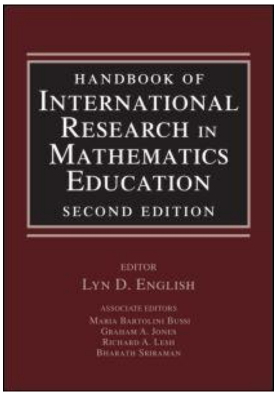

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Lyn D. English, Maria Bartolini Bussi, Graham A. Jones, Richard A. Lesh, Bharath Sriraman, Dina Tirosh

Democratic access to powerful mathematical ideas

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Ole Skovsmose, Paola Valero
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17 Democratic access to powerful mathematical ideas

Ole Skovsmose and Paola Valero
Aalborg University, Denmark

Carlos and his family have to move out of his home. His mother lost her job, and the money she made through great effort to pay for their small house is in the hands of the bank. Carlos, a 10th-grade student, is one of the many Colombian youngsters who will finish high school at the beginning of the 21st century. Many of these students seem to be confused about their future. Teachers insist on the importance of schooling and learning, especially in mathematics. Yet how could that help in the real-life circumstances of children like Carlos? On the other side of the world, in Denmark, Nicolai became seriously sick after eating a homemade ice cream. Contracting salmonella via eggs, chicken, or meat products is a relatively common occurrence in Denmark. People blame quality control, but do they, in fact, know what quality control is about? Does the education that Nicolai gains in school help him understand the dangers of his apparently "safe" society?

These are cases of real students in two different countries, and, as we argue, their life experiences are significant to mathematics education. In the task of advancing the field of research on the phenomena connected to the learning and teaching of mathematics, we start with a consideration of the global informational society in a complex social, political, cultural, and economic context. Within this context, both world and local trends intermesh, and new challenges to mathematics education practices and research emerge. Based on the contradictions of this current social order, we propose the paradox of inclusion and the paradox of citizenship as two central problems that mathematics education must face. With this purpose in mind, we proceed to give meaning to the term powerful mathematical ideas in four ways. We then discuss the notion of democratic access and question the simple identification of democracy with universal access. Finally, we argue that facing the paradoxes of inclusion and citizenship represents a struggle for the provision of "democratic access to powerful mathematical ideas" in mathematics education, both in practice and in research.

PARADOXES OF THE INFORMATIONAL SOCIETY

After the breakdown of the wall between East and West Berlin, Fukuyama (1989, 1992) declared "the end of history." This statement resonates with what theories of postindustrial society had been claiming since the 1970s, namely, that the world has reached a state in which the sources of value—and therefore of power—can be described not only in terms of labor and capital, but also and primarily in terms of knowledge and information. This state in the transformation of capitalism has also been called the information society (Bell, 1980). Together with the consideration of value and power, there has been a change in the kind of citizens that this new type of social order requires. People need to be able to deal with knowledge and information in continuous processes of learning. This particular shift is what has been called the "learning society" (Ranson, 1998). In what we refer to as the "informational society" (following Castells, 1999), subsuming both the information society and the learning society, the impact of technology goes beyond industrial production, and, in fact, affects political, economic, social, and cultural structures.

A discussion of the informational society cannot be separated from a consideration of globalization as the process responsible for establishing the "world village." Globalization refers to the fact that events in one part of the world may be caused by, and at the same time influence, events in others parts. Our environment—in political, sociological, economic, or ecological terms—is continuously reconstructed in a process that receives inputs from all corners of the world. Simultaneously, our actions have implications for even the most remote corners of the planet. However, globalization also relates to the apparently shared belief that a given kind of environment is desirable and that there is some kind of universal commitment to the achievement of certain ideals like democracy, market freedom, and individual competitiveness. The myth of the "end of history" can be interpreted as the legitimization of a false universalism (Eagleton, 1996). Together with the discourse of globalization comes a new discourse of colonization. In a similar way that the first European waves of colonization, from the 14th to the 18th centuries, brought new languages, religions, and social orders that trampled down indigenous cultures, the new global colonization also imposes new ways of living, producing, and thinking. D'Ambrosio (1996) saw science, including mathematics, as also playing a role in this cultural invasion; and, of course, mathematics education is not an innocent onlooker of the situation.[1]

Castells (1999) criticized some of the dominant descriptions of the postindustrial society based exclusively on the North American–European context. He emphasizes that a theory of the informational society should refer not only to the fact that certain countries and certain regions are becoming closely interrelated, but also to the fact that people, countries, and regions are excluded, apparently for not being of any relevance to the construction of the informational economy. Because access to knowledge is clearly important in the informational society, "the ability to generate new knowledge and to gather strategic information depends on access to the flows of such knowledge and information.… It follows that the power of organizations and the future of individuals depend on their positioning vis-à-vis such sources of knowledge and on their capacity to understand and process such knowledge" (p. 60). The access to the flow of knowledge and information constitutes a major division between those in the core of the informational society and those outside it. According to Castells, exclusion is devastating because "the structural logic of the information age bears the seeds of a new, fundamental barbarism" (p. 60). All the outsiders belong to structurally irrelevant areas in the informational society and constitute what Castells called the "Fourth World."

This observation draws attention to the complex dynamics of globalization. At the same time that we are becoming similar, we are also moving apart. The interplay between the global and the local is a game that connects many parts of the world in a network of flows, and simultaneously excludes regions and people from specific communities and countries in the world. The Fourth World includes not only large regions of Africa, Latin America, and Asia, but certainly also carves out large chunks of Europe, the United States, Japan, and Australia. Many people who either live in poverty or who are isolated from the centers of informational and technological production and exchange in these countries (e.g., political refugees and illegal immigrants in the United States, elderly people in rural areas in Japan, aboriginal communities in Australia, and young drug-dependent and "punk" communities in Germany) are apparently superfluous in this world order.

Nevertheless, the Fourth World has some relevant roles to play for the informational economy. First of all, it supplies spaces for dumping ecological problems and other side effects of industrial production. It also provides a market area to be flooded and a cheap source to supply the material flow of goods needed in the informational economy. The globalization linked with the informational economy seems to continue a provocative exploitation of certain parts of the world as a given. A concern for equity seems not to be part of this type of globalization.

Globalization is also responsible for determining who count as functional people in the free-flowing, informational economy. This social order is characterized by a strong capacity for renewal and flexibility in individuals and social organizations, which manifests itself through an enterprise capacity. Individuals and groups become organized under the principle of continuous learning as a mechanism of adaptation to rapid and constant environmental changes. This idea has implications for dominant current educational conceptions. Learning is conceived as a continuous "learning to learn" to fulfill societal requirements. Notions such as "constructivist teaching and learning," "active students and teachers," "rich educational environments," "technology inclusive experiences," and, more recently, "accountable and efficient educational services," together with "satisfied parent and student clientele," dominate the learning society discourse (Apple, 2000; Masschelein, 2000).

Despite the apparent suitability of some of these "learning to learn" notions, this whole discourse should be carefully questioned. There is a risk of reducing learning to a mechanism of individual survival, which opposes a conception of learning as a human activity whereby unique beings search for meaning in an attempt to initiate events that contribute to securing a sustainable, durable, common world. In other words, the possibilities of education as a questioning of the self, a judgment of the meaning of life, a construction of a common world, and a criticism of the given order of things, are highly at stake (Masschelein, 2000). Furthermore, Flecha (1999, p. 67) noted that "the knowledge prioritized by the new forms of life is distributed unevenly among individuals, according to social group, gender, ethnic group, and age. At the same time, the knowledge possessed by marginalized groups is dismissed, even if it is richer and more complex than prioritized knowledge. More is therefore given to those who have more and less to those who have less, forming a closed circle of cultural inequality."[2]

In mathematics education, the lifelong "learning to learn" ideas have been taken as a desirable goal to be reached in the 21st century. Mathematicians in the 1960s took seriously the duty of setting a mathematical education on which could rest "the ever heavier burden of the scientific and technological superstructure" (Organisation for European Economic Co-operation (OEEC, 1961, p. 18)). Nowadays, a large portion of our mathematics education community is apparently committed to the edification of competent citizens of the emerging and rapidly changing informational society. As the National Council of Teachers of Mathematics (NCTM) Standards 2000 (NCTM, 2000, pp. 3–4) state, the capacity to understand and do mathematics is more relevant than ever because it allows one to "have significantly enhanced opportunities and options" for shaping one's future. This formulation implies that acquiring mathematical competencies is a condition for being able to adapt, and therefore, both survive and help sustain this type of social development. The need and desire for more mathematically able people, as expressed in the discourse of "more mathematics for all," may contribute to spreading a utilitarian value of mathematics education that in the long run serves as a tool for the survival of the smartest. The contradiction between the social expectations emerging from this kind of discourse and actual practices where mathematics is used as a social filter determining who has access to further success (Smith, 2000; Volmink, 1994; Zevenbergen, 2000b) in fact gets resolved in favor of those who pass the gates of mathematics. Therefore, without anticipating it, mathematics education may support the dangers of the learning society.

