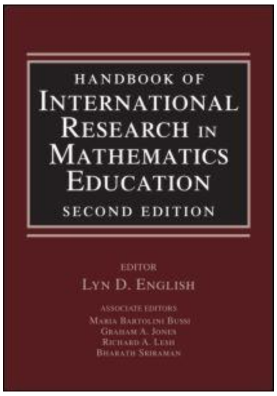

Handbook of International Research in Mathematics Education

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Democratic access and use of powerful mathematics in an emerging country

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14 Democratic access and use of powerful mathematics in an emerging country

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With the arrival of the new millennium, we are witnessing a renewed interest in education; every national educational system, in one way or another, is designing programs or taking actions to face the future with education as the sustaining foundations of what has been called the century of information and knowledge. It has become ostensible that educational systems will have to incorporate the immense scientific and technological developments of the past decades, which are, in part, responsible for our expectations for the new century. Every country, depending on its sociocultural and economic conditions, is facing new challenges in education, as education is being understood as a systematic solution to most immediate social needs. In the need to reformulate what is taught, as well as how and why it is taught, does mathematics have an important role to play in society? That is, what is the *new* role of mathematics in contemporary societies? To be precise, one would have to take into account the degree of development of the societies under study, but the pervasive presence of digital technologies reminds us that these technologies "will rock our worlds" (Baker, 2006, p. 54). Thus, we foresee important transformations in the field of curricular design and development, as well as in the application of new learning tools.

We will also need to teach students to think critically about the ongoing changes in the world and about how these changes can affect educational and national realities. Access to knowledge cannot be regarded as a politically neutral issue because there is an obvious problem of exclusion for those who are on the margins of the educational process at any of its levels. Our inclusion in the contemporary world of globalization demands that we have the critical ability to transfuse scientific and technological developments into our educational realities. However, even in countries where the use of technology has been widely supported, the route to implement its use as an instrument of mathematical work has faced obstacles since it requires not only changes in curriculum proposals but also in mathematical instruction (Guin, Ruthven, & Trouche, 2005).

We cannot forget that a pre-existing school culture has left a significant mark on students and teachers' values within the educational system. Artigue (2005) states that "these values were established, through history, in environments poor in technology, and they have only slowly come to terms with the evolution of mathematical practice linked to technological evolution" (p. 246). Thus, the school culture requires the gradual re-orientation of its practices and of its cognitive and epistemological assumptions to gain access to powerful ideas of mathematics and to new habits of mind including exploring, modeling, handling of information, and the ability to systematize. This is not always possible under the traditional teaching model that has dominated Mexican educational system until recently.

Today in Mexico, 25% of the national population (out of more than 100 millions) is concentrated in four major cities, and a population of about 10 million is dispersed in very small communities. In 2005, there were around 32 million students—all levels included.

The secondary level, with more than 11% of the school population, shows the fastest rate of growth for the next years (SEP, 2005). In this context, the educational community needs to design flexible curricular proposals around fundamental disciplinary concepts that promote critical thinking within an educational environment that enhances the use of computational technology.

In this chapter, we present different perspectives and strategies used to incorporate contemporary educational research into curricular development. One of these strategies responds to the needs of the classroom. The classroom could be considered as the central nervous system of any educational system (Moreno & Sriraman, 2006). It is possible and feasible to cultivate powerful ideas that generate different levels of mathematical thinking through the mediation of computing instruments. Another approach refers to innovation and implementation of digital technologies within the educational system. It has to be stated that any educational system is an open system and thus, subjected to the multidimensional influence of its social and cultural environment.

From an international perspective, curriculum innovation projects reveal three essential commonalities (Black, & Atkins, 1996): (1) the importance of the students' practical work; (2) the relevance of making explicit linkages among disciplines or scientific fields, in terms of identifying connections and relationships among powerful ideas embedded in those domains, and (3) the importance of recognizing that mathematics and science are ways of knowing and explaining the world.

MATHEMATICS LEARNING IN SCHOOL: RELEVANT FEATURES

Essential changes in basic education programs are central in current curriculum reforms (Moreno & Block, 2002). In Mexican secondary school, it is expected that students (12- to 18-year-olds) will strengthen their problem-solving skills and be able to transfer these skills to other domains. In other words, they are expected to initiate the complex process of decontextualizing their own knowledge. We understand this process as one that enables students to establish rich connections among situated versions of their knowledge.

Geometry is regaining the place it once had. For example, the current approach to geometry focuses on students' development of reasoning skills that involves the construction and verification of hypothesis or conjectures by the students themselves. Similarly, algebra is studied from modeling and problem-solving perspectives. Change and variation problems are also studied. It is within this general background that we will describe strategies whose purpose is to improve students' mathematical competences. These should be able to make feasible democratic access to powerful mathematics.

At this point, we begin elucidating the notion of *access to powerful mathematical ideas* through school culture: It mainly has to do with providing students with the opportunity of (a) experiencing the construction and development of mathematical knowledge; (b) developing their creativity through exploration of different approaches that emerge from discussing questions posed within the classroom; and (c) developing their own computing techniques or procedures. In other words, successful access to powerful mathematical ideas should reflect a tangible gain in conceptual understanding and computational fluency.

