

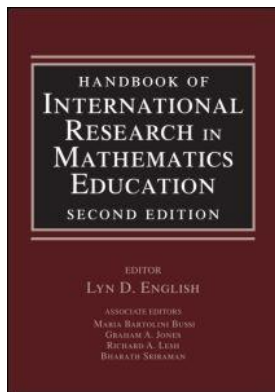
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Lyn D. English, Maria Bartolini Bussi, Graham A. Jones, Richard A. Lesh, Bharath Sriraman, Dina Tirosch

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Nicolina A. Malara, Rosetta Zan

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21 The complex interplay between theory in mathematics education and teachers' practice

Reflections and examples

Nicolina A. Malara

Universita di Modena & Reggio Emilia

Rosetta Zan

University of Pisa

This chapter addresses the complex issue of the relationship between theory, considered as the corpus of knowledge constituting mathematics education, and practice, seen as the act of teaching. Clearly, this analysis is influenced by our model of teaching, which views the teacher as decision maker whose decisions are conditioned by his/her knowledge, beliefs, and emotions. This model has strong implications for research, including reorganizing quality criteria such as reproducibility and relevance, and also for teacher training. In this framework we discuss as an example some aspects of an Early Algebra Project (ArAl), which is also an in-service education process. In particular, we sketch the main features of the Project, and describe the complex methodology of work with/for the teachers based on the sharing of reflections, made at different levels, about the processes enacted in different classes on the same didactical path. We conclude by stressing the importance for teachers and researchers of a common reflection on teachers' practice, which both leads to a greater awareness of the complexity of teaching, and suggests some issues for future research.

There are many ways of looking at the relationship between theory and practice, depending on the point of view from which you look at the two poles in question. There exists a researchers' theory, a teachers' theory, and even a mathematicians' theory, just as there is a researchers' practice, a teachers' practice, and a mathematicians' practice. Each of these different combinations provides a different key for reading this relationship. For instance, some scholars analyze the link between the researcher's knowledge and beliefs and the methods he/she uses on researching (Burton 1994); others consider the relationship between mathematics as a mathematician's practical activity, and mathematics as his/her theoretical body of knowledge (Vergnaud, 1998); others identify and analyze the gap between teachers' theory and teachers' practice (Jaworski 1994, 1998; Mason, 1998).

Here we concentrate on the most common combination, that is, we refer to the relationship between Mathematics Education (ME) researchers' theory, and teachers' practice, looking at theory as a body of knowledge in the hands of researchers, and at practice, as carried out by teachers.

This approach forces us first of all to clarify our conceptions about what is knowledge in ME and what is teachers' practice.

We conceive ME as a discipline essentially constituted by problem-driven research (Bishop 1998; Zan 1999b; Arcavi, 2000), and as a *Science of Practice*—which studies the concrete action of teaching through a mediation between mathematics (with its history and epistemology), pedagogy, and other disciplines (psychology, anthropology, sociology, etc.), and, from the integration of which, it acquires its own peculiarity and authenticity (Wittmann, 1995;

Pellerey, 1997; Speranza, 1997). Using the Stokes-Schoenfeld metaphor¹ (Schoenfeld, 2002), we see research in ME in the Pasteur's Quadrant.

In the international arena, in some cases research in ME has been criticized because of its little connection to social reality and to the most pressing needs of teachers. In the 1980s, some important scholars had pinpointed this separation (Kilpatrick 1981; Freudenthal 1983). Later, Howson (1992), in his plenary conference at ICME 7 (Quebec, 1992), underlined the necessity of overcoming it:

I have written elsewhere of the danger that parts of “mathematics education” will detach themselves from mathematics teaching in much the same way that ‘philosophy of mathematics’ has drifted well away from ‘mathematics’ itself. (...) The importance of such studies is not to be denied, but where does that leave the mathematics educator who wants to serve and help teachers, not just to study, count, or assess them? Perhaps it would be a useful check for all of us contributing to this congress to ask of our contribution: How will/could it help teachers, under what conditions and within what timescale? (pp. 22–23)

This thought-provoking claim highlights that most studies are *about* teachers but not *with* and *for* teachers. This is witnessed by the fact that teachers and educational practitioners are often quite skeptical and not interested in theorization (Silver, 1997; Margolinas, 1998), even if we don't ignore the risk of falling into the trap of naïve pragmatism.

Wittmann (2001), also quoting others, argued in favor of a re-orientation of research towards practice; moreover, other scholars have underlined that communication and spreading of research results must be increased among teachers (Bishop 1998; Lester 1998; Lester & Wiliam, 2002). In particular, Lester and Wiliam (p. 496) wrote:

We promote a renewal of a sense of purpose for our research activity that seems to be disappearing, namely, a concern for making real, positive, lasting changes in what goes on in classrooms. We suggest that such changes will occur only when we become more aware of and concerned with sharing of meaning across researchers and practitioners.

We entirely agree with these scholars. We believe that research in ME is naturally validated in practice, which means not only in the daily managing of classroom activities, but also in teachers' wholeness as living human beings.

In order to face the analysis of the relationship between theory, seen as a corpus of knowledge about ME in the hands of researchers, and practice, the actual teaching carried out by teachers, we discuss the following points:

1. A way of looking at practice: teaching as decision making.
2. The impact of teachers' knowledge, beliefs, awareness and emotions on their practice.
3. Consequences for theory (criteria for the quality of research: reproducibility, relevance, communicability).
4. An example of merging theory with practice: the joint work of researchers and teachers in the implementation of ArAl Project.
5. Conclusions.

A WAY OF LOOKING AT PRACTICE: TEACHING AS DECISION MAKING

The multiplicity of variables involved in a pupil's learning process, highlighted by research in ME, and the context constraints (whether a class has many pupils or not, the syllabuses to be

followed, the interaction with colleagues, families, etc.) imply that the teaching process, too, is a rather complex activity. There are several ways of looking at this complexity, depending on the theoretical lenses through which the teacher and his/her actions are seen. Here we want to stress the fact that the teacher repeatedly faces situations forcing him/her to decision making (Shulman, 1985; Cobb, 1988; Carpenter, 1988; Cooney, 1988; Mason, 1994a; Simon, 1995; Crawford & Adler, 1996; Brown & Coles, 2000). These decisions do not only involve the solution of problems arising in the classroom, but also their identification (Thompson 1992; Cooney & Krainer 1996; Jaworski 1998; Schoenfeld, 1999). Therefore, teaching can be seen also as an activity of problem solving and problem posing.

Moreover, the constructivist approach to the learning of mathematics, particularly in its social context (Lerman, 1992), has two important implications for teaching. The first implication is that teachers, as well as pupils, have an individual interpretation of reality, and in particular of texts, syllabuses, teaching aims in their discipline (Carpenter, 1988; Arsac, Balacheff, & Mante 1992; Cooney, 1994). This interpretation, as in the case of pupils, is not only influenced by the teachers' knowledge, but also by their beliefs and values.

The second implication is that in teaching mathematics teachers must do much more than merely convey knowledge through the "right" words or actions (Cobb 1988; Simon & Shifter, 1991; Cobb, Yackel, & Wood, 1992; Jaworski 1994, 1998; Pirie & Kieren, 1992; Steffe & Kieren, 1994; Simon, 1995). They have the responsibility of creating an environment that allows pupils to build up mathematical understanding, but also of making hypotheses about pupils' conceptual constructs and the possible didactical strategies helping them to modify such constructs.

The teacher's role in the construction of mathematical understanding is particularly emphasized by researchers who adopt a Vygotskian perspective, and see teachers as a guide in the "zone of proximal development": this role is then crucial in making decisions not only about the tasks but also in choosing the communicative strategies to be adopted in classroom interaction (Bartolini Bussi, 1998; Bauersfeld, Krummheuer, & Voigt, 1988).

These elements explain what happened in the mid-1970s: research about teaching shifted from studies on the only observable phenomena such as the teacher's behavior, to studies about the teacher's decision processes (see the reviews of Shavelson & Stern, 1981; Clark & Peterson, 1986). In these studies, teachers are seen as "thoughtful professionals" (Shulman & Elstein, 1975)—like physicians or lawyers rather than like technicians—who make judgments and carry out decisions in a complex environment.

The following problems become important in research on teaching:

- Locating teachers' decisions that influence pupils' learning and the nature of this influence;
- Finding out which factors influence these decisions.

The teachers' decisions can be classified according to various criteria, the most significant of which is related to time and to the crucial moment of the interaction with students. This criterion distinguishes the decisions made in the pre-active phase from those made in the interactive phase and in the post-active phase (Jackson, 1968, in Brown & Borko, 1992). However, decisions can also be classified by typology: Cooney (1988) mentions *cognitive decisions* (related to content), *affective decisions* (related to the more interpersonal aspects of teaching) and *managerial decisions* (including allocation of time).

Of course, these two criteria can be combined: for example, a content decision that occurs in the preactive phase consists in deciding which issues to present and which to exclude from the instructional program.

As underlined by Carpenter (1988), the model of the teacher as problem solver or decision maker, shared today by most of researchers in ME, suggests the need to put together research on teaching and research on problem solving.

In fact, the factors recently taken into account by research on teaching are also relevant to research on problem solving. Stress was formerly given to the teacher's *knowledge*, whereas the evolution of research on problem solving and on learning processes—and particularly the importance given to metacognition—encouraged investigation on other aspects, such as teachers' *beliefs, awareness, emotions*. The next paragraph focuses on these issues.

THE IMPACT OF TEACHERS' KNOWLEDGE, BELIEFS, AWARENESS, AND EMOTIONS ON THEIR PRACTICE

As acknowledged by many research findings, teachers' knowledge, beliefs, awareness and emotions have a very strong impact on practice. We analyze these factors individually, but this distinction is purely theoretical and artificial, since in actual fact they always interact and intermingle very closely.

Teachers' knowledge

Although the strong influence of a teacher's knowledge on his/her decisions is generally acknowledged, there is no agreement on *what* fundamental knowledge is. Initially, research on teachers' knowledge mainly concentrated on the specific knowledge of mathematics as subject matter. The first studies that tried to show the existence of a cause-effect relationship between teachers' subject knowledge and pupils' learning (School Mathematics Study Group, 1972; Eisenberg, 1977) were not very successful. In order to interpret such failure, however, it must be observed that criteria used to assess teachers' knowledge (number of courses completed or performance on a standardized test), as well as those used to assess pupils' achievements (performance on standard exams), were not very fine.