We find that the "informational society" is a contested concept.[3] It contains contradictions, and it can develop in different directions. We attempt to summarize this fact by formulating two paradoxes of particular importance for mathematics education. The paradox of inclusion refers to the fact that the current globalization model of social organization, which embraces universal access and inclusion as a stated principle, is also conducive to a deep exclusion of certain social sectors. The paradox of citizenship alludes to the fact that the learning society, claiming the need of relevant, meaningful education for current social challenges, at the same time reduces learning to a matter of necessity for adapting the individual to social demands. The paradox of citizenship concerns in particular the notion of Bildung,[4] which refers to the development of general competencies for citizenship, especially the capacity to act critically in society, and in this way have an impact on it. This paradox refers to the fact that, on the one hand, education seems ready to prepare for active citizenship, but, on the other hand, it seems to ensure adaptation of the individual to the given social order.

Although from our field of research and practice (mathematics education), we cannot solve the paradoxes, we find it necessary to face them. If not, mathematics education could act blindly in the further development of current society. We engage in the task of exploring the significance of these two paradoxes from the particular perspective of mathematics education by examining the notions of "powerful mathematical ideas" and "democratic access." We do so by referencing two examples.

TWO EXAMPLES

Terrible small numbers

Salmonella poisoning is an everyday danger in Denmark. In one way or another, Nicolai knew that an "innocent" homemade ice cream, prepared with infected eggs, could be enough to make him sick. Students in school hear and can read about salmonella infection. A newspaper article under the headline "We have to live with salmonella" reads as follows:

> Experts estimate that a steady number of more than 1,000 Danes will be sick with salmonella each year. The Minister of Food, Henrik Dam Kristensen (Social Democracy) says that we won't succeed in wiping it out. Danes have to live with the permanent risk of getting sick from salmonella via Danish meat and egg products.... This was one of the conclusions from the report given to the Minister by the Danish Zoonosis Center, the advisory institution in these matters. According to the Minister, the paper does not lead to any changes in the strategy against salmonella, but Danes must learn to live with the infection risk. We will still prepare tests and investigations so that we can come as close as we can to zero risk. However, that is not the same as ensuring that salmonella infected eggs, chicken, and pork will not pass the control. Today it is impossible to make people believe that a 100-percent secure control is in place. (Politiken, 2000, our translation)

If risks, as stated by Beck (1992, 1999), are an essential constituent of our current world, how could school and especially mathematics teaching and learning provide tools for analyzing those risks in a meaningful way? The project Terrible Small Numbers tries to address this question.[5] Together with their students, the teachers participating in the project collected 500 black film cases to simulate eggs; film cases were selected because they resemble eggs in size, they lack transparency, and they could be opened for examination. Inside each egg there was a yellow centicube, in some of them a blue centicube was placed. The blue "yolk" represented a salmonella-infected egg.

During the first sequence of activities, the mix of healthy and salmonella-infected eggs was made in front of the whole class. Everybody knew that out of the 500 eggs, 50 were infected. The students then had to take samples consisting of 10 eggs, and to count the number of infected eggs. Intuitively, the students expected to get one blue egg in each sample, but after some experiments they found that in some cases they could get 3 blue eggs—or even more—out of 10. How could that be? Was it because the mix was not done in a proper way? Was it bad luck? The basic question to be addressed by this experiment has to do with the reliability of information provided by samples. How can it be that a sample does not always tell the "truth" about the whole population? And how should we operate in a situation in which we do not know anything about the whole population, except from what a sample might tell? How can we, in this case, evaluate the reliability of numerical information?

In a second sequence of experiments, students were presented with two types of eggs, Spanish and Greek, to buy for retail sale in shops. In both types, there were some infected eggs, but this time the students did not know how many. To make a decision about which type of eggs to buy for retailing, they needed to run a quality-control test. It was impossible to test all eggs because eggs opened in the quality control could not be sold. Furthermore, it was expensive to check eggs for salmonella, so the students' (acting as the retailers) budget was affected by control costs. They had to consider carefully how many Spanish and Greek eggs needed to be sampled to make a decision about which type to buy. The concern for making a responsible decision was confronted with the interest of making a healthy business.

A third sequence of activities dealt with the evaluation of the risks of getting salmonella from food products with eggs in the ingredients. The starting point for the preparation of the products was a mixture of 500 eggs, 5 of which were infected. A preliminary question was to calculate the probability of finding a blue egg; it was not difficult to arrive at 5/500 = 0.01. Now, if we want to make an ice cream portion out of six eggs—and, of course, we would like them all to be healthy—the probability of getting a salmonella-free portion is (1 – 0.01) 6 and therefore the risk of infection is 1 – (1 – 0.01).[6] To get to this formula was not simple. The students began by suggesting that if the probability of getting an infected egg is 0.01, then, when picking 6 eggs, the probability must be 0.06. However, by finding the proper formula, the students had an opportunity to contrast mathematical calculations with empirical experimentation.

The project tried to provide ground for a discussion of the difference between ideal mathematical calculations and empirically obtained figures, as well as a debate about the possibility of calculating risks in general. The notion of risk can be summarized in mathematical terms by the equation

$$R(A) = P(A)C(A).$$

Here A represents an event. The risk, R(A), is the product of the probability that A happens, P(A), and the consequences of A happening, C(A). In other words, the risk of eating an ice cream dessert equals the probability of being infected times the "cost" of being infected, and naturally the "cost" increases with the size of the dessert because many more people may taste it.

Macro-figures becoming macro-dangers

A country in an unstable economic and political situation is a perfect scenario for witnessing the macro-dangers of macro-figures. Colombia, in the last decade of the 20th century, represented a deeply troubled society, in conflict between democratic consolidation and international globalization demands. In this scenario, where almost premodern, modern, and postmodern living conditions coexist, students struggle to find good reasons for finishing school—if, of course, they have a chance of doing so. Carlos certainly finds it difficult to see the role of so much studying in his future. It is even more difficult now that his family had to leave the house that his mother began paying for some years ago. Recently, the monthly mortgage payments became so high that she had to give up trying to make the payments. When she tried to sell the house, she could not recover a single cent of what she had invested, and her best solution was to give it back to the bank as part of the debt payment.

Carlos was not the only student who, between 1998 and 1999, lost his home. Such was the story that many people lived in Colombia. For the first time, people were concerned about what the UPAC (Unidad de Poder Adquisitivo Constante [Unit of Constant Buying Power]), introduced in 1971, could mean in their lives. Certainly a mathematical investigation in the classroom could be of help. In what follows, we imagine the general guidelines of a project, Macro-figures becoming Macro-dangers, with 10th- or 11th-grade students.[7] The project may allow students to reflect about the use of mathematics as a power resource through economic and social models.

Where do we start? We could ask students to ask their families and friends about the UPAC and its predicaments. We want the project to be of relevance for the students' actual situation. As one of the essential inquiry sources, we can collect receipts for mortgage payments during the last one or two years. We can ask for help from the social science teachers to get information about the UPAC system and the reasons why the government adopted it. Is it possible to discover the assumptions of the system? Are they still valid? The UPAC system, which was intended to promote private savings and housing acquisition, was designed under the assumption that, on the one hand, devaluation, inflation, and interest rates could be controlled by the government (Currie, 1984; Perry, 1989), and, on the other hand, that the country would have a steady economic growth.

