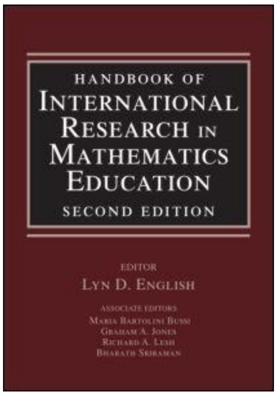

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The ordering of discontinuities in school algebra

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Richard Noss (2001) suggested that what is natural to express and how one might do so both may change as a function of the expressive tools available to a culture at a given time. This view of the power of tools and media underlies much of the enthusiasm in mathematics education for technological tools. Perhaps technology might upset the hierarchies of prerequisite skills that often seem to dictate the practice of mathematics in schools. Perhaps technology might aid teachers in making accessible to learners' powerful mathematical ideas in a different sequence and rate than has been traditionally deemed feasible (enthusiasts of such views like Papert, 1996 or Schwartz, 1999, differ often about which practices of schooling they imagine technology will up-end).

But technology does not land on a pristine curricular *tabula rasa*. Technology interacts in important ways with the desires of mathematics educators to work for what David Tall (2002) calls "long term learning." Referring to Richard Skemp's work on understanding why logically organized curricula may not understood by learners, Tall wrote:

> … He [Skemp] referred to the difference between learning which is an expansion of current knowledge—where new ideas fit easily with current schemas—and reconstruction of knowledge, where the old schemas must be reflected upon and modified to fit with the new ideas. (p.165)

In Tall's terms, when learners in school need to reconstruct their knowledge in order to make progress, then they have experienced a discontinuity in the curriculum. He goes on to chronicle such discontinuities, to suggest that they cannot be made disappear, and to argue that they are a crucial matter of curricular design and understanding.

> Many examples have been given [in this paper] of discontinuities in the curriculum in which a change in context requires considerable cognitive reconstruction from many children which may prove too difficult for many of these. These include many contexts which occur in the English National Curriculum which provoke difficulties for children and cause many of them to take the line of least resistance and learn ideas which are meaningless to them by rote. (p. 184)

Note that in Tall's terms such discontinuities are not places in a curriculum where all learners will experience substantial difficulties with the mathematics. Instead, for Tall discontinuities are places where cognitive reconstruction is called for, even if learners do not find this reconstruction difficult, or if the discontinuity is glossed over by the curriculum. Tall asserted that such discontinuities are inevitably part of mathematics teaching and learning and should be explored, rather than avoided:

> I do not believe it is sensible or practical to attempt to design a smooth long-term curriculum in which each idea builds easily on previous ones. New contexts will always demand new ways of looking at things and often require significant cognitive reconstruction. Awareness of the essential nature of cognitive reconstruction is vital in long-term curriculum design. (p. 185)
>
> Even when long-term curricula are carefully designed using a cognitive approach, some of the conflicts persist. If they are directly addressed by the teacher as mentor in a meaningful context then it is possible to give a wider spectrum of students more coherent insights into the nature of the mathematics. A standard curriculum which takes things in a steady order, avoiding difficulties till later can create a Pandora's box of different ideas that need to be rationalised (but are more probably, ignored) later on. (p. 184)

Our contribution to this volume uses Tall's construct of curricular discontinuities (as distinct from, and not synonymous with, the cognitive difficulties of learners). It examines how the introduction of technology potentially alters the nature of curricular discontinuities (rather than removing them), and how different technologies can be used by curriculum designers to alter the nature of the discontinuities that students face. As we view the potential of technology to support curricular development and change, we will also raise questions about the assessment of curricula. How might societies use research to make choices about the school curriculum that they are to use? And, if there is a role for research, what sorts of research might be useful in such decision-making (see National Research Council, 2004 for a report that tackles this question)?

In our view, it is crucial that mathematics educators develop better descriptions of how curricula differ and what bets (cognitive, pedagogical, and technological) particular curricular make. For example, we think that in describing the use of technology in functions-based approaches to algebra it is useful to describe the nature of the discontinuities that students encounter as they progress through curricula (even if such discontinuities do not seem to pose learning difficulties for students). In our view, it is less important to be able to make statements about the effectiveness of a particular curriculum or the use of a particular technological innovation and more important to be able to associate evaluation data about a curriculum with key aspects of the design of the curriculum, like the discontinuities students will encounter. Such information would allow evaluation data to accumulate across different curriculum development projects in important and useful ways.

OUTLINE OF THE CHAPTER

In our contribution to the first edition of the *Handbook of International Research in Mathematics Education* (Yerushalmy & Chazan, 2002), we outlined useful distinctions in the cognitive literature on students' learning of algebra. In that chapter, we used those distinctions to:

- Describe a typical U.S. approach to school algebra,
- Outline instructional affordances of spreadsheets and graphing calculators,
- Sketch issues of teacher knowledge revealed by one preservice teacher's understandings of tensions between the curriculum she was to teach and the affordances of graphing calculators she used with her students, and

- Examine, in the context of a functions-based school algebra curriculum, issues of student learning with respect to equations in two variables.

In this contribution to the new edition of the handbook, our focus is more carefully on technology as a catalyst for rethinking algebra curricula. We will use the same distinctions from the cognitive literature to:

- Outline different potential curricular understandings of what it is that equations represent (in line with Tall, 2002 we would suggest that there are discontinuities between these views and that as curricula help students gain a deep and flexible understanding of equations the nature of the discontinuities students face are a crucial aspect of curricular decision making).
- Analyze how graphing tools, symbol manipulators, and spreadsheets offer different affordances with respect to different views of equations.
- Using these understandings of equations, describe the order in which students in a technologically based, middle school, Israeli algebra curriculum (Herskhowitz et al., 2002) meet different view of equation and the discontinuities potentially raised by this ordering.
- Using the notion of discontinuities, and based on a different Israeli algebra curriculum (Visual Math 1995), examine alternative choices at three points in this curricular sequence and the discontinuities they entail.

Our focus in raising these alternatives is not to critique the curricula we are reviewing, but rather to illustrate Noss' (2001) general point with concrete examples from function-based curricula, and to suggest that Tall's (2002) notion of curricular discontinuities is one crucial consideration in making curricular choices. With each of these alternatives, we outline what the mathematical choice is with respect to views of an equation and the relationship between these alternative choices and the technological tools available.

With these distinctions from the cognitive literature, the curricular descriptions of what an equation can represent, and our description of a particular curricula and some alternative choices, we hope to outline a way in which research and evaluation of technology based curricula can make use of the literature of student learning to do research on curricular design that can accumulate in useful ways.

Using distinctions from the cognitive literature to outline different views of equations

Over the last 15 years, research from a cognitive perspective has identified tasks that cause learners of school algebra difficulty in systematic ways (reviews of this literature include: Kieran, 1992; Bednarz, Kieran, & Lee, 1996; Wagner & Kieran, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Wenger, 1987). In this section, we outline distinctions from that literature that seem potentially useful for describing nuances in school algebra curricula. Though the labels we will use are sometimes mathematical labels, in our view these distinctions are psychological, rather than logical. With all of these distinctions, there is a fuzziness of the kind that mathematical definitions are often meant to dispel. Rather than to try to define such fuzziness away, we intend instead to emphasize that employment of these distinctions will always be a matter of interpretation.

Particular studies in the research literature have focused on the equal sign (e.g., Herscovics & Kieran, 1980; Kieran, 1989), literal symbols (e.g., Schoenfeld & Arcavi, 1988; Sleeman, 1984; Usiskin, 1988; Wagner, 1981; Kuchemann, 1978), and graphing (e.g., Bell & Janvier, 1981; Goldenberg, 1988; Monk & Nemirovsky, 1994; Nemirovsky, 1994. For a book length treatment, see Romberg, Fennema, & Carpenter, 1993.). These studies suggest psychological dimensions of learners' experience of curriculum that may not map directly onto logical distinctions one might develop in a mathematical exposition. One way to interpret many of the studies in this tradition is that they identify discontinuities that learners experience, but that the curriculum—with its focus on solution methods to particular problem types—does not address explicitly.

To illustrate the use of these distinctions, let us focus on equations. What is an equation? School algebra texts have standard answers to this question. In a standard approach, an equation is "a sentence about numbers" (Dolciani & Wooton, 1973, p. 24),[1] or "a pattern for the different statements—some true, some false—which you obtain by replacing each variable by the names for the different values of the variable" (p. 44), or "any statement of equality" (p. 583).[2] Such descriptions are true, in some sense, of all equations in an introductory algebra course. Yet, there are important differences between strings of symbols that are labeled as equations.

To help illustrate the psychological (as opposed to logical) complexity that mathematics education researchers see in school algebra, Zalman Usiskin (1988) presents five equations (in the sense of having an equal sign in them) involving literal symbols. Each has a different feel. (See Mason, 1989, for a related point.)

1. A = LW
2. 40 = 5x
3. sin x = cos x · tan x
4. 1 = n · 1/n
5. y = kx

As Usiskin observes, "We usually call (1) a formula, (2) an equation (or open sentence) to solve, (3) an identity, (4) a property, and (5) an equation of a function of direct variation (not to be solved)" (p. 9).

Though readers might agree or disagree with his classification of these strings, that is part of his point. To the enculturated, each of these symbol strings reads differently (and perhaps differently enculturated readers read them differently as well). Furthermore, Usiskin (1988) argues that each of these equations has a different feel because in each case the idea of "variable" is put to a different use.

> In (1), *A, L,* and *W* stand for the quantities area, length, and width and have the feel of knowns. In (2), we tend to think of *x* as unknown. In (3), *x* is an argument of a function. Equation (4), unlike the others, generalizes an arithmetic pattern, and *n* identifies as instance of the pattern. In (5), *x* is again an argument of a function, *y* the value, and *k* a constant (or parameter, depending on how it is used). Only with (5) is there the feel of "variability," from which the term *variable* arose. (p. 9)

Once again, any disagreements readers might have about Usiskin's description of the use of the idea of variable in any of these equations might just serve to strengthen his point; there are important complexities at the heart of school algebra. In the words of Schoenfeld and Arcavi (1988), "The mathematical meaning of a statement is determined by its context" (p. 424) and "the meaning of variable is variable" (p. 425).

Let's elaborate on Usiskin's point by adding an equation:

6. 3x+y=7

This sort of equation has a particular flavor. It fits well with Dolciani and Wooton's (1970/73) definition. If one replaces x with 2 and y with 1, one has a true statement. If one replaces x with 1 and y with 1, one has a false statement. When one graphs an equation like this one on the Cartesian plane, the plane is made up of points. The graph consists of those points for which the equation is a true statement. For the remaining values in the plane, the statement is false. Creating the graph is a matter of identifying members of the solution set.