Some years later, Shulman (1986b), reconsidering some ideas of Dewey (1902), identified seven domains of knowledge: (1) knowledge of subject matter (or content knowledge), (2) pedagogical content knowledge, (3) knowledge of other content, (4) curricular knowledge, (5) knowledge of learners, (6) knowledge of educational aims, (7) general pedagogical knowledge.

In the context of ME, three categories were viewed as particularly interesting: *content knowledge, curricular knowledge, and pedagogical content knowledge*.

The most innovative idea is that of *pedagogical content knowledge*, which merges knowledge of content and knowledge of pedagogy, including psychological and social aspects about learners, as Shulman (1986a) clarifies:

... for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing the subject that make it comprehensible to others. (...) [It] also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to learning. (p. 9)

Besides Shulman, other researchers underline that the teacher's knowledge has manifold components. Different frameworks for analyzing teachers' knowledge were proposed, for example, by Elbaz (1983), Peterson (1988), Leinhardt and Greeno (1986), Leinhardt, Putnam, Stein, and Baxter (1991) (for an overview of research about this topic see Fennema & Franke, 1992; Ball, Lubienski, & Mewborn, 2001). Leinhardt and her colleagues (1986, 1991) identify two interrelated systems in teacher knowledge, namely knowledge of lesson structure (an understanding of how to plan and implement a lesson) and content knowledge (understanding the specific mathematics to be taught). These schemata consist of sequences

of goals and actions, that correspond to what the teacher does in the classroom, and the scholars posit a direct relationship between a teacher's knowledge of a lesson and his/her behavior in the classroom. Wood (1999) stresses that teachers must not only acquire pedagogical content knowledge, in Shulman's sense, but also knowledge of interactive and discursive patterns of teaching. Doerr and Lesh (2003) underlined that the nature of teachers' knowledge is not a uniform, consistent or fixed set of constructs but rather it is characterized by complexity and variability. Furthermore they point out that the specific meaning of the concept of pedagogical content knowledge has yet to be fully elaborated although this notion has certainly gained widespread acceptance in the research literature. Davis and Simmt (2006) argued that, for teachers, knowledge of established mathematics is inseparable from knowledge of how mathematics is established; the scholars illustrate four intertwining aspects of teachers' mathematics-for-teaching: "mathematical objects," "curriculum structures," "classroom collectivity," and "subjective understanding".

In the area of research on the influence of teachers' knowledge over their decisions, some very interesting pioneer studies were carried out within a project called Cognitively Guided Instruction (CGI).

On planning this project, Carpenter, Fennema, and Peterson (1988, 1989) intended to merge the perspectives of cognitive and instructional science to study teachers' pedagogical knowledge in the area of elementary arithmetic and to analyze how that knowledge influences classroom instruction and students' learning. More precisely, as part of CGI, several studies have been conducted to determine whether knowledge about research on addition and subtraction would influence teachers' decisions (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef 1989). The results of these studies, based on Carpenter and Moser's (1983) research on young children's learning of addition and subtraction, suggest that teachers' knowledge of children's thinking can heavily influence teachers' decisions and, consequently, classroom learning: teachers prepared in CGI listened to their pupils, were able to support children individually, and spent more time in activities involving problem solving.

By demonstrating the importance of pedagogical content knowledge, these studies stress the need for teachers to have access to research results.

Of course, these results should not be seen as models to imitate, but they should rather enable teachers to make the right decisions; as underlined by Balacheff (1990, p. 269):

the aim is to construct a fundamental body of knowledge about phenomena and processes related to mathematics teaching and learning. The social purpose of such an enterprise is to enable teachers themselves to design and to control the teaching-learning situation, not to reproduce ready-made processes. This knowledge should allow teachers to solve the practical problems they meet, to adapt their practice to their actual classroom.

Teachers' beliefs

No matter which framework one chooses to analyze teachers' knowledge, it is always necessary to consider teachers' knowledge as a large, integrated, functioning system in which components are not easily isolated (Fennema & Franke, 1992). In particular, it is impossible to separate teachers' knowledge and beliefs.

Research on beliefs mainly developed in the late 1970s, simultaneously to the shift in paradigms for research on teaching, from teachers' behavior to teachers' thoughts and decisions (see the surveys by Thompson, 1992, and Hoyles, 1992).

Today, research on teachers' beliefs is a very important field in the wider area of research on teaching (beside Thompson, 1992, and Hoyles, 1992, see Pehkonen & Törner, 1996, and Leder, Pehkonen, & Törner, 2002).

Teachers' beliefs that are usually investigated in ME concern (Thompson, 1992): (1) beliefs about mathematics, (2) beliefs about mathematics teaching and learning. This categorization includes a very wide variety of inquiry fields, but other issues to be taken into account do not belong to mathematics specifically. Beliefs about mathematics teaching and learning include, for example, the causal attributions of failure and success (Fennema, Peterson, Carpenter, & Lubinski, 1990) and the theories of success (Zan, 1999a), but also beliefs about teaching that are not specific of mathematics (Thompson, 1984).

Quite relevant are beliefs that teachers develop about their pupils (Höfer, 1981). The study by Rosenthal and Jacobson (1966) on the so-called Pygmalion effect is a pioneer in this field: the Pygmalion effect shows that the idea that teachers develop about pupils strongly influences pupils' performance.

The importance of teachers' beliefs is quite evident when it comes to creating new syllabuses or experimental projects. Although teachers apparently agree with the aims of a project and its features, and in spite of the fact that many projects are very specific about the didactical choices to be made, it often happens that teachers suddenly make choices that go against the very spirit of a project: in other words, the relationship between teachers' stated conceptions (i.e., teachers' theory) and practice turns out to be very problematic. More generally, it happens that the espoused beliefs are inconsistent with practice (Furinghetti, 1997; Raymond, 1997; Nesbitt Vacc & Bright, 1999).

This mismatch between espoused beliefs and beliefs-in-practice, demonstrated by many studies on teachers' beliefs (Hoyles, 1992), confirms the results of research on problem solving (Schoenfeld, 1989): beliefs that teachers declare in the end differ deeply from those that guide their solving processes and their behavior in general.

This induces us to find out which beliefs mostly influence teachers' decisions: traditional tools like questionnaires, interviews, Likert scales seem quite appropriate to detect teachers' espoused beliefs. Lerman (1994) maintains that the context strongly influences teachers' beliefs: the shift from one setting to another allows the appearance of factors that significantly change teachers' actions from those they would profess to apply or would wish to apply. So, it appears little significant to examine teachers' beliefs through an instrument (like interviews in a laboratory or questionnaire completion) in one setting, and their impact in another setting, typically the classroom.

Generally speaking, there is a strong need for social and anthropological approaches (Eisenhart, 1988; Arzac et al., 1992; Bishop, 1998), i.e., for studying teachers' beliefs in their natural context. Therefore, many studies suggest non-traditional methods, such as narratives (Brown & Cooney, 1991; Chapman, 1997, 2002). Indeed, narratives include the 'tacit knowledge' underneath practice, which cannot be expressed in propositional or denotative form (Polanyi, 1958): this tacit knowledge embeds teachers' deep beliefs that influence practice. In particular, some researchers use metaphors in order to represent teachers' knowledge grounded in experience, and provide coherence for teachers' practice.

What seems necessary, in order to measure changes in teachers' beliefs, is to study individual teachers in depth and to provide detailed analyses of their cognitive processes (Thompson, 1992). In particular, it is possible to get teachers to research their own practice 'from the inside' rather than as objects to be studied (Mason, 1994b). These types of studies, gradually becoming more and more frequent (Brown & Cooney, 1991; Cooney & Krainer, 1996; Malara, 2005; Potari & Jaworski, 2002), highlight the importance of reflection and awareness for effective changes.

Teachers' awareness

Lester and Wiliam (2002, p. 494) stress that "the speed with which decisions have to be made means that the knowledge brought into play by teachers in making decisions is largely implicit rather than explicit." Thus, it is important that they are able to recognize and control

it. This implies that they must be able to analyze their actions and reflect on the reasons that produced them.

Recent research in mathematics teaching points out the need for teachers' reflection about their own practice (Lerman, 1990; Mason, 1990, 1998, 2002; Jaworski 1994, 1998, 2003). Jaworski (1998, p. 7) uses the following words to define the kind of practice that results from this reflection, i.e., reflective practice: "The essence of reflective practice in teaching might be seen as the making explicit of teaching approaches and processes so that they can become the objects of critical scrutiny."

This notion appears with Schön's idea of "reflective practitioner" (1983, 1987): knowledge about practice grows from knowing-in-action, through reflecting-on-action to reflecting-in-action. Reflective practice mends, according to Schön, the separation between theory and practice, and between practitioners and experts. Through reflective practice teachers become aware of *what* they are doing and *why*; awareness is therefore the product of a process of reflection.

Mason (1998, 2002) emphasized the role of awareness in teaching. More precisely, he argued that being a real teacher involves the refinement and development of complex awareness on three levels: (1) awareness-in-action, (2) awareness of awareness-in-action, or awareness-in-discipline, and (3) awareness of awareness-in-discipline, or awareness in counsel. He suggested that awareness-in-discipline is what constitutes the practice of an expert, but what supports effective teaching in that discipline is awareness in counsel.

It is interesting to observe that the strategies used by Jaworski (1994) in order to increase teachers' awareness are absolutely similar to those suggested by other researchers (Schoenfeld 1987; Garofalo, Kroll, & Lester, 1987) to develop pupils' metacognitive skills in order to become better problem solvers. In both cases, the subjects are continuously asked hard or difficult questions about their thinking processes. Over time, subjects start to anticipate questions, and to ask their own questions.

Indeed, awareness deals with metacognitive skills. More precisely, it deals with the first aspect of metacognition (Schoenfeld, 1987), while the second is self-regulation or control. The two aspects of metacognition are strictly linked, in the sense that awareness of one's resources can lead one to activate regulating processes. In fact, the influence of teachers' increasing awareness on their control processes is clearly emphasized in most research studies (Thompson, 1984; Lerman, 1990; Jaworski 1994; Borasi et al., 1999; Malara, 2005).

Teachers' emotions

Still, the shift from being aware to enacting control processes is not automatic, and it is influenced by many factors. Many studies have so far underlined the role of emotional factors. In particular, a sense of self-efficacy and an enjoyment of learning flow from individual strategic events but eventually return to energize strategy selection and monitoring decisions, i.e., executive processes (Borkowski, 1992).

In ME, the importance of emotions was initially pinpointed in the field of problem solving (McLeod & Adams, 1989), and later in the more general field of mathematics learning, even if research on affect in mathematics education has mainly focused on beliefs and attitudes rather than on emotions (McLeod, 1992; Evans, Hannula, Zan, & Brown, 2006).