In the case of mortgage payment, the UPAC system operates in the following way. To calculate nominal interest (n) on a mortgage, the system considers inflation (i), the effective interest rate (e), which is estimated at 6% annually, and a risk factor (r). The nominal interest, n, is then determined by the formula (Vélez, 1997):

$$n = (1 + i)(1 + e)(1 + r) - 1.$$

For instance, in normal conditions, if $i = 0.06$, $e = 0.06$, and $r = 0.01$, then $n = 0.13$, which would be a reasonable case. At a time of deep economic crisis, the nominal interest gets out of control due to variations in inflation, the effective interest rate and the risk factor, as actually happened in Colombia in the period between 1997 and 2000. For instance, in a crisis situation, if $i = 0.18$, $e = 0.20$, and $r = 0.10$, then $n = 0.55$, generating an aberrant situation. In the Colombian case at the end of the 1990s, when people could not afford to cover the payments (e.g., due to unemployment) and were forced to sell their property, they lost all their savings because the value of real estate decreased as part of the crisis itself.

After a first exploration, we could start looking at specific cases—that of Carlos' family if he and his family agree—and make groups based on students who have similar cases. The main purpose of the work could be to advise specific families in the process of negotiating new payment systems with the bank. Based on the payment receipts gathered, students can study the connections between the different figures in a given period of time—the total amount of the mortgage, the interest rate, the proportion of the debt that has actually been repaid, the payments for interest, and so forth. We could go as deep as needed into the mathematical exploration of the situation.

Then, we could enter into a discussion about the consequences of the model. We could prepare a report for the families, explaining what happened during the time they paid their mortgages and proposing suitable alternatives to deal with bank proposals about the renegotiation of their mortgage and the adoption of the new model proposed by the government.[7] As one of the aims of the project, we would like to grasp the potential that a mathematics class investigation could have for initiating changes in the students' lives. However, is what we all gain during the development of the project enough to act politically around the families in trouble?

Does this experiment illustrate essential aspects of what to consider in an inquiry in the mathematics classroom? Is it important to make this project a reality? Likewise, what can we say about "Terrible small numbers"? If mathematics education should face the paradoxes of inclusion and citizenship of the informational society, such questions become important. To discuss in more detail the possibilities of tackling the paradoxes, however, we must first explore what could be the meaning of "powerful mathematical ideas." We then discuss the different aspects of providing "democratic access" and then return to the paradoxes.

POWERFUL MATHEMATICAL IDEAS …

To say that something is powerful is tantamount to affirming that it can exercise power. If we state that mathematical ideas can exercise power, we should try to clarify the following questions: What do we mean by "power"? What is the source of the power of mathematical ideas? What are the consequences of that power? In what follows, we put forward different possible interpretations.

…LOGICALLY SPEAKING

Mathematical ideas can be seen as powerful from a logical point of view. In this sense, power refers to the characteristic of some key ideas that enable us to establish new links among theories and provide new meaning to previously defined concepts. In this sense, one can certainly assert that plenty of powerful mathematical ideas have emerged throughout the history of the discipline.

In particular, we can associate the notion of powerful mathematical ideas with abstraction. A concept may be interpreted as powerful to the extent that it provides new insight into a different set of concepts. The notion of group illustrates the logical power of making abstractions. A group can be defined as a set, M, consisting of certain elements, and an operation, *, which to any pair of elements from M associates an element from M, and which fulfills certain properties. Exemplars of groups are then recognized all over mathematics, a basic one being the set of integers together with the operation "addition." A wide range of other mathematical structures, besides group, are recognized, such as ring, vector space, metric space, topological space, all defined solely by their formal properties and not by any qualities of their elements. Such formal structures make it possible to bring an understanding obtained in one area of mathematics to apply in a seemingly completely different area. In this way, abstractions have led to a class of powerful mathematical ideas, logically speaking. The power or strength of those ideas, then, can be defined as an intrinsic and essential characteristic of their position in the hierarchy of mathematics, which allowed them to influence other ideas so as to reaccommodate and redefine them. Once a more abstract mathematical idea provides a new conceptualization for previously existing notions, the building of mathematics is restructured via the legitimacy of the new ruling and organizing principle. In this sense, powerful mathematical ideas, logically speaking, have an intrinsic power exercised within the realm of mathematics.

Such powerful mathematical ideas can be expressed in a logical architecture, as exemplified by the work of the Bourbaki group. A closer look at this strictly modernist edifice reveals also what "powerful," in a logical sense, could mean for mathematics education. If mathematics education is conceived as having the role of enculturating students into established mathematical knowledge and its ways of working, then it is easy—as the proponents of the New Math Movement in the 1960s thought—to generate a list of powerful mathematical ideas around which to organize the curriculum. By means of such logically basic ideas, all other ideas could be defined.

Although the particular approach and aims of the modern mathematics education wave have almost disappeared from school curricula, there is still a dominance in practice of the idea that mathematics curricula consist of a list of essential, powerful mathematical ideas and topics to be learned. The amazing similarity and stability in the structure of national mathematics curricula across the world (Kilpatrick, 1996) show the strength of the shared belief in the logical power of mathematical ideas. Independently from the orientation of the approach to school mathematics, such as "back-to-basics" in the United States and National Numeracy Strategy in the United Kingdom, which stress the traditional priorities of mathematical topics, or the NCTM *Standards*, which represent a more progressive curricular proposal, all these views try to grasp the essence of powerful mathematical ideas from this logical point of view. Much mathematics education simply assumes that mathematical ideas are powerful primarily in a logical sense. This justifies that mathematics teaching can concentrate on providing students access to "real" mathematics, either by following the school mathematics tradition or even by a progressive establishment of a scaffolding, which makes it possible for the students to construct mathematics by and for themselves.

From this perspective, what can we make of the projects Macro-Figures Becoming Macro-Dangers and Terrible Small Numbers? Could they lead to powerful mathematical ideas, logically speaking? Certainly we could imagine possible ways of strengthening a mathematical focus. In the case of Terrible Small Numbers, one could have gone deeper into the mathematical significance of expressions such as $1 - (1 - p)^x$ and into other probability notions and the connections among them. This could have brought the students into a whole exploration of probability theory. In the planning of the Macro-Figures Becoming Macro-Dangers project, one could start considering the equations:

$$n = (1 + i)(1 + e)(1 + r) - 1 = (i + e + r) + (ie + ir + er) + ier.$$

In particular, by making this algebraic reduction, it becomes clear that $n > i + e + r$. Furthermore, the project could provide a nice entrance to algebra, and once more it can be illustrated that abstraction is an essential element of powerful mathematical ideas. The calculations could also open a route directly into the exploration of exponential functions because the project makes it relevant to consider how a function like $f(t) = (1 + n)^t$, with t referring to time, operates.

The logically based interpretation of powerful mathematical ideas legitimates doing mathematics for the sake of the internal characteristics of mathematics. It supports the desirability of allowing students to experiment and play with ideas and ways of working that in themselves appear powerful. Nevertheless, this perspective embraces some risks. It could accentuate the paradox of inclusion because it will justify the provision of an abstract curriculum that, as much research has documented, systematically closes the possibility for the majority of students of participating in a meaningful mathematics education experience (Boaler, 1997). This perspective could also contribute to exacerbating the paradox of citizenship because mathematics education could end up offering knowledge that appears relevant for students to their further career opportunities, but for which the relevance beyond this is limited.

...PSYCHOLOGICALLY SPEAKING

We could also associate power with the individual's experience in learning mathematical ideas. In this sense, power is determined in relation to learning potentialities. From this perspective, what counts as significant ideas is what students can grasp and make meaning of in the process of developing mathematical thinking. In fact, the majority of research in mathematics education in the 1980s and 1990s is an important source for the identification of this kind of powerful ideas.