There is an intriguing contrast between equation 6 and equation 2. With equation 6, the xs and ys in an equation in two "variables" seem less fixed than the xs in equation 2. If one is given a value for x in 3x+y = 7, one then has an equation in one variable that can be solved. One can solve for the related value for y, or for x if one is given y. On the other hand, as compared to the independent variable in a function of one variable (e.g., a variant of equation 6, f(x) = 7–3x, that is similar to Usiskin's equation 5), x and y in 3x+y = 7 have a different flavor, as does the equal sign. When the focus is solely on the values of the letters for which the equation is true, x and y might be thought of as of yet unknown numbers, rather than quantities that vary, or indeterminate objects (Bell, 1995; Schoenfeld & Arcavi, 1988; Usiskin, 1988). The xs and ys that make the statement true, taken in pairs, constitute members of the solution set; whereas in a function of one variable with an explicit correspondence rule, values of the independent variable are substituted into a computation that determines the dependent variable. Thus, in the function of one variable truth somehow does not seem directly relevant (though of course one can force it to be).

In the context of solving 3x+y = 7, the equal sign between two sides of the equation can be thought of in two different ways. On the one hand, the equal sign can be thought of as what Matz (1982) calls equality as constraint—the equality of two expressions constrains the values that a letter can take. A second approach takes for granted the condition of truth. It suggests that when the statement is true, the equal sign indicates that two different expressions, 3x+y and 7, are names for the same number; they can be used interchangeably (Kieran, 1981 calls this use of the equal sign equivalence relations).

By way of contrast, there are similar looking equations in school algebra that have quite a different "feeling" to them (following Usiskin, 1988), for example y = 7–3x (a variant of equation 6). From one perspective, nothing has changed. We still have an equation. The solution set to this equation is identical to the solution set to 3x+y = 7. These are arguably different representations of the same cognitive construct (as is suggested in Crowley & Tall, 1999).

There are subtle changes to the equation and its constituents, however. Whereas 3x+y = 7 was an implicit function—if one substituted a value for x, only one y value would result—y = 7-3x is explicitly a function. This form of the equation indicates that given any x value, one does not need to solve for y, one can merely compute it. In this sense, the value of y can be said to depend on the value of x. To capture this sense of the equation, one might use the notation y = f(x) = 7–3x—y is a function of x.

This notation helps capture the notion that the equation has changed. This equation no longer fits the definition "a pattern for the different statements—some true, some false—which one obtain by replacing each variable by the names for the different values of the variable" (as suggested in Dolciani & Wooton, 1970/73, p. 44) as nicely. Instead this "equation" feels like a correspondence rule that "assigns to each number in the domain of the variable x another number, the value of y" (p. 146). Similarly, the letters and equal sign in the equation have a different feel (as discussed in Mason, 1989). In an attempt to capture this difference, one might now refer to x and y as variables rather than unknown numbers (Bell, 1995; Schoenfeld & Arcavi, 1988; Usiskin, 1988). The form of this equation allows one to keep track of how y varies as x varies. The equal sign no longer constrains the values of the variables. It can be thought of, instead, as assigning a label to the outcome of a computational process (discussed in Freudenthal, 1973; Matz, 1982). When one graphs this function on the Cartesian plane, though the graph appears to be identical to that of the previous equation, the natures of the axes and of the plane have changed. x is now the independent variable and y the dependent. For any value of x, there can only be one value of y; the whole plane is not active in the way that it previously was. And, as mentioned earlier, truth seems less relevant. The points on the graph are a set, but what creates them seems less a statement that is true or false than a defining rule. As a result, the domain structures one's view of the function; one can create the graph by systematically substituting the values of x into the function and computing the y values.

Stepping back from the points we've just made about the nature of the xs and ys, of the equal sign, and of the use of the Cartesian plane, from our point of view, the definition of an equation as a string of symbols with an equal sign encompasses at least the following four different ways of thinking about equations of one variable, as well as two other ways of thinking of equations in two variables. We suggest that all of these views of equations are present in introductory school algebra (Usiskin, 1988 suggests other categories as well, ours are elaborations of distinctions inside his categories of (2) an equation or open sentence to solve or (5) an equation of a function of direct variation (not to be solved).) and that movement between these views of equations are curricular discontinuities. We will illustrate the different perspectives on equations in one and two variables by using an illustrative equation in one variable (3x+2 = 7), in addition to equation 6.

If one thinks of 3x+2 = 7 as a representation of a set, the solution set to this equation, then implicitly one is thinking of a conditional statement. This statement is of the form, if x takes on the values in the specified solution set, then, 3x+2 will equal 7. The question one asks when solving an equation is: what set of numbers can be used in the "if" part of the statement as replacements for x to make the conditional statement a true statement? One might then represent this solution set on a number line.

In contrast, one can think of this same equation 3x+2 = 7 as a template for producing sentences about numbers, sentences that can be true or false depending on the values used to replace x in this template. One then asks for what replacements of x is this statement numerically true, and for what replacements of x is this statement numerically false. Unlike the previous way of thinking about equations, the solution to the equation does not determine the replaces for x, one determines the replacement set for x. To represent this way of thinking, one might make a table keeping track of true/false by value of x (though this way of thinking is not often explicitly made part of the school curriculum).

While these two ways of thinking about 3x+2 = 7 are relatively close to each other, there are ways that involve functions that seem quite different. First, one might view 3x+2 as the defining expression for a function and ask for what input will this function produce the output 7? Such a question might lead to the creation of a table that lists input values and their outputs.

What does an equation in one variable represent?	Questions asked when solving an equation in one variable	Linked tabular and graphical representations
• a 1 d solution set	In a conditional statement of the form "For this set of replacement values for x, the equation generates true statements," what numbers can be used to make the conditional statement a true statement?	• No table. • Graph on number line.
• A set of arithmetic sentences	Which arithmetic sentences generated from the template provided by this equation are true and which are false?	• Two column table: values of x, T/F. • No graphs.
• a coordinate pair with an unknown input	What input to the expression on one side will produce the output on the other?	• Two column table: values of x, output of the expression in x. • A point on a graph in the Cartesian plane.
• A comparison of two functions of one variable	For what input to the expressions on either side of the equality, will the outputs of each be the same?	• Three column table: common values of x, results of the two expressions. • Graph of two functions in the Cartesian plane.

Figure 30.1 Ways of thinking about equations in one variable, e.g., 3x+2 = 7.

Alternatively, one could think of 3x+2 = 7 as a comparison of two functions where 3x+2 is being compared with the constant function g(x) = 7 (this way of thinking might seem more plausible for equations like 3x+2 = 7x where neither of the functions is a constant function; see Figure 30.1). Such a comparison might lead to the graphing of these two functions on a Cartesian plane and to the development of a table that lists the outputs of either side as a function of their shared independent variable. As suggested in Chazan (1993), this latter perspective shifts qualitatively how one models simple word problems; and as hypothesized in Yerushalmy and Gilead (1999) some standard word problem tasks become simpler when this perspective is adopted, while others become more difficult.

When equations include two variables, these four ways of thinking continue though in slightly altered forms, and there are two new possibilities that are introduced by the second variable. In an analogy to 3x+2 = 7, 3x+y = 7 can be thought of as a representation of a two-dimensional solution set, the coordinated pairs of values for x and y that will make the equation a true statement. Then, rather than simply considering the equation as a representation of a solution set of ordered pairs, 3x+y = 7 can be viewed as representing a relation between the values that the two variables take on for members of this solution set. This sort of relation between two variables is one new wrinkle that enters with the move to two variables. Such relations are typically graphed in the Cartesian plane as the points whose values make the equation a true statement. And, like 3x+2 = 7, 3x+y = 7 can be viewed as representing a set of mathematical sentences about numbers.

Turning to views that involve functions, in an analogy to 3x+2 = 7, 3x+y = 7 can be thought of as a representation of a set of coordinate triples whose two independent variable values are not known; one is looking at the function 3x+y and asking for what values of x and y does this expression produce the number 7. In addition, an equation in two variables can be a representation of a function of one variable; some equations in two variables can be manipulated to show that they are explicitly a function of one. Such functions, like relations are a new perspective that becomes available with the new variable. And, like relations, these functions of one variable are similarly graphed on the plane, though the nature of that plane is slightly different than the plane in which relations are graphed (Yerushalmy & Chazan, 2002). Finally, like 3x+2 = 7, 3x+y = 7 can be viewed as a comparison of two functions of two variables (see Figure 30.2) [f(x,y) = 3x+y & g(x,y) = 7, though again it is useful to also think of equations, like 3x+y = 7x, where neither of the functions is a constant function].

In our view, fluency in algebra involves the capacity to move flexibly between these different views of equations.

UNDERSTANDING THE INSTRUCTIONAL AFFORDANCES OF SPREADSHEETS, SYMBOLIC MANIPULATORS, AND GRAPHING TOOLS WITH RESPECT TO EQUATIONS

Elsewhere we have argued (Yerushalmy, 1999) that technological tools are not neutral with respect to the different views of x, meanings of the equal sign, uses of the Cartesian coordinate system, and the meanings of equation outlined earlier. As a result, it does not seem fruitful to discuss the capacity of technological tools to support algebra instruction in general. Instead, it seems more useful to describe how particular technological tools, or perhaps even specific parts of a particular tool, have the capacity to contribute to particular curricular approaches by supporting particular views of what equations represent. Doing so, once again, involves us with the distinctions about the role of x, the nature of the equal sign, and the use of the Cartesian coordinate system outlined in the second section of this chapter.

To illustrate relationships between technology and curricular approaches to school algebra, in this section, we will discuss spreadsheets, computer algebra systems (CAS) and graphing calculators, three technological tools that are touted for algebra instruction. The tools were

What does an equation in two variables represent?	Questions asked when solving an equation in two variables	Linked tabular and graphical representations
• a 2 d solution set	In a conditional statement of the form "For this set of replacement values for x and y, the equation generates true statements," what coordinated pairs of numbers can be used to make the conditional statement a true statement?	• Two column table of coordinated x's and y's. • Graph of an equation in the Cartesian plane (any point can be on or off).
• a relation between replacement sets for the two variables	What coordinated pairs of values are related in the way indicated by this equation?	• Two column table: x and y, where one is computed from the other. • Graph of an equation in the Cartesian plane (any point can be on or off).
• a set of arithmetic sentences	Which arithmetic sentences generated from the template provided by this equation are true and which are false?	• Two-dimensional table with x values on one dimension, y on another, and T or F in the cells. • Graph in the Cartesian plane (any point can be on or off).
• a coordinate triple with unknown inputs	What inputs to the expression on one side will produce the output on the other?	• Two-dimensional table with x values on one dimension, y on another, and output of the expression in the cells. • A point on a graph in 3 space.
• a function of one variable	What coordinated pairs of values are generated by replacing x with members of a given set (the domain)?	• Tables with x, and f(x) computed from x. • Graph of a function in the plane (only one point a particular x value)
• a comparison of two functions of two variables	For what inputs to the expressions on either side of the equality, will the outputs of each be the same?	• Table with x values on one dimension, y on another, and output of the two expressions in each cell. • Graph of two functions in 3 space.