The most frequent approach to emotions in ME, and in particular in problem solving, is borrowed from cognitivist psychologists (Mandler, 1984). This approach can help us understand how teachers' decisions are influenced by their emotions, which research on teaching often tends to forget. Moreover, this approach is significant to describe the model of practice that we use.

According to Mandler (1984) the emotional experience is the result of a combination of cognitive analyses and physiological responses:

- If a sequence of actions gets interrupted, or if a cognitive or perceptive discrepancy occurs between facts and expectations, the consequence is visceral arousal;
- The subjective experience of emotion is a combination of visceral arousal and a cognitive assessment of the experience.

Therefore it is not the experience itself that causes emotion, but rather the interpretation that one gives to the experience. This interpretation is influenced by an individual's beliefs. Still, beliefs play an important role also in causing perceptive or cognitive discrepancies.

As McLeod (1992) suggested, Mandler's theory is particularly interesting, since among the cognitive theorists he has done the most to apply his ideas to problems in mathematics education (Mandler, 1989). But the efforts by Ortony, Clore, and Collins (1988), though not regarding mathematics, are interesting too, since they attempt to categorize the various emotional responses.²

Despite the different approaches to emotions in mathematics education, there is some measure of agreement: emotions are seen to involve physiological reactions, and to affect cognitive processing in several ways (biasing attention and memory and activating action tendencies). Emotions are also seen to be functional, with a key role in human coping, adaptation, and decision making (Evans, 2000; De Bellis & Goldin, 2006).

The importance acknowledged to emotional aspects in problem solving and decision processes suggests that we consider emotions to be relevant in the teaching process, too. Many researchers state that teachers must take into account emotional aspects in their teaching (McLeod & Adams, 1989; Simon & Schifter, 1991; Middleton & Spanias, 1999). However, it is not enough to consider pupils' emotions, which of course are important: teachers' emotions, too, play a fundamental role in the teaching/learning interaction, since they influence teachers' decisions, exactly as it happens with pupils (Shulman, 1985).

As a consequence to the approach described, context constraints have a double influence on teachers' decisions. They certainly influence them directly since they are objective constraints (see for instance: time needed to deal with a topic, syllabus prescriptions, number of pupils). These constraints, however, are also perceived and interpreted by teachers according to their aims, values and beliefs, and this interpretation elicits emotions, which influence their decision processes. Time is quite a typical example in this respect, because it influences teachers' decisions not only by imposing an objective constraint, but also by arousing anxiety, which, too, has a strong influence on decisions.

Even if emotional aspects are very seldom the direct object of research on teaching, many studies demonstrate the important role they play. Arzac et al. (1992), for example, facing the issue of reproducibility of didactical situations, considered the problem of teachers' role in the class, when they have to follow a predefined scenario. Through two case studies the researchers discover two types of factors that tend to hamper fidelity in reproducing the given scenario: constraints resulting from the didactical system, like time constraints, and teacher's conceptions about mathematics and learning. They observe that a teacher's decisions for coping with these constraints tend to oppose the devolution of the problem situation to students. But in our opinion, some teachers' behavior, opposite to the planned scenario (such as making questions which induce direct answers, not writing false statements on the blackboard, bypassing the processes they considered too uncertain), seems to derive from context-provoked emotions, and not directly from the context, in particular from the anxiety elicited by time constraints and by the difficulty to manage uncertainty.

If we observe the decisions made in the pre-active, interactive, and post-active phases, we can see that most context constraints belong to the interactive phase, in which time for decision-making is short, and there is no possibility of pondering before deciding. That is why teachers' decisions are strongly influenced by emotions in this phase.

In particular, the emotions connected to the interaction between teachers and pupils are very important in this phase. Salzberger-Wittenberg, Williams Polacco, and Osborne (1983)

face this problem from a psychoanalytic point of view. They state that in front of pupils teachers can feel “fears” (e.g., fear of criticism, fear of hostility, fear of loss of control). Moreover, the authors underline that the attitude and the expectations that teachers have towards pupils can also deeply influence their perception and interpretation of pupils' behavior, as well as their reactions to such behavior.

Here, too, awareness appears to be crucial in order to minimize the consequences of this influence (Salzberger-Wittenberg et al., 1983). If we assume this point of view, it is clear how important the studies that consider teachers' emotions are, in particular those that examine their influence on decision processes. Aspects analyzed here (knowledge, beliefs, awareness, emotions) must be seen in their interconnections, but also in the more general frame of values (mental, moral, and aesthetical), on which research in ME has not yet focused enough (Vinner, 1997).

The comprehensive theory about teachers' decision making developed by the Teacher Model Group at Berkeley (Schoenfeld 1998, 1999), named KGB theory (Knowledge – Goals – Beliefs), exemplifies the attempt to merge these aspects. As underlined by Sriraman and English (2005) the systemic work engaged by Schoenfeld's research group ended up in a teacher's decision-making model of “teaching in context” which provides a fine-grained characterization of the teacher's decision making processes, grounded in the analysis of teacher's knowledge, goals and beliefs. This model has been constructed for both theoretical and pragmatic reasons, since a better understanding of the underpinnings of teaching competence and teachers' developmental trajectories is likely to favor an enhancement of their professional development.

CONSEQUENCES FOR THEORY

Seeing the teacher as decision maker or problem solver, rather than an executor of algorithms, strongly influences both the theory-practice relationship, and theory itself. In ME, theory was born with studies that were strongly characterized by the positivist paradigm, considered as a synonym of scientificity, especially in the science of education. Therefore, up to the 1970s, the predominant methodology was statistical in nature. However, this approach turned out to be unsatisfactory as soon as the complexity of learning processes got acknowledged. Since learning is a complex activity, we have a sort of ‘uncertainty principle of didactic variables’ (Arzarello, 1999); being able to have all variables under control is an illusion:

(...) education may not be best served by continuing to employ a solely cause - and - effect perspective. (...) In scientific enquiry, all factors are held as constant as is possible; in education, no factor remains stable when another is perturbed. (Mason, 1994b, p. 194)

This raises doubts about the positivist paradigm as synonym for scientific method and questions it³ (Kilpatrick, 1993; Schoenfeld, 1994; Sierpiska & Kilpatrick, 1998). In particular, it becomes more and more important to acknowledge the influence of teachers' decisions on pupils' learning processes, as well as the complexity of this influence and of these decisions.

For a long time, teachers were treated as a ‘constant’ in classroom studies (Chapman, 1997) or when curricula were developed (Fennema & Franke, 1992). However, as we have seen in the previous paragraph, the failure of many innovative programs, although extremely careful in foreseeing most of the important decisions for the teacher (e.g., regarding content, activities, and also assessment), and the difficulties in reproducing experimental situations (Balacheff, 1990; Artigue & Perrin-Glorian, 1991; Arsac et al., 1992) underline the dramatic importance of the ‘teacher’ as variable.

Although researchers adopted the language of “treatments” and “variables,” the objects they named so often failed to have the requested properties, as underlined by Schoenfeld (1994, p. 701):

oftimes, for example, an instructional ‘treatment’ was not a univalent entity but was very different in the hands of two different experimenters or teachers. Similarly, if an instructional experiment used different teachers for the treatment and control groups, the teacher variation (rather than the instructional treatments) might account for observed differences; if the same teacher taught both groups, there still might be a difference in enthusiasm, or in student selection. In short, many factors other than ones in the statistical model—the variables of record—could and often did account for important aspects of the situation being modeled.

In other words, in ME a typical phenomenon for complexity takes place, known as the butterfly effect: *if a butterfly flaps its wings on the Caribbean sea, in North America the weather could change*. Therefore “microscopic” teachers’ decisions can have “macroscopic” effects in the dynamics of situations, as stressed by Artigue and Perrin-Glorian (1991, p. 14):

Various recent researches, for instance (Arsac, 1989), have highlighted the macroscopic effect of decisions which can be qualified as microscopic if one refers to the level of observation, and the bifurcation in the dynamics of a classroom which can be caused by an apparently innocent remark, or even a simple movement or expression by the teacher. They clearly prove that the teacher can exert a close control over the dynamics of situations, at this microscopic, nearly invisible level, in order to reproduce what he perceives as necessary, or at least important, to reproduce through the description given to him.

Criteria of quality for research

Once complexity is acknowledged, many different approaches and methods are needed, because only the choice of many different points of view can help describe a complex situation (Steiner, 1985; Kilpatrick, 1993; Bartolini Bussi, 1994; Schoenfeld, 1994, 2002; Mason, 1994b; Lester, 1998; Arzarello, 1999; Doerr & Lesh 2003).

Opening to a multiplicity of methods borrowed from other disciplines (psychology, sociology, linguistics) means, of course, that the quality of research must be always kept under strict control (see Zan, 1999b and the relative references).

Since the 1990s, this issue has been explicitly addressed by several researchers, even at international conferences.⁴ In particular, the meaning of constructs like relevance, validity, objectivity, originality, rigor and precision, predictability, reproducibility, and relatedness in the context of different kinds of research methodologies is discussed; moreover, it is discussed whether, and possibly in which form, these constructs should continue to be regarded as fundamental criteria for assessing mathematics education research.

The discussion of some of these criteria is strongly influenced by the model of teacher as decision maker. The fundamental role of teachers’ decisions, and the butterfly effect observed when one tries to reproduce a teaching experiment, generates a long series of questions about reproducibility.

Reproducibility

The problem of reproducibility has often been studied by French scholars within research on didactical engineering (Balacheff, 1990; Artigue & Perrin-Glorian, 1991; Arzac et al., 1992). Even if Artigue and Perrin-Glorian spoke of internal and external reproducibility and underlined the need of “rejecting an over-simple assimilation between internal and external reproducibility,” they acknowledged the existence of obstacles created by teachers’ unforeseen decisions:

Even when the teachers carrying out the experimentation have closely participated in its development, when experimenting they frequently take unforeseen initiatives which disturb the functioning of the research process. (Artigue & Perrin-Glorian, 1991, p. 14)

As we have already observed, these unforeseen initiatives are influenced not only by the constraints resulting from the didactical system and by the teachers' conceptions about mathematics and learning (Arsac et al., 1992), but also by the emotions elicited by the interaction between these elements.

So, teachers play a fundamental role in facing the problem of the reproducibility of a teaching experiment. The teacher variable must be considered with and among all other variables. Furthermore, in order to allow research to be reproduced, it is extremely important that teachers undergo a preliminary training about all the aspects that, as we said, influence decision processes in doing research (knowledge, but also metacognitive skills, beliefs, emotions). From this perspective, teacher training is no longer the last link of a chain—unessential to the previous ones—that researchers can neglect or delegate to others. On the contrary, it is related to the quality of research itself.