Influenced by the work of Piaget, and more recently of Vygotsky, on the development of human cognition, mathematics educators have formulated different theoretical frameworks to describe what the learning of mathematics is about.[8] These theories have also served the purpose of describing basic principles for what should be achieved through the mathematical schooling experience. Verschaffel and De Corte (1996) offered an example in the case of arithmetic. First, they stated the leading principles for arithmetic learning and teaching in school in terms of learning mathematics as a social and cooperative constructive activity, the role of meaningful contexts, and the progression toward higher levels of abstraction and formalization (pp. 102–103). Then, they formulated some major aspects that need to be given more attention in connection with, for example, the acquisition of number concept and number sense (pp. 105–111). These aspects include counting at the expense of logical operational skills in the early grades, allowing an awareness of multiple uses of numbers, promoting number sense and estimation, and going beyond whole numbers. In contrast to a logical interpretation of powerful mathematical ideas, such items do not emphasize the mathematical content involved in the learning process, but focus instead on the mental operations that go together with the acquisition of the mathematical notions. In the case of algebra, Kieran (1992) provided a list of similarly powerful mathematical ideas.

One important notion emphasized in the learning of algebra, and of more complex mathematics, is that of the duality between conceptions of mathematics as processes and as objects (Sfard, 1991), which in the French *didactique des mathématiques* version is formulated as the dialectic between mathematics as tools and as objects (Douady, 1987). This discussion, which has certainly influenced the understanding of mathematics learning and teaching,[9] combines a mathematical analysis about the nature of mathematical objects with an analysis of learning processes. In this way, the point of the power of mathematical ideas is connected to the degree to which they can be integrated into the students' understanding through processes of interiorization, condensation, and reification. The whole issue of understanding (Sierpinska, 1994) is, therefore, the key to defining the potential that mathematical ideas can have once located in the domain of human learning.

In addition, an emphasis on affective, motivational, and idiosyncratic aspects of both students' and teachers' understanding of mathematics—and of its learning and teaching—is also considered a central part of the generation of powerful mathematical ideas, psychologically speaking. The realization that meaningful mathematical ideas are only acquired—or constructed—if the individual has a favorable mental disposition to engage in the process of learning generated a complementary set of ideas such as the importance of students' and teachers' attitudes and beliefs toward mathematics and its teaching and learning. In this sense, some metamathematical thinking notions, such as competencies in problem solving, metacognition, and sense making (Schoenfeld, 1992), came to go hand in hand with mathematical ideas. This combination constitutes powerful clusters in a psychological sense.

For mathematics education all these principles implied the advance in ideas of reform along the lines of, for example, the NCTM *Standards* proposals, which represent a combination of powerful mathematical ideas in both a logical and psychological sense. To illustrate this combination, we can see how the description of the *Standards* (NCTM, 2000) plays with the identification and integration of mathematical topics (e.g., number and operations, algebra, data analysis, and probability), mathematically related activities (e.g., problem solving and communication), and competencies in those topics and activities (i.e., understand numbers, ways of representing numbers, relationships among numbers, and number systems, use mathematical models to represent and understand quantitative relationships, monitor and reflect on the process of mathematical problem solving, and communicate their mathematical thinking coherently and clearly to peers, teachers, and others).

Considering our two projects, following the psychological interpretation, we could discuss the role of the contextualization on which the projects are based. The projects bring into the classroom concrete situations that the students can use as a basis for understanding. In this sense, each project provides a frame for the students to become familiar with mathematical notions that intervene in the situation. Its main role is to bring students into mathematics as a facilitator and as a motivational device. In the case of Terrible Small Numbers, the students are familiar with the issue of salmonella. The proximity of the topic to their lives can provide the possibility of making connections between already internalized concepts and new ideas to come. The experimentation with the samples of eggs opens further links to which the ideas of probability and risk can be connected. In Macro-Figures Becoming Macro-Dangers, the extraction of basic data for the mathematical analysis from real sources can be viewed as an especially engaging activity, which can motivate students to learn the mathematical aspects behind the real cases. In particular, the students could observe a new significance of making algebraic reductions. They can reveal connections that are not so easy to identify if only numerical calculations are used.

In most cases, the psychological interpretation of powerful mathematical ideas rests on the assumption that human learning processes are universal, even though strong cultural and social differences may affect meaning construction. It also assumes that those ideas are therefore transferable into diverse situations and that, given this transferability, they constitute useful knowledge. We find this interpretation problematic in light of recent studies that have evidenced and developed a radically different view of knowledge and human cognition. First of all, recent studies have shown (Lerman, 2000) that the individual's social and cultural situatedness—in particular ethnic, social, or gender groups at a given historical moment—has an impact on her cognitive development. Secondly, it has been suggested that learning is not a mental process but participation in communities of practice (Lave, 1988; Lave & Wenger, 1991). From this perspective, there is no possible knowledge transfer but different types of participation and action in different contextualized situations (Boaler, 1997; Wedege, 1999). A view of mathematical ideas from a broader perspective is necessary.

...CULTURALLY SPEAKING

If students should experience the relevance and meaningfulness of their learning in relation to their sociocultural experience, it is necessary to consider what counts as powerful from the situated learners' perspective. We could then try to relate powerful mathematical ideas to the opportunities for students to participate in the practices of a smaller community or of the society at large. These possibilities have to do with the students' foreground, which refers to the way students interpret and conceptualize—explicitly or implicitly, consciously or unconsciously—their future life conditions given the social, cultural, economic, and political environment in which they live.[10] It also refers to the students' interpretations and conceptualizations of their possibilities to engage in meaningful action. Naturally, the foreground is modulated by the background of the students, that is, their "socially constructed network of relationships and meanings" (Skovsmose, 1994, p. 179) that belongs to their personal history, but the foreground provides resources and reasons for the students to get involved—or not—in their learning as acting persons. In other words, the foreground allows students to focus their intentions on the activities connected to learning. We see intentions as primarily constructed from the person's foreground. So, mathematical ideas can become powerful to students in as much as they provide opportunities to envision a desirable range of future possibilities.

Many studies have tried to identify what "powerful mathematical ideas" could mean from a cultural perspective. In this context, "cultural perspective" refers to radical and political interpretations, for instance, as described in Frankenstein (1995).[11] She tried to identify issues that specifically concern the political situation of working-class, urban adults involved in remedial mathematics programs and showed how questions about issues such as unemployment, military expenditure, taxation, and economic policy can be dealt with as a central part of mathematics education. Being able to handle such questions means developing a relevant competence for acting politically as critical citizens. In this way, powerful mathematical ideas become defined first of all with reference to the situation of the learner in a given sociocultural situation. This radical perspective is also present in many of the chapters in Powell and Frankenstein (1997).

Knijnik (1996, 1997) provided another example of culturally and politically powerful mathematical ideas in the case of the Brazilian landless movement. From an ethnomathematical approach, she, together with teachers from the community, found ways of bridging the gap between academic mathematics and people's popular mathematical knowledge as a way of enhancing possibilities of social change. The emergence of a "synthesis-knowledge" that rescues and values popular understandings but also raises awareness about its limitations, is one of the results of relevant pedagogical work in mathematics for the community.

Mukhopadhyay (1998) also presented an interpretation of mathematics education as a tool for adopting a critical stance toward current popular culture. She exemplified her point of view with a mathematical investigation in the classroom about Barbie dolls. This investigation, starting from making a model of Barbie of "normal" height, can promote the adoption of a critical attitude toward the stereotypes with which we are confronted and which have an influence on youth behavior, such as women wanting to have a body like Barbie but having serious eating disorders in an attempt to accomplish this goal. Generally speaking, mathematics education becomes powerful in a cultural sense when it supports people's empowerment in relation to their life conditions.

More recently, attention has been paid to the way in which the increasing multicultural composition of many mathematics classrooms in the world poses challenges to teachers and, in general, to educational systems about how to relate mathematics learning to students' lives and culture. Abreu, Bishop, and Presmeg (2002) present a collection of articles which addresses the issue of transition between different mathematical practices. This issue concerns the connection (or lack of connection) between practices of the mathematics classroom and every day practices including mathematics. It also concerns the transition between different cultural contexts including mathematics.