Figure 30.2 Ways of thinking about equations in two variables: e.g., $3x+y = 7$.

not created initially with the teaching of school algebra in mind. Instead they were designed to support the doing of mathematics by educated users, users already comfortable with the multiplicity of meanings of the representational systems used in school algebra. Spreadsheets and graphing calculators are both tools that support users in making observations about relations between quantities, given either with algebraic symbols or developed from the user's own mathematical ideas. Symbolic manipulators are the core of CAS. Like spreadsheets, they were originally designed for users of mathematics, scientists and engineers who are users of symbolic procedures rather than for learners of mathematics. A key strength of symbol manipulators is their capability to perform algebraic symbolic procedures quickly and correctly. This capability when integrated with accessible graphic and numeric representations of expressions potentially support changes to algebra and calculus curricula. All three tools support multiple representations of functions (Heid, Choate, Sheets, & Zbiek, 1995) and aim to reduce the cognitive load for the user of interaction with some aspects of mathematical symbol systems.

But, in our view, these kinds of technological tools are different with respect to some of the distinctions outlined in section two. As a result, they support different views of equations differentially. We think it is important to analyze relationships between technological tools and the views of mathematical objects that they support in order to understand the nature of the discontinuities they may lead to, before integrating tools not explicitly designed for educational purposes into teaching practice (for a similar sort of analysis, see Philipp, Martin, & Richgels, 1993 and for one teacher's appreciation of this issue, see our earlier chapter, Yerushalmy & Chazan, 2002.).

In this section, we will make some remarks about how each tool supports:

- Work with multiple quantities,
- Expression of equality as comparison, and
- Equivalence of equations, rather than expressions.

Equations and spreadsheets

Filloy, Rojano, and Rubio (2000), Sutherland and Balacheff (1999), and Rojano (1996) described a spreadsheet-supported solution process of a word problem. The problem asks students to compute the length and width of a rectangular field, if its perimeter 102 meters, and its length is double its width. We will use this particular problem, and the solution strategy they outline as a tool to discuss the affordances of spreadsheets with respect to equations.

As Rojano (1996) suggested in her conclusions, students who are beginners at algebra do not necessarily use "algebraic methods" to solve problems of this kind. Quite often they will use reverse arithmetic operations, "undoing," or what Rojano describes as a "whole/part" method (they compute the measure of each "part" where the whole is made in this problem of 6 parts).

For students who have not yet had any formal algebra studies, however, all three studies suggest that spreadsheets can support an "algebraic" strategy.[3] By this they mean that students use spreadsheet "formulas" to write an explicit rule for the perimeter of the rectangle based on a particular cell, specified by its location, and representing the width of the rectangle. As Sutherland & Balacheff (1999) indicated, the problem "… was crafted to provoke pupils to use a spreadsheet cell to represent an unknown quantity and to build up relationships with reference to the unknown" (p.11). In the given problem, the notion is that students will begin with a cell containing a value for the width and build up cells in other columns, eventually building up a cell with information on the perimeter as a function of the value in the original "width" cell. Students will then vary the values of the width by typing values into the original cell or by incrementing its values down a column. Regardless, the task is to search for values of a width cell that generate a perimeter of 102.

In such an approach, the correct width value is unknown. In that sense, students using the spreadsheet in this way are working analytically on known numbers, as opposed to synthetically, by reasoning on an unknown to make it known. This sort of work is different from traditional algebraic methods to solve for an unknown in the sense that there are numbers in the width column. Other columns are built from the specific numbers in the width cell; and, they refer to cells in the width column with symbols representing a location, rather than a letter as a name for a quantity. Finally, there are many values for the width that can be present simultaneously, conveying a sense of variation.

If one views columns as representing quantities in the problem, with cells being values of this quantity, the spreadsheet potentially affords some unique opportunities for working with many quantities, especially those that are related to each other. For example, in this problem, length depends on width. Thus, while the spreadsheet allows users to compute the perimeter solely from width, as well as from both length and width, its syntax easily supports the latter. Users can define the length column as double the width and then create the perimeter column from double the sum of length and width. When used in this way, the columns are linked and are not free in the sense of the unknowns in the equation $2(x+y) = 102$. Instead the perimeter column represents $2(x+(2x)) = 102$ with the substitution of $2x$ for y already carried out. Thus $2(X+Y) = 102$ does not mean "all the Cartesian combinations of X and Y whose sum is 102" but "the sum of all the same row pairs in the two columns representing length and width, where the width is already defined as twice the length." The capacity to write expressions involving linked variables is a powerful property of the spreadsheet and is indicated in the cognitive research (e.g., Rojano, 1996) as a way to overcome the complexity of writing an analytic model of two equations for two unknowns and substituting one equation in the other. Instead, work with spreadsheets provides a way to construct a single equation without ever needing to write a model of the problem involving two separate equations, or perhaps in another sense, the spreadsheet model is both a system of two equations in two unknowns and a single equation in one unknown at the same time.

In order to write an equation in two variables, and not a system, users would need to leave each quantity independent and unlinked. Only then could one represent the equivalent of twice the sum of the length and width is 102. One potential error in the use of spreadsheets would be students who do not initially encode the relationship between length and width in their spreadsheet representation of the problem and thus represent a single equation in two variables, rather than a system of two equations in two variables. In this context, substitution would involve the creation of new columns to take the role previously played by other columns.

Returning to the original problem and writing it as a single equation in width, note that the equation $w+2w = 102$ is a "start-unknown" (Nathan & Koedinger, 2000) problem whose goal is to determine the width that will produce 102 as a perimeter (of course, spreadsheets can be used to represent other sorts of equations as well). What is fascinating here is the equal sign. In fact, in the spreadsheet table, while the width column represents the sought unknown quantity, there is no direct representation of the unknown number, or of the equation itself. One brings this to the tool. The equality in the equation in one variable recorded above does not appear in the spreadsheet. Instead the equal sign that appears in the spreadsheet is one of assignment of values.

Perhaps the closest one can get with a spreadsheet to a representation of the equation as written above is that one could make a column that is always 102 and then create another column to compare the always 102 column with the column that computes the perimeter based on an initial width value. The formula of the comparison column would be something of the form "= <name of the cell in the computed perimeter column> = <name of the cell in the always 102 column>." That "formula" (with its two equal signs) would create outputs of either true or false. Other ways to convey this sense of the sought equality would include creating another column that would be the difference between the computed perimeter and 102. The user would then look for a value of 0 in this column. Or, the user could create a column that would be the ratio of the computed perimeter and 102. The user would look for a 1 in this column. Regardless of the lengths to which the user goes in identifying the solution set of the equation, it is clear that these solution strategies are not based on operating on both sides of an equal sign in mathematically sanctioned ways. Instead, they are based on successive computations while observing input and output relationships, a systematic "trial and error," or "guess and test" method, using a different sense of the word "analytic," an analytic, rather than algebraic approach.

Describing the types of conceptions present in this sort of work with a spreadsheet is a complicated affair. When students are working with symbols representing locations in the spreadsheet table (like A3), these symbols are neither unknowns, nor variables. They represent particular locations and in that sense seem too particular to be variables, though of course the values in cells to which they refer can change; the cells to which they refer either do or do not have values, when they do, it seems funny to call them unknowns. Haspekian (2005), who takes an "instrumental approach" to the analysis of spreadsheets, offers four meanings to what she calls "cell variable": Numerical content, address (geographic reference), compartment of the sheet (a material reference), and abstract variable. The last one is the only one Haspekian sees as corresponding to pencil-paper writing, when the user refers to a cell in the definition of the value of another cell. Yet, somehow the reasoning involved with the columns suggests notions of variables (the cells in a column seem something like the points along an axis representing a quantity). The formulas users develop to compute one column from another are an explicit function rule on these variables. Copying rules down the spreadsheet's columns makes use of the tool's capacity to carry out recursive operations with these variables and thus creates a sense of the varying of the variable, even though the perimeter task asked students to find an unknown.

As an aside, it is worthwhile thinking about how students would approach the task if the perimeter of the field were supposed to be 100 (rather than 102). In many descriptions of students work with spreadsheets (including the one in Rojano, 1996), the input is increased by a unit step, thus creating a sequence of integer inputs. This might be supported by an important resource for students provided by the tool.[4] It might be that a tendency to view the entries in a column as terms of a sequence is supported by the fact that each row is labeled with a ordinal number. A question for further research is whether non-integer solutions make it more difficult for students to solve tasks like the field task above.

To complete our view of the solving of an equation in one variable with a spreadsheet, we turn to the notion of equivalent equations. Although an equal sign with its assignment, or naming, sense is common in the spreadsheet work we described above, the equal sign in its comparative sense is not an object that is built in the spreadsheet; as suggested above, the equal sign in the equation $2(x+2x) = 102$ is not explicitly present in the spreadsheet environment.

Since this is true for the comparison sense of the equal sign (though it is present in the action of the user by visual comparison of two columns, or by the creation of a difference column, a ratio column, or a T/F function), one might assume that operations on equations to produce equivalent equations are not a natural integral activity in spreadsheet environments. While comparisons of pairs of equations and reflection on their equivalence are not supported in such an environment, there is support for the examination of equivalence of expressions, rather than equations. For example: If column A is the width and column B is the length (2*A#), column C that represents the perimeter could be 2*(A#+B#) or A#+A#+B#+B# or 2*A#+2*B#. A subtask of the problem can then be to obtain identical columns to column C by a different defining expression than that used in Column C. Tabach and Friedlander (2006) offered such activities to their students as a way to use the numerical support of the SpreadSheet to explore and generalize algebraic identities; but this is not the same as exploring what it means for $2(x+2x) = 102$ to be equivalent to $x+2x = 51$.

Equations and symbol manipulators

While the symbols in a spreadsheet may be used to express relationships between variables, the symbols in symbol manipulators are variables in expressions and unknowns in equations or system of equations. The symbol manipulators embedded in commonly used CAS are designed to carry out symbolic manipulations efficiently; this design goal shapes the capabilities and the constraints of the tool. The goal is to have as accurate a result with as much efficiency as possible while making the tool transparent. The algorithms often stay hidden. Thus, the growing body of research on the impact of symbol manipulators on the algebraic thinking (summarized in Guin, Ruthven, & Trouche, 2005) is centered on the instrumental genesis—the development of mental schemes that appear as new techniques in manipulating expressions and solving equations.

As a way to illustrate the instrumentation processes of CAS in relationship to the algebra curriculum, Drijvers and Gravenmeijer (2005) discuss the solution of a problem similar to the Field problem: the right-angled triangle problem. The problem asks students to find the length of each leg of a right triangle, when the length of the hypotenuse is 25 and the sum of the lengths of the two legs is 31. A model for the problem is $x^2+y^2 = 25^2$ for $x+y = 31$. As with the Field problem we described earlier, with pencil and paper, this problem would typi-

Command	Result
Solve(x+y=31, y)	y=31–x
Substitute (x^2+y^2=25^2/y=31–x)	2x^2–62x+961=625
Or:	
Solve (x^2+y^2=25^2 | y=31–x, x)	2x^2–62x+961=625
Solve (2x^2–62x+961=625, x)	x1=24 x2=7
Or:	
Solve (x^2+y^2=25^2 | x=31–y, y)	2y^2–62y+961=625
Solve (2y^2–62y+961=625, y)	y1=24 y2=7

Figure 30.3 Solution procedures for the Right Triangle problem using a symbol manipulator.

cally be modeled using a system of two equations. We chose this problem to describe work with CAS because it involves more elaborate symbolic procedures than the linear models of the Field problem.