Therefore, the issue of the theory-practice relationship becomes unavoidable: researchers' theory cannot exist without teachers' practice.

Relevance

In this frame, reproducibility is not the only quality standard influenced by the model viewing the teacher as decision maker. This model implies also a revision of the standard of *relevance*. This criterion, regarded as fundamental for research in any discipline (Polanyi, 1958), is linked to the ultimate goal of research, and therefore it is developed differently, according to the typical values of each discipline. Because mathematics education is a relatively young discipline, it seems difficult to identify typical values, actually shared by all researchers; still, improving the teaching practice seems to be quite unanimously considered as the ultimate goal (Vinner, 2000).

Even assuming that this is the ultimate goal, "relevance" is a very ambiguous term. One possible reason is that the criterion of relevance—unlike other criteria, such as validity—can be referred to the various components of research: so we can speak of relevant research problem, but also of relevant method and relevant results.

Sierpiska (1993) suggested that we make a distinction between "pragmatic relevance" and "theoretical relevance":

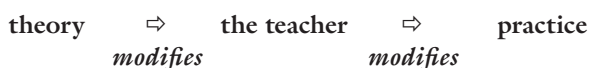
something is pragmatically relevant in the domain of mathematics education if it has some positive impact on the practice of teaching; it is cognitively relevant if it broadens and deepens our understanding of the teaching and learning phenomena. (p. 38)

She observed that if we accept the idea that the ultimate goal of research is the improvement of the practice of teaching, each theoretically relevant research must be pragmatically relevant too: the only distinction, in this case, is between more direct and less direct pragmatic relevance. Kilpatrick (1993) also mentioned direct relevance, noting that a particular research study might be of direct relevance to teachers, but is frequently of more direct relevance to other researchers.

The meaning of the term "direct relevance" is not self-evident. We suggest a distinction between direct relevance for *teaching* and direct relevance for *teachers*. If direct relevance for teaching can mean a direct usability of some parts of research in practice (but then, in this sense, there is very little relevant research), direct relevance for teachers is a subtler issue.

Teachers' role in mediating between theory and practice does not necessarily consist in properly modifying experience deriving from research in order to adapt it to the classroom.

The teachers' role is quite different: since theory modifies teachers' knowledge, metacognitive skills, beliefs, and emotions, it modifies teachers *directly*. In particular, theory modifies teachers' decision processes and consequently their practice. This change does not take place through external intervention (where someone says to the teacher "do this, not that" or "think differently"): this change happens as a progressive growth in teachers' awareness, as induced by the theory and by reflection on it. From this point of view, the model of teacher as decision maker knits together the break between pragmatic and theoretically relevant research. As a matter of fact, teachers' decisions are influenced—over rather a long time—by a change in teachers' knowledge, metacognitive skills, beliefs, and emotions:



Still, in order to make sure that theoretically relevant research has a direct influence on teachers, two conditions need to be met:

- Teachers must be able to absorb this research: in particular, they must be aware of their role as decision makers;
- The research itself must be conveyed in forms which are accessible also to practitioners.

The first point is particularly important. Without an appropriate education, teachers (but more in general those who do not do research in ME) tend to prefer research results that seem *immediately* applicable in the teaching practice.

Boero and Szendrei (1998) observed that for many teachers, but for many mathematicians too, the most useful research results are those offering innovative patterns or quantitative information. Still, they underlined that results offering qualitative information and theoretical perspectives are important not only as such, but also because they allow teachers (and researchers) to keep the other kinds of results under control.

And indeed, without proper warning there is a risk of giving naïve interpretations. Artigue and Perrin Glorian (1991, p. 14) stressed that this risk exists as to internal and external reproducibility:

(...) it is then, for obvious reasons of communicability, accompanied by a flattening-out of scientific didactic language into the common language of teaching. It is not at all certain that, by doing this, we really reduce the problem of transmission. We give an illusion of communicability—but only an illusion. In fact we encourage naïve interpretations and therefore possibly make internal reproducibility more difficult to obtain.

Communicability

These considerations lead us to the problem of communicating research and to a revision of the quality standards for research reports: in this perspective, standards like clearness, organization, synthesis, which belong specifically to the phase of communication, become very important also for the quality of the whole research.

One thing that should not be forgotten when it comes to communication is the wide variety of methods that are used in ME research, and above all the variety of disciplines to which ME research refers to (mathematics, epistemology, psychology, linguistics, sociology, anthropology, etc.). The co-existence of different, sometimes contradictory, paradigms is complicated by the fact that sometimes researchers do not declare their choices explicitly (Dörfler, 1993; Mason & Waywood, 1996). Moreover, these choices often derive from very personal beliefs, which should also be made explicit (Burton, 1994; Schoenfeld, 1994; Mason, 1994b).

Of course, it is not enough to merely give such information, because the way the information is conveyed is extremely important. In order to be appreciated and have some feedback, research must be communicated. Sometimes the quality of reports makes communication very difficult even among researchers, but most of all between researchers and practitioners. Many fundamental details are often given for granted, and language is understandable only by 'initiates' (Bishop, 1998; Lester & Lambdin, 1998; Hanna, 1998; Lester, 1998). According to Mason (1998, p. 370):

the more familiar and overt products of research, namely reports, articles, books, professional development materials, and classroom materials all suffer from what might be called a 'research transposition', following Chevallard's (1985) *transposition didactique*.

Lester (1998, pp. 203–205), highlighting the failure of research in mathematics education to talk with teachers, sees one of the many explanations in the fact that

(...) researchers and teachers have accepted different ways to frame their discourse about what they know and believe about mathematics teaching and learning. By and large, teachers communicate their ideas through, what Schwandt (1995) calls 'the lens of dialogic, communicative rationalism'. By contrast, researchers typically communicate their ideas in terms of (monologic) scientific rationalism. (...) To accept dialogic rationalism involves accepting that reason is communicative: "It is concerned with the construction and maintenance of conversational reality in terms of which people influence each other not just in their ideas but in their being" (Schwandt, 1995, p. 7). It aims to actually move people to action, in addition to giving them good ideas. Dialogic rationalism, then, has something to say to mathematics educators about how we make and justify claims in our research. In particular, dialogic rationalism attempts to avoid treating students and teachers as objects of thought in order to make claims about them that will guide future deliberative actions. Instead, it aims to include teachers (and students?) in dialogic conversations in order to generate practical knowledge in specific situations.

But in order to move people to action, it is necessary to create a social practice and speak directly to people's experience (Mason, 1998). So, it is important to present research in forms which promote personal construal, in which readers find themselves seeing their past experiences in a fresh light, and are sensitized to potential incidents to notice in the future (Mason, 1994b).

This kind of communication does not consist only in research reports. Communication can also go other ways. Conferences where researchers and practitioners can meet (like CIEAEM, for example) allow a real two-way exchange, and not only from researchers to practitioners, as it usually happens with reviews.

AN EXAMPLE OF MERGING THEORY AND PRACTICE: THE ARAL PROJECT

In order to give an example of the complex interplay between theory and practice, we will now examine ArAl Project: Arithmetic Pathways Towards Favours Pre-algebraic Thinking (Malara & Navarra 2003, <http://www.aralweb.it>) and the multifaced methodology through which it develops.

The ArAl Project

The ArAl Project was born in 1998, within the framework of previous studies by Malara and collaborators, carried out between 1992 and 1997 and devoted to the *renewal of the teaching of arithmetic and algebra in scuola media* (Grades 6–8). Among the results of their experimentations, the strong potential of an approach to algebra as a language to be used in modeling, solving problems, and proving was pointed out (Malara & Iaderosa, 1999), but they also highlighted, as also discussed in the literature (see, for instance, Kieran, 1992), the negative influence of the type of teaching received in primary school, which is essentially procedural and concentrated on computations' results. This led the authors to consider a revision of the teaching of arithmetic in primary schools in an algebraic perspective (Davis, 1985; Linchevski, 1995); in 1998 they started to work with and for primary teachers. This work, and the ArAl project that followed, now constitutes an important means of teaching early algebra in Italian schools.

The hypothesis

The specific hypothesis on which the ArAl Project is based is that there is an analogy between ways of learning natural language and ways of learning algebraic language. The *babbling* metaphor can be useful to clarify this point of view. While learning a language, the child gradually appropriates its meanings and rules, developing them through imitation and adjustments up to school age, when he will learn to read and reflect on grammatical and syntactical aspects of language.

In the traditional teaching and learning of algebraic language, the study of rules is generally privileged, as if formal manipulation could precede the understanding of meanings. On the contrary, mental models characterizing algebraic thinking should rather be constructed within an arithmetical environment—starting from early years of primary school—through initial forms of algebraic babbling, teaching the child *how to think arithmetic algebraically*. In other words, algebraic thinking should be progressively constructed in the child as both an instrument and an object of thinking, strictly interweaved with arithmetic, starting from its meanings.

For this purpose, it is necessary to construct an environment able to stimulate an autonomous elaboration of algebraic babbling and consequently to favor the experimental appropriation of a new language in which the rules may find their place just as gradually, within a teaching situation which is tolerant of initial, syntactically “shaky” moments, and which stimulates a sensitive awareness of formal aspects of the mathematical language.

A fundamental role in promoting a conscious approach to the use of letters for formal coding is played by the didactical contract, centered on the task *first represent and then solve* that forces a shift from results to processes and reduces calculative attitudes.

A view of arithmetic-algebraic teaching like the one described above requires a metacognitive approach, in which—through a game of translation and interpretation of expressions in natural and formal languages—the teachers can make pupils aware of the meaning of used signs and symbols as well as of the representational strength of formal writings.

In relation to this, in our project, a central role is played by *Brioshi*, a hypothetical Japanese boy who does not know Italian but can communicate through mathematical language. Turning to Brioshi prepares pupils to appropriate the problem of making formal representations and motivates them, through forms of algebraic babbling, to gradually acquire the correct way to realize them (for further discussion see Malara & Navarra, 2001).

The ArAl Project and teachers

Preparing to teach with these modalities requires a remarkable competence on the part of the teacher, a competence that must be completely built up. Then, teachers who intend to embrace

these innovative teaching approaches must be prepared to combine their existing knowledge, competencies, and beliefs with a mix of far-from-marginal methodological and organizational aspects to stimulate activities with a high metacognitive content, to favor the reflection on language, to promote verbalization and argumentation, and to reach a fine analysis of protocols. All these aspects operatively support an actual *culture of change*.