Both Terrible Small Numbers and Macro-Figures Becoming Macro-Dangers illustrate what it could mean to consider the political dimension of the students' culture. Danish students know about salmonella poisoning, and many Colombian students may have experienced the consequences of the disturbance in the logic of the UPAC-system. Therefore, we know that it is possible to relate the content of mathematics education to the students' background. Nevertheless, it might be easy to miss the relation with their foreground. How could our two projects touch students' learning intentions by touching their foreground? We imagine that for some Danish students experimenting with the meaning of quality control in food products could generate a learning intention related to their capacity for making decisions about types of aliments appropriate for consumption. For Colombian students, especially for those who actually lived through the negative consequences of the break down of the UPAC system, we can imagine at least two significant ways in which their foreground is touched. For some, talking about the issue itself can be so painful that a resistance to get engaged in the topic will dominate. In this case, learning intentions could emerge in opposition to the proposed learning environment.[12] On the other hand, for some students the project could generate learning intentions in relation to their capacity for helping their families make a critical decision about housing and real estate acquisition.

The issue of touching students' foreground is a delicate point in some of the ethnomathematical approaches. Some studies identify mathematical competencies built into the students' culture (for instance, competencies related to basket or fabric weaving and ornamentation in some Mozambican communities; Gerdes, 1996, 1997) as a starting point for mathematics education. However, there is no guarantee that, although belonging to the cultural background of a particular group of students, these geometric competencies will be considered relevant, engaging or motivating.[13] Students' intentions for learning might be related, first of all, to their foreground. Can ethnomathematics be criticized for providing restricted access to mathematical ideas or access to mathematical ideas without sufficient potential for touching the students' foreground? We need to point in the direction of the potential of mathematical ideas for developing critical citizenship and mathemacy as efficient tools for a critical reading of mathematics and of how mathematics may operate in the social environment.

…SOCIOLOGICALLY SPEAKING

Powerful mathematical ideas can be investigated from a sociological perspective as well. Such ideas can be defined in relation to the extent to which they are used as a resource for action in society.

Mathematics does not exist as independent knowledge in society. Social actors, not only mathematicians, use mathematics as a descriptive and a prescriptive tool. Mathematics, including its applied forms such as engineering mathematics and mathematical economics, is part of the available resources for technological action, involving planning and decision making. We use the term *technological action* (Skovsmose, 1994) in the broadest possible sense, including making decisions about, for example, how to manage the economy of the family, establishing a new security system for electronic communications, investigating traffic regulations, organizing insurance policies, instituting quality control of mechanical constructions, providing a booking system for airlines, testing of algorithms and computer programs, and many other activities that today are present in most working places. In various ways, mathematics constitutes a resource for such actions.

To express this rapidly growing multitude of "actions through mathematics," we highlight three characteristics of the way in which society operates with and through mathematics in technological enterprises. First, using of mathematics, it is possible to establish a space of (technological) alternatives to a situation. Mathematics also provides a limitation on this space of alternatives, however. In this sense, mathematics serves as a source of technological imagination, which is limited by many blind spots. Second, mathematics allows us to investigate particular details of a not-yet-realized plan. However, hypothetical reasoning about details of imagined constructions, supported by mathematics, also lays a trap because mathematics imposes a limitation of the perspectives from which hypothetical situations are investigated. In particular, risks can emerge from the gaps in hypothetical reasoning, which might overlook whole sets of consequences of certain technological implementations. Finally, as a resource for technological action and decision making, mathematics becomes an inseparable part of our present reality and of other aspects of society. We come to live in an environment created and supported by means of mathematics. In particular, the development of the informational society is closely linked to the spread of mathematical based technologies.[14]

Talking about powerful mathematical ideas, sociologically speaking, we do not simply have in mind a long list of mathematical modeling examples or applications of mathematics that have been presented in many textbooks serving the purpose of motivating students and illustrating that mathematics can be an useful subject.[15] We want to draw attention to the fact that mathematics operates as an integrated part of many technological actions and that such actions, as any other, may have unpredictable positive or negative consequences. The quality of the consequences of a technological action is not guaranteed by the quality of the mathematical base behind it. The three aforementioned characteristics of how we operate with and through mathematics may help one to grasp the complexity of mathematics in action and draw attention to the basic uncertainty associated with any mathematical idea put into operation in any technological context.[16] Therefore, a critique of mathematics in action is necessary.

We can illustrate some of the aspects of powerful mathematical ideas, sociologically speaking, by considering Macro-Figures Becoming Macro-Dangers. This project builds on an actual situation in which a mathematical model has been the basis of an economic policy that has great social impact. A primary task for the students could be to reveal the connections between the blind spots of the hypothetical reasoning of the model and the emergence of certain social uncertainties. A particular issue is to consider the relationship between the growth on the one hand of the functions $f_i(t) = (1 + i)^t$, $f_e(t) = (1 + e)^t$ and $f_r(t) = (1 + r)^t$, and, on the other hand, the growth of the function $f_n(t) = (1 + n)^t$, where n, e, i, and r are connected by the formula $n = (1 + i)(1 + e)(1 + r) - 1$. By studying these functions, we witness some elements of the logic of the UPAC system. We experience the drama of "actions through mathematics." The system of mortgage payment is determined by this logic. In particular, it becomes relevant to clarify to what extent the growth of $f_n(t)$ gets out of control (economically speaking), even though the growth of $f_i(t)$, $f_e(t)$ and $f_r(t)$ seems "reasonable." Thus, the project can illustrate how mathematical calculations used for social decision making can provoke new risk structures for certain groups of people.

Terrible Small Numbers also shows the relationship between risks and mathematics. In this project, students were brought into a situation in which economic and epistemic interests were confronted. This contradiction is exemplary in many technological design processes. How much additional investigation is needed to make an educated decision? How large a sample of eggs must be investigated to decide whether to put the eggs on the market? In many cases, mathematical modeling makes it tempting—and possible—to jump into conclusions about what to do; such conclusions may bring new risk structures to our future.

Thus far in our analysis, we have tried to show different interpretations of "powerful mathematical ideas." Each one can be related to central notions such as the level of abstraction in the mathematical architecture, the meaningfulness of acquired mathematical notions, the way learners can experience an empowerment as citizens, and the critical concern about how mathematics operates as a resource for action in a technological environment. Each one of these interpretations can suggest a response to the paradoxes of inclusion and citizenship. Thus, if the logical and the psychological interpretation dominate, the paradoxes seem to disappear. What could be more important in the mathematics classroom than bringing students to master the highest level of abstraction in a meaningful way? Considering the sociological and the cultural interpretation of "powerful," both paradoxes reappear in a strong version. Mathematics education cannot ignore them. The discussion of providing democratic access to powerful mathematical ideas becomes more complex when we also consider the notion of democracy, however. We discuss this issue in the following section.

DEMOCRATIC ACCESS

All students, everywhere in the world, have the right to education. We can go further and say that all students in the world should have the chance to learn mathematics. Democratic access, in this sense, refers to the actual possibility of providing mathematics for all. However, the idea of democratic access, understood as the right to participate in mathematics education, is more complex. Here we discuss in more detail what the expression can represent.

Democratic access designates the possibility of entering a kind of mathematics education that contributes to the consolidation of democratic social relations. As we have argued elsewhere (Skovsmose & Valero, 2001), a critical view on the connection between mathematics education and democracy situates the notion of democracy in the sphere of everyday social interactions and redefines it as purposeful, open political action undertaken by a group of people. This action is collective, has the purpose of transforming the living conditions of those involved, allows people to engage in a deliberative communication process for problem solving, and promotes coflection—that is, the collective reflection or thinking process by which people "bend back" on each other's thoughts and actions in a conscious way (Valero, 1999).

Democratic access in mathematics education, in the sense we have indicated, is played out in many different arenas in which the practices of mathematics education take place. We comment on three such arenas that we believe are fundamental: the classroom, the school organization, and the local and global society.

In the classroom

The mathematics classroom is a microsociety in which democratic relationships between students and teacher and among students must be present if education is to provide any form of democratic access. Democratic relationships that allow collaboration, transformation, deliberation, and coflection are central in opening possibilities for a critique of mathematical contents in the class and of their significance in social actions based on them.