The design intent of the CAS is that the work to solve such a problem with a CAS should be similar to the paper and pencil procedure, for example: Isolate one of the variables in the linear equation, substituting for this variable in the quadratic equation, solve the resulting equation in one variable, and substitute to find the value of the second variable (see Figure 30.3).

There is an interesting contrast here with the spreadsheet in terms of the naming of variables. The variables in the CAS model are two independent unknowns. While the work with spreadsheets often involves defining the connection between variables (columns) while constructing the second column, the equations in a CAS can be written as two equations without having them linked. Linking them is a subtle thing.

Drijvers and Gravenmeijer (2005, p. 181) observed in their study that some students carry out the isolation of y mentally and then immediately proceeded to substitution. Others have difficulty understanding what it means to solve an equation in two unknowns. Here the work with the tool involves a discontinuity that is removed by the structure of the spreadsheet. With the isolation of a variable, the CAS involves a departure from the assumption that the two letters x and y are just independent variables. It asks users to define one as unknown and another as known but not specified quantity. Drijvers and Gravenmeijer suggested that the inherent difficulty is in moving from the consideration of substitutions to be "filling in numerical values" to substitution being "pasting expressions" (p. 178). The communication with the tool via the Solve command further complicates the non-trivial substitution. Solve (x+y = 31,y) means providing a functional expression that allows the computation of *x*s from known *y*s. While the choice of y is arbitrary (it could have been x) in the resulted expression x depends on y. Thus, while in the context of equations in one variable, solve reveals the unknown value, solving here on one hand simplifies the situation by providing an operational rule to compute one unknown: x = 31-y, but on the other hand presents a "lack of closure" (p. 178). Another type of dramatic change occurs here; while in the beginning the two equations were written as two independent equations now the two have different roles: one is the equation (the quadratic) and the other is the expression that has to be pasted in. The command could also integrate the two operations—the substitution and the solution: *Solve (x^2+y^2=25^2 | y=31–x, x)* with the intention of solving the equation by substituting for y with an expression of x and solving for x. Using this nested Solve command, one can get the solution of the second unknown just by *Solve (x^2+y^2=25^2 | x=31–y, y)*. The efficiency of the CAS leads to a strategy that is different than pencil and paper, because technically, with paper and pencil, it is easier to substitute the numerical results in the x = 31–y for each potential value of y. With a CAS, it might be easier to type in (or even copy-paste) and solve for each unknown independently. Beyond the difference in technique, these two options have a different flavor; the second seems to preserve the interpretation of x and y specified as two independent unknowns, while the first involves a stage where one variable depends on the other.

In contrast to our description of spreadsheets, solving an equation in a single variable with a CAS requires typing in a string of expressions that includes an equal sign. The tool is completely non-committal in terms of the interpretation of the letter or of the equality. The letter could be a variable or an unknown and the equality could state comparison of expressions or a statement of truth. However, once an attempt to solve the equation by the Solve command is made, the solution is in respect to an unknown. Thus, even when the equation includes a single letter the syntax (*equation, unknown*) requires specifying the unknown. The equation then is a conditional statement "waiting" for a letter to be replaced by a number. This stands in a clear contrast to the view of equation in the spreadsheet. There an equation is not an object but is identified by a process of comparing two columns representing two expressions. Equation is a central basic object (among 3: number, expression, and equation) in a CAS and when using the *Solve* command to solve a single variable equation the numerical solutions are the only feedback one gets from the machine.

Finally, to explore the notion of equivalent equations with the CAS, let's return to the Right Triangle problem. Let's imagine that a student (as suggested earlier) replaced the substitution done through the CAS by isolating y in their head. They might proceed now by typing into the CAS *Solve (x^2+(31–x)^2=625,x)* and getting x = 7 and x = 24 as solutions. The equation they are solving will not be the same one as their classmates get using the CAS; their classmates will be working with the equation 2x^2–62x+961 = 625, though solving for *x* will give the same solutions 7 and 24. Watching students errors Drijvers and Gravenmeijer (2005) suspected that if the result of the substitution command would have been x^2+(31–x)^2 = 625 instead of 2x^2–62x+961 = 625 it could provide insight into the meaning of the substitution and prevent or make it possible for students to explain and detect errors.

We wonder at this point how do students know that both solutions correctly solved the system of equations? How might they know that the two equations are equivalent equations? Students might deem the two equations equivalent because they get the same solutions. Using the CAS, they could *Simplify* the "in the head" equation to get the equation produced by the CAS. Another way to produce equivalent equations would be to operate manually on both sides of the equation (rather than simplifying each expression); for example: *(x^2+(31–x)^2 = 625)-625* → *x^2+(31–x)^2–625 = 0.* Imagining this scenario brings out a key feature of the CAS environment. Whether the machine simplifies automatically or simplifies according to a specific request, the feedback received from the machine when using this simplify command is always another equivalent equation. The user cannot start from an equation and create another that is not equivalent to it.

The always-correct nature of the CAS work on equations in the automatic mode, fits the tool's design imperatives, but it raises some questions for learning. In order to replicate the mistakes that students might make on paper, they have to be encouraged to use the CAS in a "manual" mode (see below in the description of the CompuMath materials). By contrast, graphing tools graph expressions that are functions on numbers. When we describe below how equations are represented in such tools, it will be clear that, as in the manual CAS mode, students can operate on one side of an equation at a time, and thus not preserve equivalence.

Equations and graphing tools

To explore the affordances of graphing tools, we will use the same two problems: the Field problem and the Right Triangle problem and refer to studies on learning algebra with graphing tools. We start by returning to the Rojano and Sutherland's study of the Field problem. We would like to conjecture about posing this task to beginning algebra learners with a graphing calculator. Utilizing capacities of the calculator, a user could insert ordered pairs created from the problem description (like (10,60), (20,120), (30,180), (40,240)) and use a regression line to find an expression that might fit this "data" (as described in Hershkowitz & Kieran, 2001 perhaps algebra beginners would choose this option in specific circumstances).

However, beginning algebra students who are successful in solving the task might start by typing in a function rule (like, $y_1 = 2x+4x$). Such a rule describes all possible perimeters as a function of the width, x. Students might then graph this function and search for an x for which the value of the function is 102, perhaps by tracing the graph and generating values. It is possible, although less likely, that students would start with an analysis of tables of values of the function generated automatically from the expression. Still another option is to enter two functions (rather than just the one that computes the perimeter from given widths): $y_1 = 6x$ and $y_2 = 102$. The x value of the intersection point of the two functions' graphs would indicate the x value that solves the width value (as argued in Chazan, 2000, pp. 78–85).

Stepping back from the particular problem, like the spreadsheet, the graphing calculator here supports a view of x as variable, even though the problem wants a particular x value as the solution to the equation. As with the spreadsheet, with the graphing calculator, there also is no explicit representation of the equation present, but rather expressions for two functions. Whatever functions one graph, one has to formulate an algebraic formula without the support of the tool. Since the only equations involve as objects in graphing calculators are *equations of functions* which are essentially symbolic expressions of functions in the format of y = f(x) solving an equation with this sort of graphing support (of course, this would be different with tools that graph equations rather than functions) has to involve a process of comparison similar to the one users employ when using a spreadsheet. Numerical [almost-correct] solutions can be read on the screen by reading intersecting points of two graphs or by reading values of zeros of the difference function of the two expressions. However, the major strength of 2D graphing of the two sides of the equation as two functions is its support in viewing the processes involved rather than viewing the solution (Yerushalmy & Gilead 1997, 1999). The tool's support is in freeing the student from the need to manipulate the comparison of two expressions to find a solution.

However, there are some concerns to consider with beginning learners. Researchers, like Stacey and MacGregor (2000), and Herscovics and Linchevski (1994), reported that beginning algebra students tend to avoid algebraic expressions in their problem solving. In such cases, it seems unlikely that the graphing calculator, with its need for an explicit formula in x as a starting point, would support beginning learners of algebra.

While spreadsheets and symbol manipulators let the number of quantities grow as you like, and often models are written with multiple variables or unknowns, it is less likely that users of graphing calculators will write multiple variables models and use the graphing tool as a tool for representing multi-variable models. If, however, one wants to stay with the metaphor of comparison for equations of two variables, there is another way to proceed, one that is not typically supported in graphing calculators. One could imagine graphing calculators that would allow the definition of functions of two variables. In such an environment, equations in two variables could be described as comparisons of two 3D graphs of two functions of two variables (see Figure 30.4). Attempting to consider the possibility of graphing the system of equations in 2 variables of the Right Triangle problem, four surfaces would be graphed: the planes z1 = u+v and z2 = 31 and their intersection line and its projection on the x–y plane y = 31–x; the parabolic surface z3 = u^2+v^2 and the plane z4 = 625 their intersection and its projection on the x-y plane: x^2+y^2 = 625. Reading the intersection values of the two projected lines: x+y = 31 and x^2+y^2 = 625 provides the two possible solutions for the problem (7,24) or (24,7).

This construction and representation of the solutions involves identifying two 3D surfaces as the graphs functions in two variables, identifying intersection lines, understanding the relations between the intersections and the projections require good graphing calculators and cognitive processes that would support the visualization of 3D in 2D screens and connecting

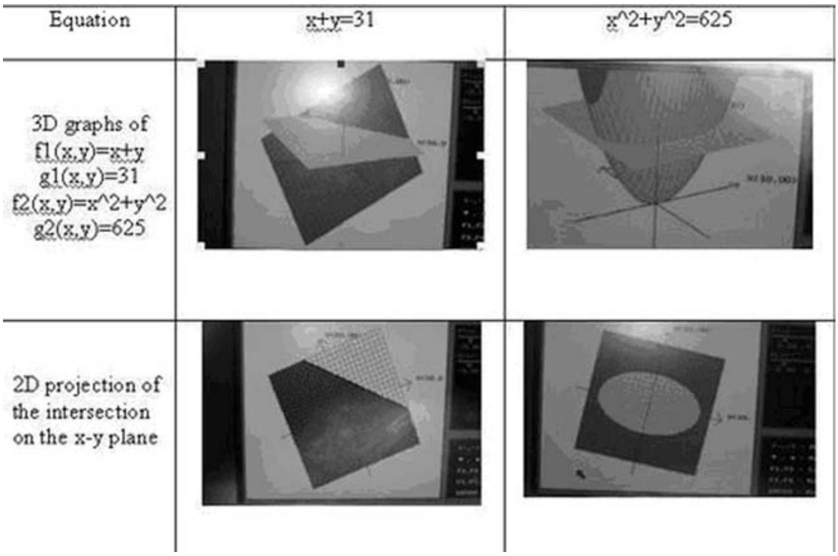


Figure 30.4 3D graphing of the system of equation for the Right Triangle problem (from the 3DFunctionGrapher, Yerushalmy & Shternberg 1995).

all that to the symbolic model. It is likely that most users would prefer to eliminate the 3D graphing. The complexities involved in attempting to address system of two equations make it clear that any attempt to extend the comparison of functions view of equation to systems of more variables and equations is certainly impossible. Thus, the view of an equation as comparison of multi variable functions is one that can be symbolized by algebraic expressions, but cannot be supported by graphs.