But this approach is made more difficult by the fact that in our country almost all compulsory school teachers do not have a university background in mathematics, as the majority either come from areas of humanistic and pedagogical education (primary teachers) or from a generic scientific education (junior secondary teachers). Their cultural background and beliefs naturally influence the ways through which they give pupils basic knowledge. Therefore, one of the main aims of the project is the rephrasing of the teachers' conceptions so that they can get a deeper competence and acquire a new, more adequate professional identity.

Methodology

The methodology for accomplishing the project is complex and is based on a net of relationships among the several actors involved: the university researcher, the teachers (who play different roles: researcher, trainee researcher, experimenter), and the pupils. In this net, two relationships are privileged: that between university researcher and (trainee) teachers-researchers and that between teacher researcher and teachers experimenters.

The relationship between university researcher and teachers-researchers, especially trainees, is not only practice-oriented but also theory-oriented. The university researcher stimulates teachers about some didactical issues presenting theoretical questions and results from the literature, pointing to readings and proposing possible research themes. These themes, and related hypotheses to be tested, are chosen and organized on the basis of teachers-researchers' interests, also in relation to their classroom planning. The teachers—trainee researchers—take charge of designing tasks to be experimented in the frame of an hypothesized research plan (tasks strongly discussed in the group before the given to the pupils), and of transcribing the classroom records for a fine analysis of the classroom process.

This methodology, though having its roots in the Italian model of research for innovation (Arzarello & Bartolini Bussi, 1998), represents an important and complex evolution of the model itself. It fits the model of co-learning partnerships by Jaworski (2003, 2004), although it differs from it as to elements concerning the study of the students' ways of learning, the relationship with the teachers, and, most of all, the conceptions underlying the roles played by the partners.⁵

A big part of the work made in the project tackles the problem of weaving together experimental studies focused on the students' learning and difficulties (Malara & Navarra 2001, 2003) with training paths that put the teachers in the condition of critically thinking over their own personal epistemology (Malara 2003, 2005; Malara, Incerti, Fiorini, & Nasi, 2004).

The studies focused on the teachers' side are more complex and could be defined as *second level studies* (Arzarello & Bartolini Bussi, 1998). Their main aim is to bring the teacher-researchers (mainly trainee) to:

- Having a more and more refined control on their own behavior and ways of communication in the classroom;
- Pointing out the impact on classroom interactions of micro-variables linked to individual attitudes or to emotional-relational dynamics;
- Becoming aware of the incidence of knowledge, beliefs and emotions on their own decisions.

This aim is realized through the analysis of the transcripts of classroom processes which is focused on the interrelations among the knowledge built by students and the teacher's behavior in guiding students to construct the concepts in question.

In these second level studies, aspects of the analysis made and the relapses of such analysis on the teachers are highlighted as far as they concern the teachers' ability of critical reflection: on their own actions (identification of proactive attitude or, on the contrary, of negligence or approach mistakes); on the incidence of their behavior on the collective construction of knowledge; and on their ability in assessing pupils' contributions and performances.

In more detail, the analytical process develops along the following steps: the teacher's autonomous reflection, the teacher and university researcher's joint reflection, the teachers' common reflection, and the teachers' reflection in interaction with the researcher. Each step brings a new contribution to the *teacher's awareness*.

In the first phase, concerning the autonomous reflection on what happened in the classroom, teachers are requested to transcribe the recorded classroom discussions and to make explicit comments about points they consider problematic. This forces them to observe their action with detachment, in order to monitor the consequences of their ways to communicate with pupils, to pose questions, to give hints and to make decisions. It also allows them to observe their ability to orchestrate the classroom activity and to assess the level of participation.

In the second phase, the joint analysis, the researcher guides the teacher to make local reflections by asking him/her to explain the meaning of some interventions/steps, the researcher indicates potential strategies for overcoming dead-ends and provides explanations about (sometime subtle) mathematical questions arisen. The researcher also triggers global reflections on what has been done and objectifies significant steps in the development of the mathematical construction.

Through a lively confrontation where the teacher is encouraged to express his/her points of view, doubts, perplexities (important indicators of his/her beliefs), this joint analysis provides an opportunity to make the teacher's habits, stereotypes, beliefs, misconceptions explicit and to disclose possible conceptualization gaps in his/her mathematics knowledge.

This moment turns out to be of particular importance for the teacher's awareness of his/her way of being in class and for a first assessment of his/her decisions (didactical choices, interventions/silences, word turns to the pupils, reintroductions, timings, etc).

The third phase, consisting of an exchange involving all teachers, represents a moment of free sharing of the events, useful to express any possible fear or doubt, as well as to look for the roots of possible common questions. This phase is also characterized by a cross reading of protocols related to classroom processes and an initial getting aware of the divergences of the individual action developments. This leads to further reflections on one's actions and formulation of some possible hypotheses.

In the final phase, the whole group reflection, a global revision of the teachers' transcripts is made and this turns out to be the climax of the whole experience. The sharing of transcriptions, aimed at constituting the Units (see below) and gathering the different classroom discussions arisen about the same problem situation, allows spotting out and objectifying the reasons that have determined them. Comparing their own developmental path with what colleagues did in the same steps of a teaching sequence, the teachers detect important distinctive elements and reflect on the effectiveness or limitations of their work (personal hasty and decisive interventions, little attention to listening, non understanding of potentially fruitful interventions, scarce ability to orchestrate voices, difficulty in managing leaders or minimizing effects of tacit alliances, etc). All this leads teachers to acquire deeper awareness of their way of being in class, of better controlling their behavior and also of conceiving new views on their teaching and possibly, to *assume a new identity* over a lengthy time.

Of course, this methodology is sharply dependent on teachers' involvement and cannot be used at large, for instance in short refreshing courses.⁶ Nevertheless, our hypothesis is that it

is possible to promote the spreading of these ideas and methods through the direct influence of the teachers involved in the project on their colleagues at school, according to a wave-style model, adopted for the renewal of teaching in our country since the 1970s. Moreover, work done with these teachers allows us to construct tools capable of presenting teachers' actions in paradigmatic classroom scenes, as we will show in the next section.

The Units and the glossary of the ArAl Project

A first important result of this analytical process is construction over time (about 3 years) of various teaching sequences called "Units" to facilitate communication among teachers- which can be seen as *models of teaching processes* for arithmetic in an algebraic perspective.

These Units are not theoretical tools for researchers (Schoenfeld, 1999), but tools for the renewal of classroom practice, even if they cannot be immediately used in the classroom but require a theoretical study before being set up. To this end, two key tools for the project have been created: the Theoretical Reference Framework and the Glossary, which includes more than 90 terms and enables a clarification of specific mathematical concepts or methodological theoretical constructs. In the glossary, each term is described by a text containing other terms, to which it is cross-referenced for a wider and deeper analysis.

Through the combined use of these tools, teachers can attain a double goal: the first one, immediate and local, concerns guiding pupils in the collective exploration of proposed problems; the second one, more general and attainable in the longer term, concerns the objectification of "hypothetical learning trajectories" (Simon, 1997) as to the subject in question, according to the spirit of the project. Therefore the Units constitute a versatile study and guidance instrument for those who aim to renew their own teaching on the arithmetic-algebraic side (teachers can choose didactical situations that most suit their own needs and they are guided towards understanding how to develop a discussion and which scenarios can shape up). But the main aim is to offer teachers the opportunity to reflect on their own knowledge and *modus operandi* in the classroom, before actually providing them with didactical pathways that they should follow.

Some remarks

Our research experience both with and for teachers, aimed at the actual development of a socio-constructive approach to the teaching of algebra, made us aware of the difficulties teachers meet both in designing and in managing whole class discussions. Many cases we have analyzed show that teachers either do not grasp a pupil's reasoning or fail to give due value to significant contributions and let them drop, or are conditioned by some pupils' invasiveness, or are even unable to use appropriate silent pauses (a wide documentation of this can be found in Malara et. al., 2004). This clearly highlights how rich and at the same time how dangerously delicate a classroom discussion is: in the midst of the overwhelming energy of a participating class, "traps" for the teacher lie everywhere (unforeseeable diverging solutions, potentially fruitful but perhaps not too clearly expressed; time constraints; the need to keep pupils' general attention high; the need to consolidate achievements, rather than disperse them, etc.).

All this shows very clearly the importance of a *fine* education for teachers to listen to their pupils. This condition poses the hard challenge of how to best help them "fine-tune their antennas" and acquire that local flexibility to enable them to adapt to the flow of thoughts which emerges from the class, to grasp potentialities, to develop them and adequately insert them into the working context. The task is far from being easy, since it is not a matter of dialogue on mathematical or pedagogical knowledge, but on the more complex and delicate level of behavior—mostly subconscious—that is rooted in the teacher's past life experiences. Furthermore, it is not a matter of making teachers aware of what is wrong with the way they

operate (what they tend to anticipate or, on the contrary, to omit in the midst of actual classroom action), but rather a matter of increasing this awareness, in order to create a new, more appropriate behavior.

As to this end, in countries with a long tradition in mathematics education, video recordings of classroom processes are used for bringing teachers to reflect on their own decisions and actions in the class and to compare them to the ones of their colleagues involved in the same class work (Jaworski & Gates, 1987). In the 1980s, some scholars promoted debates on the facilities that this tool offers to research as well as to teachers' education (see Pirie & Breen 1998; Mousley, Breen, & Frederick, 1999). Recently, other scholars have pointed out how the video is an efficient tool for understanding teachers' beliefs underlying their decisions and actions in the class (Sherin 2004; Kuntze & Reiss 2005, 2006; Törner, Sriraman, Sherin, Heinze, & Jablonka, 2005). But in our country videos are still not used in mathematics education and are only marginally used in Italian research (due to laws protecting individual privacy).

In our opinion, videos of classroom processes are important for teacher education, mainly because they might help teachers analyze the use and incidence of non-verbal language, as well as the participation of the whole class. Nevertheless, we believe that watching the video as such does not enable teachers to capture details fully, relate them to one another and to the context in which they actually take place. Transcripts, instead, are able to crystallize interactive processes: in this way the single expressions used can be analyzed line by line, related to one another, and checked globally. The latter type of analysis often makes it possible to highlight gaps, crucial decision making moments and also omissions, oversights, carelessness. As a consequence, they enable teachers to review their own and others' behavior, acknowledge possible mistakes and conjecture alternative moves.

Another follow up of the joint work with teachers in the context of ArAl project concerns the design of tools for initial teacher training. These are actually a number of "teaching simulations," a type of individual laboratory-based activity we purposefully designed and centered around the critical analysis of and reflection on scenes composing a given classroom process, adapted for e-learning as well (see Malara & Navarra, 2007).