Communication in the classroom can follow many patterns, but to establish a spirit of democracy, dialogue and critique are indispensable components. Thus, a mathematics classroom governed by bureaucratic absolutism or by a communication that does not incorporate possibilities for politicizing the mathematical education experience does not represent democracy. Alrø and Skovsmose (2002) provided an example of a communicative model with a democratic concern in mathematics education. The inquiry cooperation model refers to a variety of communicative acts supporting an inquiry process. Such a process cannot simply be guided by the teacher; rather the students act in the process of investigation in cooperation with the teacher. The elements of the inquiry cooperation model are: getting in contact with, discovering, identifying, thinking aloud, challenging, reformulating, negotiating, and evaluating.

The nature of some of these acts can be clarified with reference to the project Terrible Small Numbers. When the students carry out the experiment related to the quality control of eggs, the process is not reduced to a set of exercises organized in a certain sequence. The openness allows students to "own" the learning process and to experience what it could mean to be responsible for making decisions. When the students work on their own and the teacher wants to intervene, students should not feel threatened in their ownership of the process of investigation. The teacher has to get in contact with students and then she can challenge them: How could it be that in some of the samples there is more that one egg with salmonella? The students can try to identify sources for explanation: Maybe the teacher did not mix the eggs sufficiently? Discoveries can be made: Could it be that samples do not always tell the truth about the whole population? During the process of negotiation in which different possible explanations are considered, thinking aloud is possible. Thinking aloud is a way of providing public access to a line of thought, and it can be open for negotiations and reformulations. Any result of such a process can be evaluated.

In the section "Sociologically Speaking ...," we outlined three aspects of actions through mathematics: Mathematics helps to open possibilities by providing the basis for a technological imagination; mathematics supports investigation of particular aspects of not-yet-realized constructions, and, when realized, mathematics operates as an integrated part of the technological device. If such aspects of mathematics in action should be addressed critically in the mathematics classroom, then mathematical content needs to be contextualized, not only in terms of the provision of a "task context" but mainly in terms of a "situation context" (Wedege, 1999). In Terrible Small Numbers, it would not be possible to introduce a discussion about reliability and make students experience what it would mean to make decisions if the figures were not strongly related to actual situations happening in the social, political, economic, or cultural environment in which learning takes place. In general, we find that contextualization is a precondition for problematizing "trust in numbers." Such problematizing is essential for establishing a critical citizenship.

The contextualization is not simply a motivational device—although it might be motivating. It is a condition for establishing a discussion of how mathematics can operate as a source of power in a sociological sense because it invites a critical examination of how mathematics in fact is put into operation. A rich contextualization could help an inquiry cooperation model enter the classroom, and in this way influence the structure and content of the discussion.

We are aware that Terrible Small Numbers took place in a school situation and that this particular context provides a frame for interpreting the activities. Students worked with eggs that were not real, and their calculations had no actual consequences. Although they calculated the risk of producing an ice cream dessert out of six eggs, there was no real ice cream production in the classroom. Still, there is an important difference from a traditional exercise context in which a problem can refer to prices, goods, and amounts to be bought, but in which these prices, goods, and quantities operate in a completely different way from real prices, goods, and quantities. The traditional school mathematics exercise is accompanied by a set of metaphysical assumptions, notably that the description provided by the text is exactly true and it cannot be challenged. If a problem makes us buy apples and their price is set at $3.10 per kilo, it will not make sense if one student knows that the same apples can be bought around the corner for $2.30. If we are asked to buy 3 kg of apples, it does not make sense to question whether we can expect the scale to show exactly 3 kg—although we all know that apples are big units and it is difficult to have a weight of exactly 3 kg. The information provided by the text of the exercise is what we need for solving the problem, and the problem has one—and only one—correct answer (Mukhopadhyay, 1998; Skovsmose, 2000).

An essential task of the contextualization is to crack the metaphysical assumptions of the exercise paradigm. This metaphysics was challenged by Terrible Small Numbers, and this is essential for a critique to make sense. Opening the classroom for in-depth reflections is a condition for mathematics education to be part of a democratic endeavor.

In the school organization

Although many opportunities for establishing democratic access to powerful mathematical ideas are present in the classroom, this is not the only or the main arena for establishing such an access. In fact, recent research has acknowledged the importance and necessity of understanding classroom practices in connection with the whole context of the school organization and, more generally, the educational institution (Krainer, 1999; Perry, Valero, Castro, Gómez, & Agudelo, 1998; Stein & Brown, 1997; Valero, 2002). Teachers and students in the classroom are not isolated from the way mathematics teachers and school leaders work to shape mathematics education through curriculum planning and teachers' professional development. So, when we discuss democratic access, we must also consider how mathematics education practices in the school as a whole operate and create opportunities for, or obstacles to, this endeavor.

In the context of the school organization, we draw attention to the importance of who organizes the curriculum and how it gets organized. Let us assume that we, as well-intended policy makers, want to provide a curriculum that ensures students get democratic access to powerful mathematical ideas—independently of what interpretation of "powerful" we have in mind—and that we, as a result, offer a detailed plan including topics and ways of working in the classroom. Let us assume, furthermore, that this curriculum is put into operation. The detailed planning itself will obstruct the realization of our democratic intentions because the top-down model closes possibilities for the people involved in the actual curriculum development to own the process.

This conflict points to the basic complementarity[17] in curriculum thinking. The very process of planning, carefully and in detail, access to any kind of ideas obstructs the possibility of making this access democratic. The latter presupposes that teachers, students, and school leaders are acting subjects in identifying, planning, and implementing the curriculum. (Naturally, other groups, such as parents, could be considered as well.) This view implies that certain curricular decisions need to be taken in the community of participants in the school mathematics practices. The way a curriculum is organized, then, depends on the relations inside a network of school mathematics practices (Valero, 2002), where teachers as individuals, mathematics teachers as a group, the students, and school leaders can share their expectations about the meaningfulness of a mathematics educational experience. Who participates in formulating a curriculum and how such a formulation is implemented are constantly implicated in an unsolvable and necessary tension between specificity and freedom.

The planning of Macro-Figures Becoming Macro-Dangers is a microcurriculum design process in which basic components are identified and developed in close connection to the classroom. Practicing teachers, as a response to their own and their students' experiences, identified the idea of the project. It was not a suggestion from a textbook or an external authority, but it emerged from a situation that needed understanding because it was affecting the life of the school community. The potential for collaboration among teachers and students in the development of the project, as well as for transforming their understandings, and eventually their situation concerning mortgages, are key elements in the project. Were it predetermined that the projects should serve as an introduction to, say, algebraic calculations, then the significance of the projects could easily be lost. It would be impossible for the students, or for the teachers, to maintain ownership of the project. The experimental character of the curriculum design process exemplified by macro-Figures Becoming Macro-Dangers highlights the importance of creating a "laboratory for curriculum development" (Vithal, 2003). This notion refers to cooperation between different participants in the network of school mathematics practices to build an open curriculum planning process that acknowledges democratic concerns.

In the local and global society

There is a contradiction between establishing mathematics education in terms of democratic access to powerful ideas and, at the same time, letting that education serve differentiation functions in society by, for instance, ranking students in a way that significantly influences their future career possibilities. The emphasis on high-stakes assessment or classroom assessment in most countries can clearly contribute to the paradoxes of inclusion and citizenship (Morgan, 2000). Furthermore, mathematics education—at least in certain forms—generates different situations of exclusion according to gender, race, language, class, and ableness (e.g., Grevholm & Hanna, 1995; Keitel, 1999; Khuzwayo, 1998; Rogers & Kaiser, 1995; Secada, Fennema, & Adajian, 1996; Zevenbergen, 2000a). If we want to end this exclusion, then we should allow entry into mathematical learning to all.

The difficulty of establishing mathematics education as a democratic resource can be illustrated clearly by the following dilemma. Ethnomathematics (D'Ambrosio, 1996; Powell & Frankenstein, 1997) has represented a challenge to Eurocentrism first by demonstrating that all cultures, not least indigenous ones, demonstrate a deep mathematical insight and second, by showing that building on this knowledge makes it possible to reconstruct a mathematics education that does not recapitulate the priorities of colonization. This has led to the formulation of ethnomathematical curricula for disadvantaged populations such as, for example, the landless people in Brazil (Knijknik, 1997) or Mozambican peasants (Gerdes, 1997). Could it be, however, that offering ethnomathematical education to certain disadvantaged groups prevents them from being active members of the informational society and therefore dooms them to a life in the Fourth World?