Using a two-dimensional graphing calculator to graph an equation in two variables or a function in a single variable will both take the same format: *y =expression in x*. As with spreadsheets, an equation in a single variable is not recognized by the tool as an object. Rather the user who requests the plotting of two functions' expressions can interpret the two independent graphs of functions $y_1 = f_1(x)$ and $y_2 = f_2(x)$, as graphs of two sides of an equation in a single variable, as two independent graphs each represents a solution set of an equation in two variables or as a system of two equations in two variables.

The rich repertoire of interpretations of the graphs and their relationship to equations could be an opportunity for learning or a source of confusion. This ambiguity might lead a curriculum in three directions:

- Making the ambiguity of reference equations an explicit topic of discussion: In this case, the curricular sequence and activities lead to discussions that elaborate the mathematical similarities and differences between treating the graphs as representations of a system or as representations of a single equation in one variable.
- Separating the explorations of functions and comparisons that make use of graphing tools from equations. This would leave equations to be studied with symbol manipulators where an equation is an explicitly recognized object of the tool.
- Making the interpretive choices of the user visible through the environment: For example, one might design an interface for a graphing calculator that will explicitly require the users to define the type of object to be graphed. While such a design might not be desirable for educated users, it might be helpful for students who use it for learning mathematics. In such tool x+y = 31 could be entered as the function f(x) = 31–x, or as the equation x+y = 31. The first graph represents the dependency of f(x) in x and the second graph would be of the locus of the ordered (x,y) pairs that solve the equation. Such an interface would insist that

users specify whether the equation x^2+(31–x)^2 = 625 should be graphed as a *single function* f(x) = x^2+(31–x)^2 and the output 625 would be looked for in a table of values for the function, or as comparison of pair of functions: f(x) = x^2+(31–x)^2 and g(x) = 625.

As with spreadsheets, graphing calculators do not support direct manipulation of equations (since an equation in a single variable is not even an object that the tool recognizes). However, Graphing Tools could offer a model for thinking about equivalent equations—one of particular importance for the algebraic formation of processes and phenomena (such as appear in algebra word problems and modeling tasks, as described by Chazan, 1993). Each equation is represented by two function-expressions and graphs and the x values of the intersection points of the two pairs of functions. As embedded in various curricular agenda (Heid et al., 1995), the National Council of Teachers of Mathematics (NCTM) Standards recommend using this graphing capability for forming a new view of the manipulations of equations in a single variable, either to simply one of its two expressions or to create a new equivalent equation.

Beyond what is typically done by graphing two expressions, graphing tools can be used to demonstrate each step of a symbolic procedure graphically. This visual information can help explain a structural change caused by a manipulation and offers ways to make equivalent equations be an important concept in learning symbolic manipulations. (In order to be tenable, technological tools need to use color or some other mechanism to indicate which pairs of graphs belong to which equations, see the figures below.)

For example, when the procedure involves manipulations of each side of the equation separately then each of the two graphs should stay unchanged and so does the intersection. When the procedure involves operation on two sides than both graphs should change (unless the operation is null: multiply by 1 or adding 0), the intersection point would change however the x-values of the intersection points would stay unchanged[5].

With this sort of technology, the computer does not carry out the manipulations; students must decide what an equivalent equation will be. The graphical feedback supports their work by indicating either the change or equivalency of each of the two expressions and whether the x-values of the intersection points have changed. The type of change to or equivalency of the comparison suggests a different kind of curricular engagement with symbolic manipulations. An extended discussion of such design options with graphing tools designed for learning algebra can be found in Yerushalmy (1999). We will return to the Right Triangle problem to illustrate such options:

For example, a student might make a standard error when manipulating an equation. The student might begin with x^2+(31–x)^2 = 625, then they might write x^2+31^2–x^2 = 625 and then conclude that there are no solutions because the two graphs *f(x)* = 961 and *g(x)* = 625 are parallel horizontal lines.

Alternatively, students might make correct steps, however ones that do not work strategically toward the goal of isolating the single variable. Figure 30.5 illustrates different actions a student might take when manipulating the equation x^2+(31–x)^2 = 625 while solving the Right Triangle problem. The graphs are changing radically but the correctness of the manipulations is visible by the common solution lines produced by the tool at the intersections' x values.

As with a CAS, if one operates equivalently on both sides of the equation, it results in a correct equivalent equation. However, by looking at the different pairs of graphs at each stage one may start to think about the two processes that the equation compares; how and why they are changing the way they are? How does the equivalent equation reflect the original given situation or a different situation? What does it tell us about "equivalent situations"? How would a simpler equivalent equation look like? Such questions may complicate the process of reaching a numerical solution but have the potential to enrich the meaning of the equation for the solver.

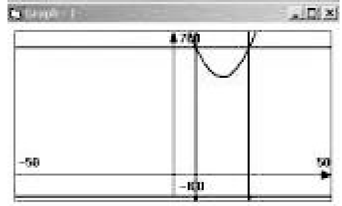
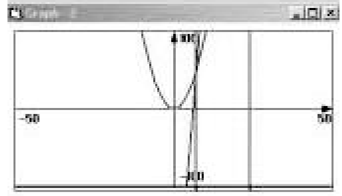
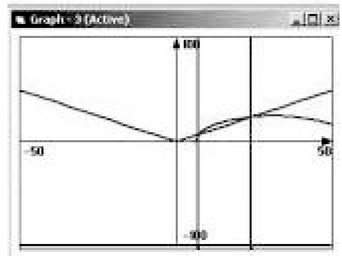
$(31 \quad x)^2 \quad 625$	
$625 \quad (31 \quad x)^2$	
$\overline{}2 \quad \sqrt{625 \quad (31 \quad x\ )^2}$	

Figure 30.5 Graphs describing correct equivalent equations produced by modifications on both sides of the equation using Calculus UnLimited (Schwartz & Yerushalmy 1996).

Didactical design combinations

As our review of the affordances of these three tools suggests, each of these tools carries different views of equation, of variables, and of manipulations. Since such tools are available on the web and on different types of hardware, we may imagine using this diversity as a lever to teach different views of "algebra" or a fuller view of algebra while using different tools. One belief is that taking such an approach requires making the differences and the distinctions between the tools an explicit part of a curricular sequence. Others may suggest that the openness of students to different syntactic structures and various visual representations by itself (without necessarily restructuring the curricular sequence to explicitly reflect and discuss the differences) would elevate the meaning making usually absent in learning about equations and manipulations.

Another trend to ease what seems to be a deficiency in using a specific tool is to develop component architecture (Kynigos, Koutlis, & Hadzilacos, 1997) that would allow educators to construct tools that serve their own agenda. While such capabilities may turn out to be an optimal future microworld, we will now use our analysis to speculate about potential for a convergence of these tools in the future.

Appreciation for the power of visual representation has led to the design of graphs as modules in both spreadsheets and CASs. Based on the way in which spreadsheets seem to support beginning learners, what if one wanted to be able to use graphs with a spreadsheet, just like in the graphing calculator? To use the graphing capacities of a spreadsheet, one must choose appropriate input and output columns (there are usually more than two present since students insert intermediate computation columns on their way to the final column). Then, one must choose from among a series of graphic representations, if one wants a continuous graph that represents dependency. The menu terms and icons to choose type of graph from are sometimes hard to interpret and are not necessarily in line with the appearance of Cartesian graphs in algebra textbooks.

#1: $x + y = 31$

#2: $x^2 + y^2 = 625$

#3: $y = \sqrt{(625 - y^2)}$

#4: $2 \cdot x \cdot y = 961 - 625$

#5: $x^2 + (31 - x)^2 = 625$

#6: $x^2 + (31 - x)^2$

#7: 625

#8: $x^2 + (31 - x)^2 - 625 = 0$

Figure 30.6 A list of symbolic statements with Derive6 (1995–2006).

There is a similar issue with CAS. With a CAS like Derive, expressions, equations and inequalities in one or two variables are all legitimate input under "Author's Expressions." The graphing tool interprets each mathematical object and provides an appropriate 2D graph for solutions of equations or inequalities and as descriptions of expressions interpreted as functions. Figure 30.6 displays equations and functions that describe the solution of the Right Triangle Problem in different ways.

When graphing solutions of equations in two variables in the Cartesian plane (Figure 30.6, expressions 1–4), the graph is the representation of the (x,y) values that solve the equation (Figure 30.7).

When graphing equations in a single variable and functions in a single variable (expressions 5–8) the graphic plane (Figure 30.8) serves both for describing the values of the functions (Figure 30.6: #6, # 7) and the values of the solutions for the single variable equation (Figure 30.6: #5, #8).

Thus, although such tools integrate the manipulation of symbols with graphical information, the load on the user is heavy; users must switch among different views of equation.

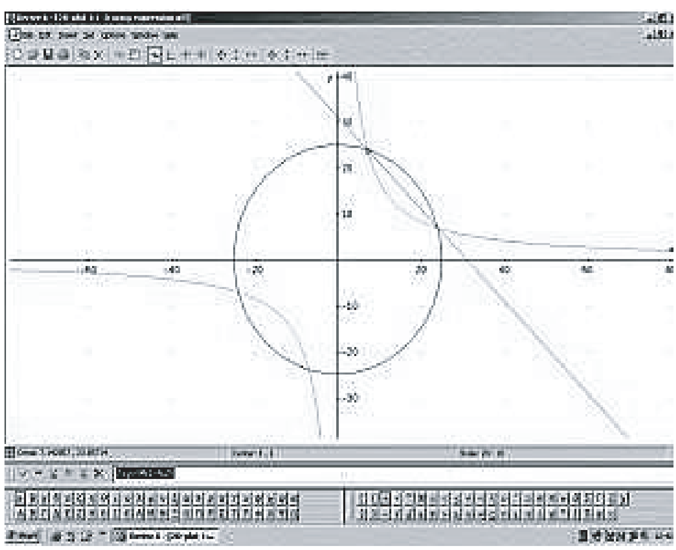


Figure 30.7 Graphs with Derive6 (1995–2006) describing the solution of the Triangle Problem as intersection of x+y = 31 and 2xy = 336 and as intersection of x+y = 31 and x^2+y^2 = 625.