A further, completely different, and important ground for reflection is for us the incidence of the network of socio-emotional relationships within the classroom (leadership, power groups, median roles, individuals) in the development of discussions. In many cases, we observed rivalries between different gender groups, complicities between individuals, or even a refusal by some pupils to get involved. These observations have brought us to consider pupils' participation as an important research issue. We currently promote the study of this issue among teachers-researchers.

CONCLUSIONS

Let us get back to the complex interplay between theory and practice. Among the various ways of looking at the relationship between theory and practice, all of them relevant within ME, in this chapter we have chosen to analyze it in the most traditional way, i.e. by considering researchers' theory and practitioners' practice, even if, as observed by Brown and Cooney (1991, p.112):

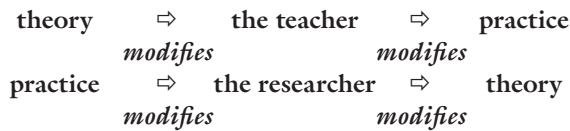
intelligent reflection on the actual and potential relationships between researchers and practitioners may be better achieved by locating the place of both theory and practice in each of these communities rather than by dichotomizing them.

Of course, the analysis of this relationship is strongly influenced by the model of practice that one chooses. The one we have suggested is *teaching as decision making*. Not only does

this model influence the theory-practice relationship, it even has consequences on theory itself, on influencing its standards of quality such as reproducibility and relevance. In particular, teacher education seen as an action on knowledge, but also on beliefs, emotions and awareness, becomes an important aspect of certain kinds of research, in order to inform practice.

The model we have chosen suggests that research influences teaching, if it does influence teachers' decisions. Teachers must educate their awareness in order to broaden their sensitivity to the possibility of making decisions moment by moment. Theory has a crucial role in this process, since it changes teachers' knowledge, and therefore beliefs and emotions: so, on one hand, research results must be communicated in a way that even nonresearchers can appreciate, on the other hand, teachers must have constructed a positive attitude towards theory.

We have characterized teachers' practice as decision making. But research activity, too, implies making continuous decisions about aims and users, and as a consequence about the choice of research problems, theoretical frame of reference, methodology and modes of communication. On the other hand, the interaction between the researcher and the teacher influences not only the choice of the research problems, but also the strategies to tackle them. So, if the contact with theory (slowly) changes teachers' decision processes, and therefore their practice, the contact with practice (slowly) changes researchers' decision processes, and therefore theory.



The two processes that we have considered separately, starting either from practice or from theory, related to the changes of teachers and researchers, have to be seen as connected components of a same "object," as in a Möbius strip.

For researchers, a change also happens through a renewal of knowledge, beliefs, awareness, and emotions, because these factors influence their decision processes. Emotional aspects, in particular, often neglected by research as if they only polluted thinking processes, play a crucial role in choosing what to observe, in defining goals, in directing actions. The process of knowledge is always extremely personal: "In every act of knowledge there is a passionate contribution of the person who knows what gets known, and (...) this component is not an imperfection but a vital factor of knowledge." (Polanyi, 1958, p. 8)

NOTES

1. As to the theory and practice relationship, Schoenfeld applies to educational research the perspective elaborated by Stokes (1997) for describing the tension between theory and applications in science and technology. In this perspective, basic research and utility are separate dimensions of research. The various combinations of these two dimensions are represented through a Carroll square. The Pasteur Quadrant concerns "use-inspired basic research."
2. Ortony, Clore, and Collins (1988) distinguish three main types of emotions, which they classify as reactions to:
 - Objects: the emotions resulting from reactions to objects "qua" objects ("attraction" emotions) are all variations of the affective reactions of liking and disliking. They are influenced by subjects' attitudes and tastes (typical examples are love and hate).
 - Events: this is the class of affective reactions of being pleased and displeased. These affective reactions arise when a person construes the consequences of an event as being desirable or undesirable, and are influenced by the subject's goals (typical emotions are joy, hope, fear).
 - Agents: affective reactions of approving and disapproving. They are influenced by the subject's beliefs and values (typical emotions are pride, shame, admiration, reproach).

From these three classes derive more complex emotions like anger, in which the reaction to an unpleasant event is linked to a factor considered to be responsible for this event. In this sense, anger is more complex than other emotions (e.g., disgust), since the interpretation process that originates it is more complex.

3. Other disciplines, too, state the need for new approaches. In particular, in the science of education Cohen and Manion (1994, p. 106) refer to two different and complementary perspectives in research on education:

The first, based on the scientific paradigm, rests upon the creation of theoretical frameworks that can be tested by experimentation, replication and refinement. (...) Against this scientific, experimental paradigm, we posit an alternative perspective which we describe as interpretative and subjective, a focus we hasten to add that should be seen as complementing rather than competing with the experimental stance.

4. See the symposium on *Criteria for Scientific Quality and Relevance in the Didactics of Mathematics*, held in Gilleleje, Denmark, in 1992; the ICMI study conference *What is Research in Mathematics Education and What Are Its Results?*, held at the University of Maryland in 1994, and, more recently, at ICME 10 the survey team *Relations Between Research and Practice in Mathematics Education* (Sfard, 2004).
5. A difference concerns the view of the teacher as researcher. Jaworski, like other scholars, for instance Breen (quoted by Peter-Koop, 2001), claims that a teacher cannot reach the quality of researcher. The Italian academic environment is centered on the construction of this double-faceted figure, and today various teachers can be considered as full researchers both for the quality and autonomy of their research and for the international acknowledgement they have received (see, for instance, the co-leaderships offered at the 10th International Congress on Mathematics Education (Copenhagen 2004) to the teachers-researchers G. Navarra and D. Paola, respectively, for DG 18 “Current Problems and Challenges in Primary Mathematics Education” and for TG 28 “New Trends in Mathematics Education as a Discipline”).
6. Transcribing the sessions is hard work for teachers, but they are strongly motivated by: the sense of belonging to the ArAl group, the recognition of their work by both the school (which sometimes gives them some financial support) and by the Ministry of Education and related institutions (which promote and finance this kind of projects), and, last but not least, by the fact they live in small towns in Northern Italy where life-long professional learning and, more generally, cultural growth, are socially shared values.

REFERENCES

- Arcavi, A. (2000). Problem-driven research in mathematics education. *Journal of Mathematical Behavior*, 19(2), 141–173.
- Arsac, G. (1989). Les recherches actuelles sur l'apprentissage de la démonstration et les phénomènes de validation en France. *Recherches en Didactique des Mathématiques*, 9(3), 247–280.
- Arsac, G., Balacheff, N., & Mante, M. (1992). Teacher's role and reproducibility of didactical situations. *Educational Studies in Mathematics*, 23, 5–29.
- Artigue, M., & Perrin-Glorian, M. J. (1991). Didactic engineering, research and development tool: Some theoretical problems linked to this duality. *For the Learning of Mathematics*, 11(1), 13–18.
- Arzarello, F. (1999). Linee di tendenza della ricerca per l'innovazione in Italia: quadro di riferimento teorico. In J. P. Da Ponte & L. Serrasina (Eds.), *Educação Matemática em Portugal, Espanha e Itália* (Actas da Escola de Verão 1999 (pp. 197–224)). Lisboa, Portugal: Sociedade Portuguesa de Ciências de Educação & Secção de Educação Matemática.
- Arzarello, F., & Bartolini Bussi, M. (1998). Italian trends of research in mathematics education: A national case study in the international perspective. In J. Kilpatrick & A. Sierpiska (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 243–262). Dordrecht, The Netherlands: Kluwer.
- Balacheff, N. (1990). Towards a problématique for research on mathematics teaching. *Journal for Research in Mathematics Education*, 21(4), 258–272.
- Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (pp. 433–456). New York: Macmillan.
- Bartolini Bussi, M. G. (1994). Theoretical and empirical approaches to classroom interaction. In R. Biehler, R. W. Scholz, R. Strässer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 121–132). Dordrecht, The Netherlands: Kluwer.

- Bartolini Bussi, M.G. (1998). Joint activity in mathematics classroom: A Vygotskian analysis. In F. Seeger, J. Voigt, & U. Waschescio (Eds.), *The culture of the mathematics classroom* (pp. 13–49). Cambridge: Cambridge University Press.
- Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner & A. Vermandel (Eds.), *Foundations and methodology of the discipline mathematics education* (pp. 174–188). Bielefeld-Antwerp.
- Bishop, A. J. (1998). Research and practioners. In J. Kilpatrick & A. Sierpiska (Eds.), *Mathematics education as a research domain: A search for identity* (pp 33–45). Dordrecht, The Netherlands: Kluwer.
- Boero, P., & Szendrei, J. (1998). Research and results in mathematics education: Some contradictory aspects. In J. Kilpatrick & A. Sierpiska (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 197–212). Dordrecht, The Netherlands: Kluwer.
- Borasi, R., Fonzi, J., Smith, C., & Rose, B. (1999). Beginning the process of rethinking mathematics instruction: A professional development program. *Journal of Mathematics Teacher Education*, 2(1), 49–78.
- Borkowski, J. G. (1992). Metacognitive theory: A framework for teaching literacy, writing, and math skills. *Journal of Learning Disabilities*, 25(4), 253–257.
- Brown, C. A., & Borko, H. (1992). Becoming a mathematics teacher. In D. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 209–239). New York: Macmillan.
- Brown, L., & Coles, A. (2000). Complex decision-making in the classroom: The teacher as an intuitive practitioner. In T. Atkinson & G. Claxton (Eds.), *The intuitive practitioner: On the value of not always knowing what one is doing* (pp. 165–181). Buckingham, UK: The Open University Press.
- Brown, S., & Cooney, T. (1991). Stalking the dualism between theory and practice. *Zentralblatt für Didaktik der Mathematik*, 4, 112–117.
- Burton, L. (1994). Research, theory and practice: A triad. In L. Bazzini (Ed.), *Theory and practice in mathematics education* (pp. 57–66). Pavia, Italy: ISDAF.
- Carpenter, T. P. (1988). Teaching as problem solving. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem olving* (pp. 187–202). Hillsdale, NJ: Erlbaum.
- Carpenter, T. P., & Moser, J. M. (1983). The acquisition of addition and subtraction concepts. In R. Lesh & M. Landau (Eds.), *The acquisition of mathematical concepts and processes* (pp. 7–14). New York: Academic Press.
- Carpenter, T. P., Fennema, E., Peterson, P., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education*, 19(5), 385–401.
- Carpenter, T. P., Fennema, E., Peterson, P., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, 26(4), 499–532.
- Chapman, O. (1997). Metaphors in the teaching of mathematical problem solving. *Educational Studies in Mathematics*, 32(3), 201–228.
- Chapman, O. (2002). Belief structure and inservice high school mathematics teacher growth. In G. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 177–193). Dordrecht, The Netherlands: Kluwer.
- Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.), *Third handbook of research on teaching* (pp. 255–296). New York: Macmillan.
- Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist*, 23(2), 87–103.
- Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics*, 23, 99–122.
- Cohen, L., & Manion, L. (1994). *Research methods in education*. London: Routledge.
- Cooney, T. J. (1988). Teachers' decision making. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 273–286). London: Hodder and Stoughton.
- Cooney, T. J. (1994). Research and teacher education: In search of common ground. *Journal for Research in Mathematics Education*, 25(6), 608–636.
- Cooney, T. J., & Krainer, K. (1996). Inservice mathematics teacher education: The importance of listening. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education* (pp. 1155–1185). Dordrecht, The Netherlands: Kluwer.
- Crawford, K., & Adler, J. (1996). Teachers as researchers in mathematics education. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education* (pp. 1187–1205). Dordrecht, The Netherlands: Kluwer.
- Davis, R. B. (1985) ICME-5 report: Algebraic thinking in early grades. *Journal of Mathematical Behaviour*, 4, 195–208.
- Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics*, 61(3), 293–319.