In a similar way, critical mathematics education (Skovsmose, 1994; Vithal, 2003) has been proposed as an educational philosophy to address the risk of a mathematics education that contributes to the creation of citizens uncritical toward the devastating effects of mathematics in society. Nevertheless, in a research and development project intending to open possibilities for critical mathematics education with immigrant students in Catalonia (Gorgorió & Planas, 2000), the researchers perceived certain interpretations of that type of education as a "soft" program that could be suitable for these particular kinds of students. This view contrasts with the position—nonexplicit, but nonetheless easy to elicit in actions and proposals—of the educational authorities defending the need for "hard-core" mathematics education programs for those students expected to succeed within the educational system, in particular, the local students. We see that this interpretation could lead to a situation in which so-called critical mathematics programs serve as a second-rate curriculum for immigrants and political refugees because, after all, it may not provide them with the "hard" mathematical knowledge needed for climbing the ladder of social prestige in that community.

Recent studies adopting a broader sociological conception of the functioning of mathematics in society had also pointed to the relation between mathematics and mathematics education and the construction and reproduction of particular discourses about what it means to know and to master mathematics in schools (Popkewitz, 2002). Post-structuralist and postmodern points of view invite to think about the way in which mathematics education plays a role in the construction of social imaginaries about mathematical ability and thereby shapes students' identities as learners and citizens. Cotton and Hardy (2004) present two different interpretations of a series of interactions with students in an English mathematics classroom. It is evident that students' experiences in mathematics classrooms including teaching and assessment forms impact their perception of themselves and their place in society (see also William, Bartholomew, & Reay, 2004). Here, we directly face the paradox of inclusion. In this way, an attempt at inclusion through mathematics education could result in growing exclusion, and a concern for citizenship could come to establish citizenship among the excluded.[18]

In the creation of the Fourth World in the present informational society, as described by Castells (1999), the barbarism of the paradox of inclusion is associated with mathematics education. Mathematics education could help to secure access to the informational society as well as to establish and legitimize exclusion from it. For a teacher in an underresourced educational system, it is difficult to provide new possibilities in life for the students beyond what is already well known to them as their background. Thus, a fundamental discussion about mathematics education in many developing countries concerns the relocation of resources as a way of distributing possibilities in radically new ways. If this does not happen, then, for instance, historically Black schools in South Africa are doomed to belong to the Fourth World. The situation in South Africa is exemplary for the problem of how an unequal distribution of resources obstructs democratic ideals.

One particular aspect concerning mathematics education and the informational society is the use of technology in teaching and learning. We must consider how mathematics becomes installed in more and more technological devices (Wedege, 2000) and how it operates "behind the screen," making it possible to use mathematized tools without presupposing a deep understanding of their underlying mathematical structure, maybe even without being aware of the fact that a complexity of mathematics is in operation. The implication of this is that the necessary competencies to operate with these technologies has split people into two categories: those who can operate on the surface of the technology and those who can construct and reconstruct it. Both competencies are essential, and therefore it is important to discuss how mathematics education operates with regard to this split. Furthermore, we must discuss how mathematics education accomplishes its global function in a world where access to computers is still reserved for a select few. Given the nature of the information society, mathematics education occupies a sensitive position in which possibilities in the information age are distributed among students, regions, and nations. In mathematics education, the barbarism of this distribution is particularly visible.

Globalization also concerns the particular content of what is learned. Macro-Figures Becoming Macro-Dangers addresses issues that represent general aspects of how risks and uncertainties are distributed. The project tries to illustrate how large-scale economic figures and decision making can have particular effects and contribute to the creation of macro-dangers. This transformation from figures to dangers is a basic feature of globalization, where large-scale decision making distributes risk and uncertainties in formidable ways. In this sense, Macro-Figures Becoming Macro-Dangers has a particularly exemplary value.

FACING THE PARADOXES

Let us recapitulate the paradoxes of the informational society as we originally described them. The paradox of inclusion refers to the fact that the current globalization, which proclaims universal access and inclusion as a stated principle, can also be associated with processes of exclusion. As part of the development of a universal interpretations of powerful mathematical ideas framework for global connections, strong processes of exclusion and isolation are simultaneously taking place. Among other things, this brings about a "Fourth World," many new citizens of which are currently to be found in mathematics classrooms. The paradox of citizenship refers to the celebration of a learning society that emphasizes the need for relevant and meaningful education for the further development of social, political, and cultural structures, although in reality that education may have only a functional relevance for the system.

Does mathematics education face the paradoxes? Up to now, we have referred mainly to mathematics education as a field of practice, but now we concentrate on mathematics education as a research field. In Figure 17.1, we present the space for investigating democratic access to powerful mathematical ideas.

Reviewing research literature in mathematics education, there are unfortunately different ways in which the field ignores the paradoxes of inclusion and citizenship. One way of doing so is by concentrating on particular interpretations of powerful mathematical ideas, mainly the logical and the psychological ones in which the emphasis is placed on the development of mathematical thinking independent of any context in which it takes place. The second way is by ignoring that mathematics education is part of a democratic endeavor, or simply (and rhetorically) assuming that mathematics education, due to the very nature of mathematical thinking, constitutes a profound democratic enterprise. In this view, powerful mathematical ideas have an intrinsic democratic value (Skovsmose & Valero, 2000). A third and more moderate way is by considering only selected aspects of what democracy can involve. In particular, much discussion has focused on democracy in the classroom but ignored the other arenas in which meaningful participation in political action through different kinds of powerful mathematical ideas is built.

Gómez (2000) carried out a classification of the papers published in 1997 in three main international journals in mathematics education, the *Journal for Research in Mathematics Education* (JRME), *Educational Studies in Mathematics* (ESM), and *Recherches en Didactique des Mathématiques* (RDM), and in the *Proceedings of the 21st Meeting of the International Group for the Psychology of Mathematics Education* (PME). Although he acknowledged that his sample is not representative of all international research published in the area, he considered it an indicator of the type of research carried out. He concluded that "mathematics education research production is centered mainly on cognitive problems and phenomena; that it has other minor areas of interest; and that it shows very little production on those themes related to the practices that influence somehow the teaching and learning of mathematics from the institutional or national point of view" (p. 95).

Translated to our space of investigation, Gómez's results indicate that there is a high concentration of work in the lower, left-hand portion of Figure 17.1. To check these results, we classified the papers published in *JRME*, *RDM*, *ESM*, *For the Learning of Mathematics* (*FLM*), *Suma*, and the *International Journal of Mathematics Education in Science and Technology* (*IJMEST*) published between January 1999 and October 2000, in the different portions of our area of investigation. The results are indicated in Figure 17.1.

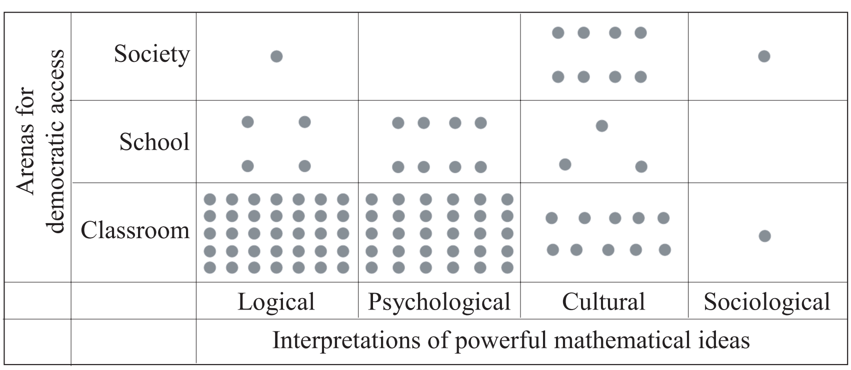

Figure 17.1 Indication of distribution of research papers in the space of investigation on democratic access to powerful mathematical ideas.