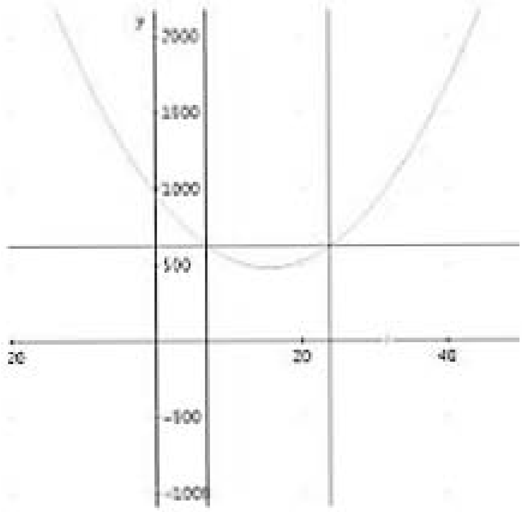


Figure 30.8 Solutions graphed with Derive6 (1995-2006) for the Triangle Problem: the lines x = 7 and x = 24 describe equations or solutions of #5 and #8. #6 and #7 described by two functions' graphs.

Finally, though we do not know of examples of this kind currently, one might imagine incorporating spreadsheet operations with a CAS in order to help students develop the equations with which they will work. We wonder about the benefits and the challenges in coordinating the spreadsheet table with the symbols. For example, is there a way that the spreadsheet treatment of the relationship between two columns would help a user to follow the substitution done by the CAS in the solution to a system? Or would the usages of these two components of the tool simply be confusing.

Whether the tools we've discussed are actually combined into a single system or whether students use them separately in a technologically intensive approach to school algebra, teaching with technology adds to existing challenges of teaching algebra. The integration of these kinds of technological tools emphasizes the need for knowledge on the part of the teacher, or support from the curriculum, about how to use these tools to support students' construction of meaning for symbols (letters, the equal sign, equations and graphs) as they negotiate the multiplicity of meanings each has.

Describing views of equation in one Israeli technologically-Intensive algebra curriculum

For a variety of reasons, Israel as a country with a centralized educational system has made a proportionally high investment in the use of technology in the curriculum. While in the United States, for example, curriculum developers are wary of integrating use of technology beyond hand-held graphing calculators into commercial textbooks (as is suggested in Fey, 2006, by the comparative lack of U.S. experience with CAS), in Israel the Ministry of Education has called for the development of school algebra curricula that are technology intensive. In the next two sections, we will describe work done in two of these curriculum development efforts. We have been observers of the curriculum discussed in this section and participants in the one described in the next. These curricula have generally taken a functions-based approach to school algebra even when these conflicted with dominant practice and the ministry's inspectors.

Against the backdrop of the description in our chapter in the previous edition of this volume (Yerushalmy & Chazan, 2002) of a typical school algebra text used in the United States from the 1970s and our description of the affordance of spreadsheets, CASs, and graphing utilities, we now describe a technologically supported approach to school algebra. At an overview level, this curriculum does not change the order in which algebraic content is introduced to students. The curriculum begins with what might be called expressions, it moves on to equations in one variable, then inequalities in one variable, equations in two variables, and finally functions as a particular kind of relation between two variables. But, at a smaller grain, in line with Noss (2001) with the support of technology there are more subtle shifts in order. In line with Tall (2002), the choices made in this curriculum manage discontinuities for students in particular ways.

The particular curriculum we will examine, CompuMath (1995–2003), is a grade 7–9 curriculum for junior high schools. As a curriculum, CompuMath has a number of key goals. As described by the developers (Hershkowitz et al., 2002), their goal was to design an intuitive functional approach that allows students to build on the arithmetic knowledge they have learned; the curriculum seeks to create a smooth transition from arithmetic to algebra. Thus, the curriculum has students investigate co-variation in quantities while making generalizations and justifying their conjectures. In this intuitive approach, there is an emphasis on keeping the formal aspects of function at "a minimal level" (Hershkowitz et al., 2002, p. 679). For the developers, it is important to introduce students to different facets of the same algebraic concepts and to encourage a variety of strategies and transition among different representations of the same concept (Hershkowitz, 2006, personal communication).

We've chosen this curriculum because it is a seriously crafted curriculum that has been described elsewhere in the literature (Hershkowitz et al., 2002), indeed also in this volume (Tabach, Hershkowitz, Arcavi, & Dreyfus, chapter 29). In this section, we focus on how this curriculum introduces students to equations and asks them to work with equations. Our description is chronological. We start with introductory material, indicate how equations are first introduced, then examine the solving of equations, and later move on to equations in two variables. Throughout, we indicate how technology is used to support curricular goals and activities. And, in line with Tall (2002), we attempt to highlight how what it is that an equation represents shifts and changes as students move along in the curriculum. To do this review, we review the published textbooks, analyzing the sequence and central activities in each of the relevant sections, as well as looking systematically at the introduction to a section and at the summary lists of what was covered at the end of each section. The examples we use for illustration are often taken from these summaries.

Introducing equations conceptually with a spreadsheet

The term *equation* is not introduced right in the beginning of Compu Math. Students learn to describe linear and non-linear phenomena using Excel. They do so by writing formulas for a cell in the spreadsheet, by dragging down a column to capture growth or decay, and sometimes by graphing one column's relationship to another. The kinds of processes examined include generalizations and comparisons of two strategies of payments (e.g., make a decision which of the two ways will be preferred by groups entering a science museum, when and why: purchasing a group ticket for fixed price or paying for each person in the group) or generalizations of numerical phenomena such as the Seal Design problem described in Hershkowitz et al. 2002 (p. 682). Constraint equations are not yet introduced, the focus is on what might be deemed expressions or functions of a single variable. The technology here works to establish variables as representing all members of a set of numbers (e.g., all members of a column), rather than only some particular unknown number.

Towards the end of the year, after students have done considerable work with processes in a spreadsheet, the term equation is formally introduced, as a statement of the equality between two expressions, or between an expression and a number:

> An equation is an equality between 2 expressions or between an expression (*tavnit* in Hebrew) and a number. The number or numbers that makes the equality true are called solutions. (CompuMath 7c[6], 2001, p.12)

This introduction is done using pencil and paper representations of equations with literal variables. Solutions are numbers that create a true statement when they are substituted into the equation. Once the term equation is introduced, students are asked to identify solutions to equations with Excel. This work is primarily like the work described in the Field problem above. There is a constraint equation involving a single process. In this curriculum, students are taught to use the T/F option to compare the outcome of the process with the constraining number. The developers summarize the role of the spreadsheet in equation solving in this part of the curriculum as follows:

> In summary, the spreadsheet's ability to produce large quantities of data by simple "dragging" of formulae provides an excellent illustration of the meaning of a variable, an algebraic expression and the pattern of a variation. On the other hand the same ability can be "abused", by preferring the extension of the numerical table to the use of a more mathematically sound strategy—such as constructing and solving an equation. (Hershkowitz et al., p. 681)

In addition to their work with Excel, on paper, students are presented with tasks ask them to compare two or more processes and evaluate values of points on graphs. Solving such problems students draw graphs or use given graphs to identify events. As in earlier units the problems involve linear and non-linear comparisons. For the linear problems, expressions (symbolic rules) are also given and students may use them to identify events such as the moment in which the cooling and heating processes reached the same temperature (an intersection of two graphs in a task that describe change of temperature) or the moment in which the temperature reached 18 degrees (identifying the input of a given output).

Solving equations in multiple ways, including with a symbol manipulator

Having introduced students to what an equation is and what the solution of an equation does, CompuMath begins to teach students ways to write equations whose solution is available by inspection, x = 5, for example.

While the symbolic trail left to document such activity may look like solving by performing operations on equivalent equations, that is not what students are doing at this stage. The curriculum uses constraint equations, where a process must equal a particular numerical value, like the one in the Field problem above, as the simplest form of equation to be solved using "considerations" about the behavior of an expression, like the symbolic manipulations that represent "undoing."

The curriculum offers three ways to solve equations of this kind: Informal methods, like solving by guessing, or considering similarities to previous equations, or undoing the expression arithmetically; solving by simplifying the expression on one side of equation using symbolic manipulations; and solving by graphing and looking for a point's values that make the equation true. Students are encouraged to do their work on paper and then to use Excel to check the correctness of their solution.

Here is one example from the overview of this unit (CompuMath 7c, 2001, p. 124):

> We learned how to solve equations such as (2(x+5)+1)/7 = 1 using "considerations." Since the quotient is 1 then the numerator should be 7.
>
> $$2(x+5)+1 = 7$$
>
> We learned to find simpler expressions to given expressions using commutative and distributive rules.

2(x+5)+1 = 7
2x+10+1 = 7
2x+11 = 7

By considerations [about the behavior of 2x+11 = 7]: 2x = –4
By considerations [about the behavior of 2x]: x = –2

Typically, this work is done in situated problems. A process is described in a story; students are asked to write an expression or rule for this process, write an equation to describe a given constraint on this process, graph the process or represent it in a spreadsheet, and look for the input to the process that will produce the desired output. For example, a linear graph is given in the book for a story problem. In order to find the values of a point on the graph (x, 148) students are asked to write the equation: 12x+2(x+4) = 148 and the challenge is to "discover" the solution to the problem using the equation. (CompuMath 7c, 2001, pp. 10–61)

Here are some other representative descriptions of solving tasks that involve early work on solving simple constraint equations with work on equations "even when the variable appears in both sides of the equation" (CompuMath 7c, 2001, p. 50):

- Solve the equation: 0.5(x+8) = 5 using a graph.
- Write an equation whose solution is x = 5, using the expression 0.5(x+8).
- Solve the following equations, using "considerations" that simplify the equation.
- Solve the following equations using a table of values.
- Use the T/F option of Excel to write an equation whose solution is not *not* available by inspection
- Suggest a strategy to solve the following equations using reverse operations on the "result."

Up until this point in CompuMath, students have been introduced to equations and have been identifying solutions to equations, but they have not been solving equations using the canonical algebraic method. With this preparation as a foundation, a central unit of the eighth-grade materials (CompuMath 8a, 2002, pp. 31–92) focuses on solving linear equations using operations on both sides, including equations with a variable on each side of the equation. Initially this work is not in situated problems.

Although equivalent equations are introduced later on (CompuMath 8a, 2002, p. 70) as a consideration in solving equations, the solution process is defined as: "Solving equation calls for operations on both sides in order to get an equation of the form x = a number, a form which makes it possible to read the solution easily" (CompuMath 8a, 2002, p. 49). To help students understand equivalent equations, recourse is made to the metaphor of balance. Solving equations is accomplished by carrying out operations on two expressions that are in each pan of the balance. Assuming that the two expressions are initially balanced, operations that leave the pans in balance are legal mathematical operations. If these operations help isolate the variable, then they are useful steps in the solution process.

