: Cambridge University Press.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114–145.

Simon, M. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 55–86). Mahwah, NJ: Erlbaum.

Simon, M. A. (2000). Research on the development of mathematics teachers: The teacher development experiment. In R. Lesh & E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 335–364). Mahwah, NJ: Erlbaum.

Simon, M. A., & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspectives: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education*, *30*, 252–264.

Skemp, R. R. (1971). *The psychology of learning mathematics.* Middlesex, UK: Penguin.

Skovsmose, O., & Valero, P. (2002). Democratic access to powerful mathematical ideas. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 383–407). Mahwah, NJ: Erlbaum.

Sowder, J. T., & Schappelle, B. P. (Eds.). (1989). *Establishing foundations for research on number sense and related topics: Report of a conference.* San Diego, CA: San Diego State University Center for Research in Mathematics and Science Education.

Stacey, K. (1994). *Calculators in primary mathematics: An analysis of classroom activities.* Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.

Steen, L. A. (1990). Patterns. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 1–10). Washington, DC: National Academy Press.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes learning trajectories of Jason and Laura: Grade 5. *Journal of Mathematical Behavior, 22*, 237–295.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning*, *6*, 129–162.

Steffe, L. P. & Olive, J. (2002). Design and use of computer tools for interactive mathematical activity (TIMA). *Journal of Educational Computing Research, 27*, 55– 76.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Praeger Scientific.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education — A paradigm of developmental research.* Dordrecht, The Netherlands: Kluwer.

Sutherland, R., & Rojano, T. (1993). A spreadsheet approach to solving algebra word problems. *Journal of Mathematical Behavior*, *12*, 351–383.

Tate, W., & Rousseau, C. (2002). Access and opportunity: The political and social context of mathematics education. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 271–300). Mahwah, NJ: Erlbaum.

The Learning Federation. (2007). *About TLF content.* Retrieved March 16, 2007 from http://www.thelearningfederation.edu.au/tlf2/showMe.asp?md=p&nodeIID=2.

Thomas, G., Tagg, A., & Wood, J. (2002). Making a difference: the Early Numeracy Project. In In B. Barton, K. Irwin, M. Pfannkuch, & M. Thomas (Eds.) *Mathematics Education in the South Pacific, Proceedings of the 25th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 49–57). Auckland, NZ: MERGA.

Thomas, N. D., Mulligan, J. T., & Goldin, G. A. (2002). Children's representation and structural development of the counting sequence 1–100. *Journal of Mathematical Behavior*, *21*, 117–133.

Thorndike, E. L. (1924). *The psychology of arithmetic.* New York: Macmillan.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction: The Wiskobas Project.* Dordrecht, The Netherlands: Reidel.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education*, *30*, 390–416.

U.S. Department of Education, National Center for Education Statistics, (2005). *The Condition of Education 2005* (NCES 2005-094). Washington, DC: U.S. Government Printing Office.

van Hiele, P. M. (1986). *Structure and insight.* Orlando, FL: Academic Press.

Verschaffel, L., & De Corte, E. (1997). Teaching realistic mathematical modeling in the elementary school: A teaching experiment with fifth graders. *Journal for Research in Mathematics Education*, *28*, 577–601.

Verschaffel, L., De Corte, E., & Vierstraete, H. (1999). Upper elementary school pupils' difficulties in modeling nonstandard additive word problems involving ordinal numbers. *Journal for Research in Mathematics Education*, *30*, 265–285.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557–628). Greenwich, CT: Information Age Publishing and National Council of Teachers of Mathematics.

vonGlasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The inverted reality* (pp. 17–40). New York: Norton.

Warren, E. (2003a). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Education Research Journal*, *15*, 122–137.

Warren, E. (2003b). Young children's understanding of equals: A longitudinal study. In N. A. Pateman, B. Dougherty, & J. Zilloux (Eds.), *Proceedings of the 27th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 379–386). Honolulu, Hawaii: University of Hawaii.

Watson, J. M., Kelly, B. A., Callingham, R. A., & Shaughnessy, J. M. (2003). The measurement of school students' understanding of statistical variation. *International Journal of Mathematical Education in Science and Technology*, *34*, 1–29.

Weber, K. (1990). Solid mathematical knowledge and ability that may be readily applied – What every student must learn: Mathematics teaching in the German Democratic Republic. In I. Wiszup and R. Streit, (Eds.), *Developments in school mathematics education around the world* (Vol. II, pp. 77–97). Reston, VA: National Council of Teachers of Mathematics.

Wheatley, C. (1980). Calculator use and problem-solving performance. *Journal for Research in Mathematics Education*, *11*, 323–334.

Wittmann, E. C. (1998). Mathematics education as a "design science." In A.Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (Part 1, pp. 87–104). Dordrecht, The Netherlands: Kluwer.

Wooton, W. (1965). *SMSG: The making of a curriculum*. New Haven: Yale University Press.

Young-Loverege, J. (2005). The impact of mathematics education reform in New Zealand: taking children's views into account. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A McDonough, R. Pierce & A. Roche. *Building Connections: Research, Theory and Practice. Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australasia (MERGA), Vol. 1* (pp. 18–32). Melbourne, Victoria: MERGA.

Zevenbergen, R. (2003). Reforming mathematics education: a case study within the context of new times. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics Education Research: Innovation, networking, opportunity: Proceedings of the 26th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 791–798). Deakin University, Geelong: MERGA.