- De Bellis, V., & Goldin, G. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics*, Special Issue, 131–147.
- Dewey, J. (1902). *The child and the curriculum*. Chicago: University of Chicago Press.
- Doerr, H. M., & Lesh, R. (2003). A modelling perspective on teacher development. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism* (pp. 125–173). Mahwah, NJ: Erlbaum.
- Dörfler, W. (1993). Quality criteria for journals in the field of didactics of mathematics. In G. Nissen & M. Bomhøj (Eds.), *Criteria for scientific quality and relevance in the didactics of mathematics* (pp. 75–87). Roskilde, Denmark: Danish Research Council for the Humanities.
- Eisenberg, T. A. (1977). Begle revisited: Teacher knowledge and student achievement in algebra. *Journal for Research in Mathematics Education*, 8, 216–222.
- Eisenhart, M. A. (1988). The ethnographic research tradition and mathematics education research. *Journal for Research in Mathematics Education*, 16, 99–114.
- Elbaz, F. (1983). *Teacher thinking: A study of practical knowledge*. New York: Nichols.
- Evans, J. (2000). *Adults' mathematical thinking and emotions: A study of numerate practices*. London: Routledge Falmer.
- Evans J., Hannula M., Zan R., & Brown L. (Eds.) (2006). *Affect in mathematics education—exploring theoretical frameworks*. Educational Studies in Mathematics, Special Issue.
- Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. Grows (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 147–164). New York: Macmillan.
- Fennema, E., Peterson, P., Carpenter, T., & Lubinski, C. (1990). Teachers' attributions and Beliefs about Girls, Boys, and Mathematics. *Educational Studies in Mathematics*, 21, 55–69.
- Freudenthal, H. (1983). Major problems of mathematics education. In M. Zweng, T. Green, J. Kilpatrick, H. Pollak, & M. Suydam (Eds.), *Proceedings of the 4th International Congress on Mathematics Education* (pp. 1–7). Boston: Birkhäuser.
- Furinghetti, F. (1997). On teachers' conceptions: From a theoretical framework to school practice. In G. A. Makrides (Ed.), *Proceedings of the First Mediterranean Conference Mathematics Education and Applications* (pp. 277–287). Nicosia, Cyprus: Cyprus Pedagogical Institute & Cyprus Mathematical Society.
- Garofalo, J, Kroll, D., & Lester, F. K. (1987). Metacognition and mathematical problem solving: Preliminary research findings. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 222–228). Montreal, Canada: University of Montreal.
- Hanna, G. (1998). Evaluating research papers in mathematics education. In A. Sierpiska & J. Kilpatrick (Eds.), *Mathematics Education as a Research Domain: A Search for Identity* (pp. 399–407). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Höfer, M. (1981). Schülergruppierungen im Urteil und Verhalten des Lehrers. In M. Höfer (Ed.), *Informationsverarbeitung und Entscheidungsverhalten von Lehrern* (pp. 192–221). München: Urban und Schwarzenberg.
- Howson, G. (1992). Teachers of mathematics. In C. Gaulin, V. R. Hodgson, D. H. Wheeler, & J. C. Eggsgard (Eds.), *Proceedings of the 7th International Congress on Mathematics Education* (pp. 9–25). Laval, Quebec: Les presses de l'Université Laval.
- Hoyles, C. (1992). Illuminations and reflections — teachers, methodologies and mathematics. In W. Greeslin & K. Graham (Eds.), *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 263–286). Durham, NH: University of New Hampshire.
- Jackson, P. W. (1968). *Life in classrooms*. New York: Holt, Rinehart, & Winston.
- Jaworski, B. (1994). *Investigating mathematics teaching. A constructivist enquiry*. London: The Falmer Press.
- Jaworski, B. (1998). Mathematics teacher research: Process, practice and the development of teaching. *Journal of Mathematics Teacher Education*, 1, 3–31.
- Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics*, 54, 249–282.
- Jaworski, B. (2004). Grappling with complexity: Co-learning in inquiry communities in mathematics teaching development. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, 17–36). Bergen, Norway: Bergen University College.
- Jaworski, B., & Gates, P. (1987). Use of classroom video for teachers in-service education. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 93–99). Montreal, Canada: University of Montreal.

- Kieran, K. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.
- Kilpatrick, J. (1981). The reasonable ineffectiveness of research in mathematics education. *For the Learning of Mathematics*, 2(2), 22–29.
- Kilpatrick, J. (1993). Beyond face value: Assessing research in mathematics education. In G. Nissen, & M. Blomhøj (Eds.), *Criteria for scientific quality and relevance in the didactics of mathematics* (pp. 15–34). Roskilde, Denmark: Danish Research Council for the Humanities.
- Kuntze, S., & Reiss, K. (2005). Situation-specific and generalized components of professional knowledge of mathematics teachers. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 225–232). Melbourne, Australia: University of Melbourne.
- Kuntze, S., & Reiss, K. (2006). Evaluational research on a video-based in service mathematics teacher training project — reported instructional quality. In J. Novotná, H. Moravová, M. Krátká, & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 1–8). Prague, Czech Republic: Charles University & Faculty of Education.
- Leder, G., Pehkonen, E., & Törner, G. (Eds.) (2002). *Beliefs: A hidden variable in mathematics education?* Dordrecht, The Netherlands: Kluwer.
- Leinhardt, G., & Greeno, J.C. (1986). The cognitive skill of teaching. *Journal of Educational Psychology*, 2, 75–95.
- Leinhardt, G., Putnam, R. T., Stein, M. K., & Baxter, J. (1991). Where subject knowledge matters. In E. Brophy (Ed.), *Advances in research on teaching: Teachers' subject matter knowledge and classroom instruction* (Vol. 2, pp. 87–113). Greenwich, CT: JAI Press.
- Lerman, S. (1990). The role of research in the practice of mathematics education. *For the Learning of Mathematics*, 10(2), 25–28.
- Lerman, S. (1992). The function of language in radical constructivism: A Vygotskian perspective. In W. Geesling & K. Graham (Eds.), *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 40–47). Durham NH: University of New Hampshire.
- Lerman, S. (1994). Towards a unified space of theory-and-practice in mathematics teaching: A research perspective. In L. Bazzini (Ed.), *Theory and practice in mathematics education* (pp.133–142). Pavia, Italy: ISDAF.
- Lester, F. K. (1998). In pursuit of practical wisdom in mathematics education research. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 199–206). Stellenbosch, South Africa: University of Stellenbosch.
- Lester, F., & Lambdin, D. (1998). The ship of Theseus and other metaphors for thinking about what we value in mathematics education research. In J. Kilpatrick, & A. Sierpiska (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 415–425). Dordrecht, The Netherlands: Kluwer.
- Lester, F., & Wiliam, D. (2002). On the purpose of mathematics education research: Making productive contributions to policy and practice. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 489–506). Mahwah, NJ: Erlbaum.
- Linchevski, L. (1995). Algebra with numbers and arithmetic with letters: A definition of pre-algebra. *Journal of Mathematical Behaviour*, 14, 113–120.
- Malara, N. A. (2003). Dialectics between theory and practice: theoretical issues and aspects of practice from an early algebra project. In N. A. Pateman, B. J. Doughert, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol.1, pp. 33–48). Honolulu, HI: Center for Research and Development Group, University of Hawaii.
- Malara, N. A. (2005). Leading in-service teachers to approach early algebra. In L. Santos, A. P. Canavarro, & J. Brocardo (Eds.), *Mathematics education: Paths and crossroads (in honour of Paulo Abrantes)* (pp. 285–304). Lisbon, Portugal: Etigrafe.
- Malara, N. A., & Iaderosa, R. (1999). Theory and practice: A case of fruitful relationship for the renewal of the teaching and learning of algebra. In F. Jacquet (Ed.), *Relationship between classroom practice and research in mathematics education* (pp. 38–54). Neuchâtel, Switzerland: University Press.
- Malara, N. A., Incerti, V., Fiorini, R., & Nasi, R. (2004). *Percorsi di insegnamento in chiave pre-algebraica: Rappresentazione di problemi e di processi, segni simboli e negoziazione dei loro significati*. Bologna, Italy: Pitagora.
- Malara, N. A., & Navarra G. (2001). “Brioshi” and other mediation tools employed in a teaching of arithmetic with the aim of approaching algebra as a language. In E. Chick, K. Stacey, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study 'The Future of the Teaching and Learning of Algebra'* (Vol. 2, pp. 412–419). Melbourne, Australia: University of Melbourne.