In this part of the curriculum, operations on both sides are done using pencil and paper or with the support of a symbol manipulator, Derive (1995–2006). The use of Derive at this stage is presented as being analogous to the operations on the balance model. The command F4 that put parentheses around the whole equation is analogous to the / sign at the right side of the equation when writing on paper. Derive is used to carry out step-by-step manipulations: operations on both sides and simplifying the resulted expressions. It is not used to look at linked graphs.

In this part of the curriculum, graphs and the term "comparison" do not appear in the student materials. While in this component of the materials, students are taught to operate manually on both sides of equations according to the balance metaphor, the term equivalent equations does not appear yet. It appears later in a separate small section.

Just a little later in the materials (p. 82), Derive is then suggested as the tool to solve equations when the equation is part of solutions of contextual problems. However, when the transition is made to contextual problem, students are encouraged to use the command *Solve* and get an answer as a solution to the problem. Students are no longer required to solve the equation step by step with Derive.

Linear inequalities: Comparing two processes with a graphing utility

To review, in CompuMath, when students are introduced to what an equation is and what it means to solve an equation, graphs and tables are used (with the support of Excel) to help students check solutions. When students begin to solve equations symbolically, new language is introduced. Equivalent equations are equations that maintain "balance." In a unit labeled "Inequalities As Well" (CompuMath 8a, 2002, pp. 93–183), having taught students to solve equations symbolically and to use Derive to identify solutions directly, CompuMath enlarges students' sphere of activity by introduction inequalities in addition to equations.

The transition from equations to inequalities is of special interest. This portion of the curriculum builds both on the early work on equations, as well as the later work on the solving of equations with manipulation. There is a return to conceptualizing equations as comparisons of two expressions, and thus related to inequalities, while the metaphor of balance is put in the background. The introduction (p. 93) and the summary (p. 183) of the unit suggest that inequalities are different from equations. The curriculum suggests that equations are an attempt to search for "balanced situations", however, in life looking for "balance" is an idealized condition and not realistic and therefore "comparing" is the more appropriate image for inequality (p. 93). Thus, this unit is not a direct continuation of the previous unit that concentrated on symbolic operations on equation; it relates back to previous units where expressions of phenomena were discussed and graphed and operations such as: difference and ratio between expressions were graphed and some tasks asked to compare tables.

In the context of this unit, students learn new ways of thinking about equations. First, students learn that equations and inequalities in one variable can be thought of as the same sort of mathematical object, that the curriculum calls "*tavniyot pasook*" (p. 110), in contrast with the earlier use of "*tavnit*" for an expression. Students learn that such objects can be thought of as comparisons of two expressions. Here, technology is brought in to help support this interpretation of equations and inequalities in one variable. Two-dimensional graphs of the expressions on each side are suggested as a way to help solve equations in one variable. Students are encouraged to graph both expressions and identify the x coordinate of the intersection point. This work builds on earlier work with Excel, but is much more explicit. MathematiX (1995) which is a graphing utility that allows users to "walk" along a graph of a function and read values, supports this sort of activity. In the context of contextual problems, students are encouraged once again to use Excel to build up the expressions on either side of the equation and to use the graphing utilities in Excel to create graphs of the sort that are given by MathematiX.

Students are taught that the goal with such objects is to identify the "truth set" which is the set of all possible solutions, where a solution is a number whose substitution results in a true statement. Students are encouraged to use the number line as a way to represent the solution set of both equations and inequalities in one variable.

Finally, toward the end of this unit, there is a small amount of work on equivalent inequalities. As appears all along when the goal is to "systematically learn to solve inequalities" the graphical representation consists of "marking the solution on the number line" following the manipulations on the inequality (CompuMath 8a, 2002, p. 183); two dimensional graphs are not used. For example, they are not used to develop an understanding why one must change the direction of the inequality when multiplying both sides by –1.

Working with equations in two variables: Ax+By = C

Having helped students see equations in one variable as a special kind of *tavnit pasook*, and thus like inequalities in one variable, CompuMath now turns students' attention to increasing the complexity of equations to equations in two variables. While students have had some experience earlier with processes of two or more independent variables, now students will be asked to consider equations in two variables.

In making this transition, the curriculum builds on the earlier introduction to equations in one variable by first working with equations that involve a constraint on an expression in x and y, equations like 5x+y = 12. However, now, rather than representing a solution set made of numbers, such an equation represents a relationship between variables, that also can be thought of as a list of coordinated values. These solution sets, or relationships between variables, can be graphed on a two dimensional Cartesian coordinate system either in Excel or with MathmatiX (though this software expects an explicit function of one variable). Students are asked to coordinate what it means for a pair of numbers to be a member of a solution set and the coordinates of a point on a graph or, for inequalities, in a region.

To change one equation in two variables into an equivalent one involves operating on both sides. The materials suggest that there is a benefit to making equations less complex, thus the equation 5(x+3)+2(y–2x) = 18-y can be turned into x+y = 1 for which a table of values and a graph can be constructed. While recourse is made to the metaphor of balance to justify the manipulation of symbols, there is little explicit attention to what it means for equations of two variables to be equivalent.

Solving an equation of this sort for one variable moves students from an implicit function to an explicit function, however the terminology of functions is not introduced at this point. While the materials do not assume this is necessary, they do suggest that an advantage in isolating a variable is that one can make a table of values where one variable can be computed directly from substitution of values for the other and then graph the points in the table.

Having moved students to equations and inequalities in two variables, the final step involving equations is the coordination of equations and inequalities in two variables into systems of two equations in two variables. The treatment of this material is pretty standard with the use of three methods: substitution, graphing, and linear combinations. The technology that is used to support learning in this unit are the graphical capabilities in Derive. These capabilities (unlike those of MathematiX) involve the graphing of equations. This allows students to get graphs of equation without rewriting them as explicit functions.[7]

THREE ALTERNATIVE CHOICES IN DEALING WITH EQUATIONS OF DIFFERENT KINDS

Having outlined how one innovative, technology intensive curriculum chooses to move between different views of equation and supports its work with technology, we use the serendipity of the existence of another similarly focused Israeli curriculum, to look at three alternative design choices to those made in the CompuMath materials. One of the reasons there is movement toward function-based approaches to school algebra is that viewing algebraic entities as real-valued functions allows curricula to use the resources of tables and graphs to help students understand algebra. Yet, in maintaining relationships between symbols and graphs, for example, there are many important and difficult transitions, of the kind Tall (2002) indicates, to negotiate; there are discontinuities between treating algebraic symbols as unknowns and variables, between treating the Cartesian coordinate system as a space of functions of one variable vs. a space for representing solution sets, between viewing equations as comparisons of functions and as representations of solution sets. Below, we examine three points where the Visual Math curriculum makes different choices than the CompuMath curriculum. All three of the points are places where the choices made influence the kinds of connections between symbols and graphs students have an opportunity to make.

Our desire here is not to stand in judgment on either curriculum, or to argue that the choices made by one are better than the choices made by the other. We cannot make such an argument. Instead our intent is to show how another curriculum with quite similar goals and desires might approach key decisions differently and how these differences in decision making might lead to a different set of discontinuities for students. While the consideration of discontinuities is not the only consideration curriculum developers must consider, we suggest that the differences between the choices made in these two curricula create fertile ground for research and exploration. It seems to us that it is articulation of competing choices that can be described in this way that can lead to productive research on curriculum, more productive than simple comparisons of student achievement.

Equations in one variable, like 3x–4 = 7

Equations where one of the two expressions is a number create a dilemma for functions-based approaches to algebra. On the one hand, it is reasonable to ask students who are used to working with functions for what inputs will a given function create a particular output. After asking such questions, it seems natural to view equations where one expression is a number as asking for what values in the domain of the variable will the expression produce the desired number. Operations on both sides to undo the non-numerical expression then link viewing the equation as about a single function with the symbolic work of solving. While such treatment seems natural, students will then be asked to view some equations as questions about one function, while equations with variables in both expressions are viewed as comparisons of two functions. This reifies the didactical cut discussed in Filloy and Rojano (1989).

On the other hand, if an expression that is a number is viewed as a constant function, then students can consistently view an equation as a comparison of two functions. However, there isn't a strong connection imagining a comparison of a non-constant and constant function and the symbolic work of undoing an equation with variables on one side only. And, one cannot connect questions about a single function to questions about equations.

This issue can be seen in how the two curricula approach equations where one expression is a number. Visual Math does not distinguish between different equations and treats equations and inequalities with an expression in one variable on one side and a number on the other as comparison of two functions. CompuMath uses equations with an expression in one variable on one side and a number on the other as the simplest form of equation to be solved using informal considerations, as well as a simple case to practice symbolic manipulations on both sides of equations.

Earlier, we outlined how the CompuMath works builds in its treatment of equations and uses Excel to support this sequence. In Visual Math, like CompuMath, prior to the unit on equations are units on qualitative descriptions of processes. However, rather than rely on spreadsheets, use is made of a tool designed for educational purposes (The Function Sketcher, Yerushalmy & Shternberg 1996). This tool emphasizes graphing, though more qualitatively than is typical. The communication with the tool is through qualitative descriptions of functions (e.g., increasing function, decreasing in constant rate) and it visual iconic representations (described in Schwartz and Yerushalmy, 1995). These terms and signs form a semiotic set for describing temporal phenomena and numerical sequences. A typical activity in this section of the curriculum has the structure: "Here are 4 stories about temporal phenomena. Here are a few sketches of graphs. Match the sketched graphs and the stories." In all such activities, a constant function graph is included (e.g., the pool was empty during the whole winter). A constant function graph is one of the 7 graphical icons built into this interface. It does not get any special treatment; it is a function like the others.

Thus, when the curriculum turns to writing expressions of functions with algebraic symbols, constant functions again receive no special treatment; they are linear functions. Using the Sketcher, linear functions are introduced and gradually constructed using the concept of "rate of change" and the notion of "stairs" to capture how two variables change in time. These ideas are then connected to the graphical slope of a line, a normalized rate of change with a one-unit change in denominator, and the coefficient of x in mx+b. There is no special treatment for a constant function; it is linear function with slope 0.

Once students are familiar with graphs, tables of values, and expressions for functions (not limited to linear functions), the curriculum introduces comparisons of functions. The activities involve either equations or inequalities. A typical activity is something like: "There are 3 ways to pay … For what values of x would process 1, be more expensive than (or less expensive than or equal in cost to) …" Transformational or analytic activities on functions do not treat the zero slope explicitly (it sometimes appear as 0x+b.) Within the unit of "Comparisons" (p. 41) there is one activity that explicitly treats equations of the form, f(x) = 0: "Present examples for equations of the type f(x) = 0, where f(x) is linear and it has one solution…. f(x) is quadratic and does not have any solution …" In this curriculum, it seems that comparing ***non-trivial*** cases is the goal in learning the concept of "comparison."

Once students have learned what an equation is, then other concepts are introduced to support work on solving equations. For example, the difference function between two functions is introduced. This is used for two purposes, to check whether two expressions are equivalent, whether an equation is an identity, and to find specific values that solve the equation. In this context, a question about two functions is turned into a question about a single function. When the comparison is with the constant-zero function, that function recedes into the background and students examine where the difference function intersects the x-axis.