This distribution shows that the majority of papers are concerned with interpretations of powerful mathematical ideas in a logical and psychological sense and in the arena of classroom interactions. There are, however, also a considerable number of papers highlighting cultural interpretations. However, there are areas, such as the society and school arenas for democratic access and the sociological interpretation of powerful mathematical ideas, that either have not been explored to any great extent or that are a low priority for publication in research journals. Naturally, we are not claiming that each and every research project in mathematics education should address the full range of issues mentioned. However, it is highly problematic that dominant research trends in mathematics education operate within a limited scope of the space of investigating democratic access to powerful mathematical ideas. Such a paradigmatic limitation effectively obstructs the possibilities for mathematics education to face the paradoxes of the informational society.

What could it mean, then, for mathematics education research to face the paradoxes of inclusion and citizenship? We attempt to offer possible answers by raising clusters of questions that point in the direction of under or nonresearched issues.[19]

1. Democracy, understood as a collective, political action for the purpose of transformation, is lived through everyday experience, including the mathematics classroom. How do particular forms of mathematics education, including interaction and communication in the classroom that they generate, acknowledge democratic values? Which are the forms of interaction in the classroom that open possibilities for politicization and critique of both the mathematical content and the interaction itself? How does mathematics education acknowledge that the microsociety of the classroom is related to the local and global society where students live? Are forms of learning in school related to forms of learning in workplaces and organizations and in everyday situations?
2. The contextualization of school mathematics is an important gateway into cultural and sociological interpretations of powerful mathematical ideas. Do we deal with a contextualization that primarily observes the metaphysics of the exercise paradigm, or must we deal with deeper, real-life references? Does the contextualization of school mathematics touch on both the students' background and foreground in significant ways? Do we try to illuminate issues in which the content of mathematics education prepares the students to operate as critical citizens in a context where mathematics and mathematically based decision making are in operation?
3. The dynamics of school mathematics practices, understood as the complex interaction among teachers, school leaders, and students in the school organization, needs exploration. A particular issue concerns who participates in the curricular decisions and where they take place. Does curriculum planning and implementation open possibilities to bring into the classroom different interpretations of what powerful mathematical ideas mean? What do teachers as a group and school leaders value as an appropriate mathematics education, given their students' backgrounds and foregrounds? In particular, it is important to consider how local curricula can operate in society. Could a particular curricular design and implementation constitute "second-rate" mathematics education, which dooms students to exclusion or to uncritical acceptance of society? How does the process of exclusion of certain social groups—defined in terms of gender, race, class, and ability—operate in the school organization as a whole?
4. It is relevant to consider how information and communication technologies (ICTs) open and reorganize new learning possibilities (Balacheff & Kaput, 1996; Borba 1997; Borba & Villareal, 2005). What do ICTs mean in boosting culturally and sociologically powerful mathematical ideas? Part of this understanding has to do with identifying the state of actual global distribution of ICT learning possibilities. Obviously, we have to work with an unequal distribution of ICT facilities around the world. What does this mean for the role of mathematics education in underresourced classrooms and schools? In particular, what does this imply for the formation of the "Fourth World"? Does the reorganization of learning possibilities also include a reorganization of inclusion, as well as exclusion,

from the informational society? Is research in mathematics education too often set up in such a way that certain social and economic resources are taken for granted, although they can be taken for granted only in certain (privileged) parts of the world?

5. As we have indicated previously, mathematics operates as a resource of power in a variety of actions and decision making in all areas of life. Does mathematics education open possibilities for students to see this resource in operation? How can "actions through mathematics" be illustrated in mathematics education? How far do we go in making mathematics education a critical activity, addressing both the wonders and the horrors of actions through mathematics? What does it mean to offer a mathematics education that tries to illustrate such contrasting aspects of powerful mathematical ideas?
6. Finally, through mathematics education in all its arenas—the classroom, the school, and society—we contribute to the construction of public images of mathematics and mathematics education. How does this process of building social images and ideologies of mathematics and mathematics education happen in the different practices of mathematics education? Which are the characteristics of the discourse that we construct so that it can actually attribute so much power and democratic relevance to our subject? What are we doing in the classroom, in schools, in society, to strengthen mathematics as a powerful form of knowledge? What are the broadest social consequences of our practices? Could one be the reproduction of a world in which the paradoxes of equality and citizenship can easily survive?

If, as mathematics educators in research and practice, we are concerned about the lives and experiences of students like Nicolai and Carlos, we should consider even more seriously the importance of broadening our interpretations of what democratic access to powerful mathematical ideas means. Furthermore, we should keep in mind the necessity of tackling the inclusion and citizenship paradoxes in our endeavor during the coming century.

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NOTES

1. For a further discussion on mathematics education and globalization see for example Atweh, Clarkson, Nebres (2003).
2. We could go further in this argument by asking who actually benefits from the expansion of the learning society discourse. Plausible explanations about the forces associated with recent reform trends in several countries can be found in, for instance, Apple (1996, 2000).
3. Here we are inspired by Young (1998), who presented the idea of "learning society" as a contested concept.
4. For a discussion of the notion of *Bildung*, see Klafki (1986) and Biesta (2000). There is no adequate English translation of the German word *Bildung*, although "liberal education" has been suggested.
5. This project is described in Alrø and Skovsmose (2002). It is a collaboration between two Danish teachers, Henning Bødtkjer and Mikael Skånstrøm, and three researchers, Helle Alrø, Morten Blomhøj, and Ole Skovsmose. Terrible Small Numbers has been tested in different classrooms, but here we primarily provide a general overview of its main ideas.
6. This example builds on discussions with Colombian teachers during the seminar "Cómo desarrollar una educación matemática crítica en el salón de clase" led by Ole Skovsmose in Bogotá, Colombia (October 8–9, 1999), on Paola Valero's presentation "Desenmascarar las matemáticas: Un reto para los profesores del próximo milenio" in Portimão (Portugal) during ProfMat 99 (November 10–14,

1999 and in particular on follow-up discussions with Jaqueline Cruz and Verónica Tocasuche, secondary school teachers in Colombia, and with Pedro Gómez. These ideas have not been implemented yet.

7. From January 2000 the UVR system (Unidad de Valor Real [Unit of Real Value]) replaced the UPAC. The UVR established a simpler index based on the national basic cost of living.
8. For a discussion of the influence of Piagetian and Vygotskian ideas on mathematics education see Skott (2000, pp. 24–39) and Lerman (2000).
9. One of the signs of the influence of this work is the extent, in quantity and quality, to which Sfard's paper has been quoted in research in mathematics education since it appeared in 1991.
10. For a discussion of the notion of foreground, see Skovsmose (1994, 2005a).
11. A much more narrow interpretation of "cultural" is found in, for instance, Seeger, Voigt, and Waschescio (1998), in which the culture of the mathematics classroom is interpreted as first of all referring to interaction and communication in the classroom. Other interpretations of culture are present in the work of Cobb and colleagues, for whom the mathematics classroom is the micro-community of practice where sociomathematical and more general social norms are built (Cobb, 2000).
12. Teachers, working from a culturally empowering perspective, may face cases of students resisting the teachers' "empowering" game because the students can envision traditional teaching as a valuable contribution to their foreground.
13. For a critique of ethnomathematics, see Vithal and Skovsmose (1997).
14. For a discussion of mathematics in action and the notion of the formatting power of mathematics, see Skovsmose (2005b) and Skovsmose and Yasukava (2004).
15. De Lange (1996) presented a discussion of applied mathematics in education. We find that his view of applied, realistic mathematics is in many respects different from what we see as sociologically relevant and powerful.
16. For a discussion of uncertainty about mathematics, see Skovsmose (1998, 2005b).
17. Vithal (2003) elaborated on the notion of complementarity in mathematics education as the necessity of bringing together irreconcilably conflicting theories to provide a better and fuller understanding of what we study in mathematics education.
18. For further details on this recent trend of studies see, for example, Walshaw (2004).
19. At this point, we could also enter into the discussion of what it means to research a situation that "does not exist" because there is also a connection between deficiencies in the area of investigation and deficiencies in actual practices in school mathematics. Because this discussion is broad and it is not our intention to tackle it here, we merely mention the work of Skovsmose and Borba (2004) and Vithal (2003, 2004), who have tried to approach this question.

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