- Malara, N. A., & Navarra, G. (2003). *ArAl Project: Arithmetic pathways towards favouring pre-algebraic thinking*. Bologna, Italy: Pitagora.
- Malara, N. A., & Navarra, G. (2007). A task aimed at leading teachers to promoting a constructive early algebra approach. *Proceedings of the 5th Conference of the European Society for Research in Mathematics Education*. Larnaka, Cyprus: ERME & University of Cyprus.
- Mandler, G. (1984). *Mind and body: Psychology of emotion and stress*. New York: Norton.
- Mandler, G. (1989). Affect and learning: Causes and consequences of emotional interactions. In D. McLeod & V. Adams (Eds.), *Affect and mathematical problem solving* (pp. 3–19). New York: Springer.
- Margolinas, C. (1998). Relations between the theoretical field and the practical field in mathematics education. In J. Kilpatrick & A. Sierpiska (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 351–357). Dordrecht, The Netherlands: Kluwer.
- Mason, J. (1990). Reflection on dialogue between theory and practice, reconciled by awareness. In F. Seeger & H. Steinbring (Eds.), *The dialogue between theory and practice in mathematics education: Overcoming the broadcast metaphor* (pp. 177–192). Bielefeld, Germany: Matherialen und Studien, Band 38, IDM.
- Mason, J. (1994a). Researching from inside in mathematics education — Locating an I-you relationship. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 176–194). Lisbon, Portugal: University of Lisbon.
- Mason, J. (1994b). Enquiry in mathematics and in mathematics education. In Paul Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematics education* (pp. 190–200). London: The Falmer Press.
- Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, 1, 243–267.
- Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: The Falmer Press.
- Mason, J., & Waywood, A. (1996). The role of theory in mathematics education and research. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education* (pp. 1055–1089). Dordrecht, The Netherlands: Kluwer.
- McLeod, D. (1992). Research on affect in mathematics education: A reconceptualization. In D.A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York: Macmillan.
- McLeod, D., & Adams, V. (Eds.) (1989). *Affect and mathematical problem solving*. New York: Springer.
- Middleton, J. A., & Spanias, P. A. (1999). Motivation for achievement in mathematics: findings, generalizations, and criticism of the research. *Journal for Research in Mathematics Education*, 30(1), 65–88.
- Mousley, J. A., Breen, C., & Frederick, H. (1999). Exploring different ways of working with videotapes in research and inservice work. In O. Zaslavski (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol.1, 253). Haifa, Israel: Israel Institute of Technology.
- Nesbitt Vacc, N., & Bright, G.W. (1999). Elementary preservice teachers' changing beliefs and instructional use of children's mathematical thinking. *Journal for Research in Mathematics Education*, 30(1), 89–110.
- Ortony, A., Clore, G. L., & Collins, A. (1988). *The cognitive structure of emotions*. Cambridge: Cambridge University Press.
- Pehkonen, E., & Törner, G. (1996). Introduction to the theme: Mathematical beliefs. *Zentralblatt für Didaktik der Mathematik*, 28, 99–100.
- Pellerey, M. (1997). Didactics of mathematics as a science of practice. In N. A. Malara (Ed.), *An international view on didactics of mathematics as a scientific discipline* (pp. 137–142). *Proceedings of WG 25 'Didactics of Mathematics as a Scientific Discipline', 8th International Congress on Mathematics Education* (Sevilla, Spain, 1996). Modena, Italy: AGUM.
- Peter-Koop, A. (2001). From “teacher researchers” to “student teacher researchers”- diagnostically enriched didactics. In M. van den Heuvel- Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 72–79). Utrecht, The Netherlands: Freudenthal Institute & Utrecht University.
- Peterson, P. L. (1988). Teachers' and students' cognitional knowledge for classroom teaching and learning. *Educational Researcher*, 17(5), 5–14.
- Pirie, S., & Breen, C. (1998). Exploring different ways of working with videotape in research and in service work. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, 282). Stellenbosch, South Africa: University of Stellenbosch.

- Pirie, S., & Kieren, T. (1992). Creating Constructivist Environments and Constructing Creative Mathematics. *Educational Studies in Mathematics*, 23, 505–528.
- Polanyi, M. (1958). *Personal knowledge*. Chicago: The University of Chicago Press.
- Potari, D., & Jaworski, B. (2002). Tackling complexity in mathematics teaching development: Using the teaching triad as a tool for reflection analysis. *Journal of Mathematics Teacher Education*, 5(4), 351–380.
- Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education*, 28, 550–576.
- Rosenthal, R., & Jacobson L. (1966). Teachers' expectancies: Determinants of pupils' IQ gains. *Psychological Reports*, 19, 115–118.
- Salzberger-Wittenberg, I., Williams Polacco, G., & Osborne, E. (1983). *The emotional experience of learning and teaching*. London: Routledge & Kegan Paul.
- Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Hillsdale, NJ: Erlbaum.
- Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education*, 20(4), 338–355.
- Schoenfeld, A. H. (1994). A discourse on methods. *Journal for Research in Mathematics Education*, 25(6), 697–710.
- Schoenfeld, A. H. (1998). Toward a theory of teaching in context. *Issues in Education*, 4 (1), 1–94.
- Schoenfeld, A. H. (1999). Models of the teaching process. *Journal of Mathematical Behavior*, 18(3), 243–261.
- Schoenfeld, A. H. (2002). Research methods in (mathematics) education. In L. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 435–487). Mahwah, NJ: Erlbaum.
- Schön, D. A. (1983). *The reflective practitioner: How professionals think in action*. New York: Basic Books.
- Schön, D. A. (1987). *Educating the reflective practitioner: Toward a new design for teaching and learning in the professions*. San Francisco: Jossey-Bass.
- School Mathematics Study Group (1972). Correlates of mathematics achievement: teacher background and opinion variables. In J. W. Wilson & E. A. Begle (Eds.), *NLSMA Reports* (23, Part a). Palo Alto, CA: Author.
- Sfard, A. (2005). What can be more practical than a good research? On mutual relations between research and practice of mathematics education. *Educational Studies in Mathematics*, 58(3), 393–413.
- Shavelson, R. J., & Stern, P. (1981). Research on teachers' pedagogical thoughts, judgments, decisions, and behavior. *Review of Educational Research*, 51, 455–498.
- Sherin, M.G. (2004). New perspective on the role of video in teacher education. *Advances in Research on Teaching*, 10, 1–28.
- Shulman, L. S. (1985). On teaching problem solving and solving the problems of teaching. In E. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 439–450). Hillsdale, NJ: Erlbaum.
- Shulman, L. S. (1986a). Paradigms and Research Programs in the Study of Teaching: a Contemporary Perspective. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 3–36). London: Macmillan.
- Shulman, L. S. (1986b). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15, 4–14.
- Shulman, L. S., & Elstein, A. S. (1975). Studies of problem solving judgment, and decision-making. In F. N. Kerlinger (Ed.), *Review of Research in Education* (Vol.3, pp. 3–42). Itasca, IL: F. E. Peacock.
- Sierpinska, A. (1993). Criteria for scientific quality and relevance in the didactics of Mathematics. In G. Nissen & M. Blomhøj (Eds), *Criteria for scientific quality and relevance in the didactics of mathematics* (pp. 35–74). Roskilde, Denmark: Danish Research Council for the Humanities.
- Sierpinska, A., & Kilpatrick, J. (Eds.). (1998). *Mathematics education as a research domain: A search for identity*. Dordrecht, The Netherlands: Kluwer.
- Silver, E. (1997). A disputation on a disciplinary dilemma: The tension between theory and practice in the didactics of mathematics. In N. A. Malara (Ed.), *An international view on didactics of mathematics as a scientific discipline* (pp. 143–149). *Proceedings of WG 25 'Didactics of Mathematics as a Scientific Discipline', 8th International Congress on Mathematics Education* (Sevilla, Spain, 1996). Modena: AGUM.
- Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics*, 26(2), 114–145.
- Simon, M.A. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema, B. Scott Nelson (Eds.), *Mathematics teachers in transition* (pp. 55–86). Mahwah, NJ: Erlbaum.

- Simon, M. A., & Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics*, 22, 309–331.
- Speranza, F. (1997). Didactics of mathematics as “design science”: An epistemological approach. In N. A. Malara (Ed.), *An international view on didactics of mathematics as a scientific discipline* (pp. 150–154). *Proceedings of WG 25 ‘Didactics of Mathematics as a Scientific Discipline’, 8th International Congress on Mathematics Education* (Sevilla, Spain, 1996). Modena, Italy: AGUM.
- Sriraman, B., & English, L. (2005). Theories of mathematics education: a global survey of theoretical frameworks/trends in mathematics education research. *Zentralblatt für Didaktik der Mathematik*, 37(6), 450–456.
- Steffe, L., & Kieren, T. (1994). Radical constructivism and mathematics education. *Journal for Research in Mathematics Education*, 25(6), 711–733.
- Steiner, H. G. (1985). Theory of Mathematics Education (TME): An Introduction. *For the Learning of Mathematics*, 5(2), 11–17.
- Stokes, D. E. (1997). *Pasteur’s quadrant: Basic science and technical innovation*. Washington, DC: Brookings Institute.
- Thompson, A. G. (1984). The relationship of teachers’ conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, 15, 105–127.
- Thompson, A.G. (1992). Teachers’ beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics learning and teaching* (pp. 127–145). New York: Macmillan.
- Törner, G., Sriraman, B., Sherin, M.G., Heinze, A., & Jablonka, E. (2005). Video-based research on mathematics teaching and learning: Research in the context of video. In G. M. Lloyd, M. R. Wilson, J. L. M. Wilkins, & S. L. Behm (Eds.), *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* [CD-ROM]. Eugene, OR: All Academic.
- Vergnaud, G. (1998). Towards a cognitive theory of practice. In J. Kilpatrick & A. Sierpiska (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227–240). Dordrecht, The Netherlands: Kluwer.
- Vinner, S. (1997). From intuition to inhibition — mathematics, education and other endangered species. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 63–78). Lathi, Finland: University of Helsinki & Lathi Research and Training Centre.
- Vinner, S. (2000). Mathematics education — procedures, rituals and man’s search for meaning. *Regular lecture at the 9th International Congress on Mathematics Education*. Makuhari, Japan.
- Wittmann, E.C. (1995). Mathematics education as a design science. *Educational Studies in Mathematics*, 29, 355–374.
- Wittmann, E.C. (2001). Developing mathematics education in a systemic process. *Educational Studies in Mathematics*, 48, 1–20.
- Wood, T. (1999). Approaching teacher development: Practice into theory. In B. Jaworski et al. (Eds.), *Mathematics teacher education: Critical international perspectives* (pp. 163–179). London: The Falmer Press.
- Zan, R. (1999a). Students’ and teachers’ theories of Success in mathematics. In G. Philippou (Ed.), *Proceedings of the MAVI - 8 Workshop* (pp. 123–132). Larnaka, Cyprus: University of Cyprus.
- Zan, R. (1999b). La qualità della ricerca. In J. P. Da Ponte & L. Serrasina (Eds.), *Educação Matemática em Portugal, Espanha e Itália*. Actas da Escola de Verão 1999 (pp. 281–292). Lisboa, Portugal: Sociedade Portuguesa de Ciências de Educação & Secção de Educação Matemática.