To reiterate, treating equations where one expression is a number as statements about a single function allows students to build on earlier work and strategies to understand some equations. However, it sets up different classes of equations. Treating numbers as constant functions allows all equations and inequalities in one variable to be treated as comparisons of functions. There is logic here. Manipulation of equations leads to a "moment" when an expression becomes only a number. There is an advantage to a view of expressions and equations that maintains continuity: if x+3 is an expression and x is an expression then 3 is also an expression. But, this gain is off-set by a cost; this choice makes students alert only to one view of equation and students must learn to treat some numbers as more than just numbers, but as constant functions.

The role of graphs in solving equations in one variable, like 3x–4 = 2x+7

How is a curriculum to make use of students' familiarity with graphs and tables to help them understand the notion of equivalent equations? How are students in a curriculum going to come to understand what operations on equations are legal ones and which operations on equations are not mathematically sound?

There is some precedent in the standard algebra curriculum for using graphs to help with solving. When solving systems of equations, students often are taught to solve graphically as well as algebraically. And, should they choose to verify that these methods generate the same solutions, students can verify that while the equations in a system may change the intersection point(s) of the two equations being graphed remain the same.

How is a functions-based curriculum in which equations are considered a particular kind of comparison of functions, just like inequalities, to approach the notion of equivalent equations? Mathematically, the situation is different from that of the system. As suggested earlier, if one is graphing the expression on each side of an equation and one manipulates both sides of the equation appropriately, both functions change, and it is the x-value of the intersection that is preserved, not both the x and y values of the intersection. The invariant amongst all of the change is relatively small. Is this invariance strong enough to build in students a sense of how to justify what operations to an equation are appropriate and inappropriate in solving an equation in one variable?

Visual Math bets on the pedagogical value of this invariance. The invariance of a solution set as the determinant of the equivalency of equations, an analytic rather than algebraic perspective, is the central driving concept for the Visual Math unit on solving equations and the rationale for the mathematical legality of manipulations on equations and inequalities in a single variable. A major concern regarding equations and inequalities is designing a representation that has the potential to turn a solution into a mathematical entity that supports meaningful operations—one that can help delivers the ideas of equivalency and invariance in equations and inequalities. One immediate metaphor for "solution" that comes to mind when considering the two-function graphical representation of equations is the solution as the *intersection point* of two lines or curves. The advantage of using this object is its tangibility. It promotes thinking about a "solution" as a somewhat concrete object which can be fixed, moved, or otherwise react to manipulations. In the Visual Math curriculum, the central activity that forms the unit on comparisons and equivalent equations is to investigate the effect of identical operations on both sides of equations in producing equivalent equations (For a more recent version of this task, see, www.cet.ac.il/math/function/english/square/comparison/comp2.htm). This task consists of writing a report on various operations and their effect on the solutions of comparisons. Students are asked to explain how the effect of algebraic operations on the solutions of a comparison depends on the type of comparison (equation, inequality), the type of operation applied to the side(s) of the comparison. Software that color codes the pairs of functions in an equation and that provides a vertical "rulers" at intersection points is used to enable easy operations on both functions. This activity is preceded by activities on the different mutual positions of two graphs and especially lines.

As this description illustrates, the Visual Math focus when teaching students to solve equations is different than the CompuMath one. CompuMath makes use of the balance beam metaphor to justify the legality of algebraic manipulations. And, it teaches students to use Derive to carry out manipulations, though students are sometimes asked to use Derive in a "manual" mode. Visual Math by comparison provides no technological aid for the writing of equivalent equations, though it does provide some non-standard graphical tools for visually tracking equivalency as students write their own attempts at equivalent equations.

In terms of issues of continuity and discontinuity, these choices have many ramifications. With choices like those made in CompuMath, there is discontinuity between the manipulations used to solve equations, and understood as techniques to accomplish this task, and the conceptual understanding of what an equation is and how it relates to inequalities. By comparison with the graphical focus of Visual Math, functions continue to operate as a cognitive root and central mathematical object in both understanding what equations are and in solving them.

As indicated earlier, neither of these choices is optimal. There are other choices to consider as well, including leaving manipulations on paper disconnected from any use of technology. But, such strategies have other disadvantages, like leaving symbolic manipulation as an activity in which meaning is set aside. Such an approach is unlikely to promote the development of what Arcavi (1995) describes as "symbol sense."

Equations in two variables, like $y = 3x–4$ and $3x–y = 4$

Farther on in a curriculum, having worked with students on the graphs of single functions, like $f(x) = 3x–4$, a question is how to approach equations like $3x–y = 4$. On the one hand, it seems that it would be useful for the graphical representation of this function and this relation to be the same. If the graphical representation of these two is the same, then students might appreciate that the operations on both sides of 3x–y = 4 that produce y = 3x–4 have not changed the underlying relation. On the other hand, such an approach elides the differences between functions and equations that have been built by conceptualizing an equation as a comparison of two functions. With such an approach, functions of one variable are a special subset of equations in two variables. Students who have come to view equations as comparisons of functions now are asked to see equations in two variables as representing a single relation, not a comparison.

What is the alternative? The alternative is to view both y = 3x–4 and 3x–y = 4 as comparisons of two functions of two variables, as 0x+y = 3x+0y–4 and 3x-y = 0x+0y+4. There are a number of potential objections to this course of action. First, how is one to graph such equations? Are they to be graphed in three dimensions with their solution sets on the x-y plane? Second, how can it be that f(x) = 3x–4 has such a different graph than y = 3x–4? Again, these are questions that the curricular designers faced.

The Visual Math curriculum generalizes its approach to equations in one variable by having equations in two variables be comparisons of two functions of two independent variables (Initial work in this direction is described in Yerushalmy, 1997). For this curriculum, an equation, like y = 3x–4 is initially a much more complex beast than f(x) = 3x–4. We have discussed this approach to the introduction of an equation in two variables in the earlier edition of this volume (Yerushalmy & Chazan 2002, pp. 743–749) and will only include a brief sketch of this work here.

To understand equations in two variables, the Visual Math curriculum starts by having students work with functions in two variables. In a unit titled "Functions in Two Variables," students are asked to match descriptions of phenomena with two independent variables and surfaces in 3D. Students learn to match descriptions of such phenomena and tables of values of two independent variables. They also learn to match such tables of values with expressions in two variables and two-dimensional surfaces. This forms an analogical sequence to the one taken as the introduction to single variable expressions. In both cases the leading terms are qualitative descriptions of functions (e.g., in two variables surfaces such terms are plane, saddle, etc.) that form the language for inquiring about the nature of the functions' expressions that form the equations in two variables.

Once students have developed familiarity with processes in two independent variables, the curriculum then proceeds to activities that require analyzing solutions of equations using tables of values. For example, a task might be: "Find an equation such that all ordered pairs in the first quadrant are solutions of the equation." At a later stage, students are taught that they can represent the solution set of comparisons of two functions in two variables on the two-dimensional Cartesian coordinate system, in the same way that the solution set of an equation or inequality in one variable can be represented on the number line.

Thus, while advantages involve building on the views of variable, equation, and inequality that have been built up in earlier stages of the curriculum the complications of such an approach are probably self-evident. Tasks involving equations in one variable and equations in two variables, tasks that involve quite similar manipulations and seemingly similar equations, are now treated in quite similar ways, but a number of intervening curricular steps have been added. And, equations that seem very similar, like f(x) = 3x–4 and y = 3x–4, are initially treated as quite different mathematical objects. And, while visual representations are helpful, comparing a pair of two dimensional surfaces increases the cognitive load in "seeing" an equation.

REFLECTIONS ON FLUX IN THE SCHOOL ALGEBRA CURRICULUM

Over the last two decades, the school algebra curriculum has become a site for innovation. Technological advances have helped curricular developers imagine that it might be possible to implement redesigned school algebra curricula. In particular, since spreadsheets and graphing calculators include representations of functions, some curriculum developers have been particularly drawn to a curriculum in which technology supports curricula in which algebraic expressions are conceptualized as representations of functions. Approaches to school algebra predicated on such notions may still maintain some goals that are similar to standard approaches. Goals may include having students learn to factor and multiply some range of polynomial expressions and to solve linear and quadratic equations. Nonetheless, such approaches may represent change to the order of introduction of material, the length of time students work with particular interpretations of symbols systems, and the explicitness with which the curriculum discusses these interpretations.

But, such function-based approaches to school algebra are not univocal. As we illustrated in section 3, the technologies currently available for use in school algebra support different views of equations and their constituent elements, letters and the equal sign. Functions-based curricula that make use of different tools and that order their use of these tools in different ways will differ in the nature of the curricular continuities and discontinuities that they offer to students.

Such differences between curricula that share many goals and features seem like a fascinating opportunity for a different sort of curricular research than a simplistic "horse-race" between two different curricula. Instead, such differences between similar curricula suggest opportunities to understand the nuanced differences between particular curricular choices and the aggregation of findings about such choices. Such findings also stand a better chance of enduring than comparisons of particular curricula that may not be extant in a few years' time.

It is an exciting time for research in the teaching and learning of school algebra. In this review, we have suggested that advances in this area can benefit from systematic study of alternative curricular approaches. We have suggested, and hopefully illustrated, that distinctions developed in the cognitive research literature on the learning of school algebra can be used to understand affordances and constraints of technological tools for teaching school algebra. Similarly, these distinctions seem useful for describing discontinuities in the perspectives that school algebra curricula offer students.

NOTES

1. We have chosen this text because it represents some of the fruit of the efforts of the School Mathematics Study Group (SMSG), an influential group in the United States. The two authors and two consultants on this text were involved in SMSG activities, and the book identifies ways in which it explicitly builds on the work of the SMSG.
2. A different approach would be to think of equations as simply representations of solution sets. In this view, an equation in two variables identifies a set of coordinated values that make the statement true. Functions-based approaches to school algebra define equations in yet a different way, as one kind of comparison of two functions (see, e.g., Chazan, 1993).
3. Filloy, Rojano, and Rubio (2000) suggest that a major contribution of the work with the spreadsheet was the shift of students' interest from solving a specific problem into thinking about a family of similar problems.
4. The "integer" aspect of spreadsheets, the notion that they are operating on a series whose starting point is an integer and whose constant difference is 1, is strongly embedded in other tools provided by the spreadsheet (e.g., newer versions of Excel allow construction of sliders to control a change of a content of a cell but only an integer change is permitted.)
5. There is still a third option where one produces equivalent equation although the manipulations were incorrect but incidentally the intersection points' values remained unchanged
6. In this notation, the number indicates the grade level of the unit and the letter indicates the placement of this order in the units of the curriculum. 7c is the third unit in the seventh grade.
7. The ninth-grade material presents a more formal study of functions assuming previous experience with equations in two variables. Thus, in order to "see" the correspondence rule of a function

students are advised to know how to manipulate equations in two variables to an explicit form y = expression(x).

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