We can assume that all students will be using calculators in school. Even when this situation results from a decision taken by authorities, it generates educational needs that must be faced as didactic and systemic problems. Dealing with these problems implies a recognition that most present curricular designs are pre-digital in their conceptions. Thus, the presence and utilization of digital tools as handheld artifacts, for instance, will generate conflicts of legitimacy when introduced in the teaching of mathematics. As this digital presence is unavoidable, it will eventually contribute to the *erosion* or transformation of our present curricular designs.

The access to powerful mathematical ideas in the context of teaching and learning with the new digital technologies can mean a change in the way we work, for instance, with decimal numbers, reinforcing and using the powerful idea of approximation. To a great extent, this is made possible by the meaning attached to the place value of digits in decimal expressions. This is undoubtedly one of the central features of the decimal representation system. Unfortunately, the uncritical use of calculators leads to the idea that all decimal expressions are finite. If we want students to reach a higher conceptual level (an enhanced conceptual competence), we must promote teaching practices that go into depth regarding numerical approximations with the intention of displacing the idea that every decimal expression is finite. This can be achieved through suitable teaching models, such as the systematic study of the change of measurement units in calculations of lengths, areas, and volumes. The activities involved would bring out another important idea, that of "better approximation." The synthesis of these (approximation and better approximation) constitutes an example of a powerful idea that is worth developing throughout all levels of the education system and with the help of various technological resources (NCTM, 2000).

The development of calculating skills combined with process of estimation and the use of hand calculators might be an invaluable resource for students to solve problems outside the classroom. In the case of geometry, the possibilities of dynamic tools such as Dynamic Geometry Environments might lead to a change in the current conception of school geometry which follows the axiomatic organization of geometric knowledge. In fact, these dynamic environments make feasible the exploration, conjecturing, systematization and justification of mathematical relations.

Thus, there is an overarching principle to be taken into consideration when using digital technologies: Dynamic representations of mathematical knowledge are instrumental to facilitate the students' construction of cognitive tools to identify and explore mathematical relationships. These tools involve ways to detect, support, and communicate those mathematical relations. Beyond the diverging perspectives of contemporary approaches, there seems to be a consensus within the research community that we must contextualize the knowledge taught to let the meaning of any mathematical situation to be first developed instead of focusing on a premature formalization. This is an explicit goal in software like SimCalc Mathworlds and Dynamic Geometry Environments like Cabri or Sketchpad.

Learning is an outcome from the interaction between the individual and the environment that is *full of interference*. Our efforts are focused on the study of the specific milieu that will favor the emergence of mathematical knowledge in the classroom (Brousseau, 1987) and its blending and eventual synthesis within the use of available digital technologies. Exploration within the classroom is one of the methodological tools we have used in our work. The design of teaching strategies that makes visible fundamental mathematical ideas is based on analyzing problems in which that idea works as a solving tool. Moreover, to support the development of problem solving strategies, the learning environment should allow students to see for themselves how they are able to solve a problem, or extent to which they can solve it. From this perspective, introducing powerful mathematical ideas into the school classroom gives students the chance to experience personal and therefore significant re-elaboration of mathematical knowledge. This provides them with appropriate tools to perform creative work by elaborating and testing conjectures and refining their calculating techniques or procedures.

SYMBOLIC AND DIGITAL TECHNOLOGIES: THE ROUTE TO POWERFUL IDEAS

In the natural world, an organism cannot *communicate* its experiences, that is, the organism lives confined within its own body. However, during their evolution, humans became able to overcome the solipsism of their ancestors, extending the reach of their cognition and developing a communal space for life. This is an important stage in the process that transformed our ancestors into *symbolic beings*. Our symbolic and mediated nature comes to the front as soon as we try to characterize our intellectual nature. Birds "know" how to build a nest; eagles "know" how to fly, fishes how to swim. As Donald (1993) has written:

> Animal brains intuit the mysteries of the world directly, allowing the universe to carve out its own image in the mind. (p. 155)

Only humans possess what can be termed *explicit* cognition that allows us to go from learning to knowledge. Explicit cognition is symbolic cognition. The symbol refers to something that although arbitrary, is *shared* and *agreed by a community*. This is particularly clear when dealing with systems of mathematical notation.

Technology emerged since early times in human history. A crucial example, from the cognitive viewpoint is the creation of artifacts to extend memory capacity (Donald, 1993). It is not difficult to value the importance of marking a bone in order to externally capture a bit of memory. Once this memory strategy is socially established, it becomes instrumental to modify the workings of individual and collective memory. The marking technique, for example, was further extended by the construction of one-to-one correspondences between arbitrary collections of concrete objects and a *model set* (a template). Earliest examples have already been found between 40000 and 10000 B.C. In 1937, a wolf bone dated to about 30000 B.C. was found in Moravia (Flegg, 1983). This reckoning technique (using a one-to-one correspondence) reflects a deeply rooted trait of human cognition. Having a set of stone bits or the marks on a bone as a modeling set constitutes perhaps, the oldest counting technique humans have designed. Between 10000 and 8000, B.C. in Mesopotamia, people used sets of pebbles (clay bits) as modeling sets. This technique was inherently limited. If, for instance, we had a collection of 20 pebbles as modeling set then, it would be possible to estimate the size of collections of 20 or less elements. Nevertheless, to deal with larger collections (for instance, of 100 or more elements), we would need increasingly larger models with evident problems of manipulation and operations. And so, the embodiment of the one-to-one technique in the set of pebbles inhibits the extension of it to further realms of experience. It is very plausible that being conscious of these difficulties, humans looked for alternative strategies that led them to think of a new technique: the idea that emerged was to replace the elements of the model set with clay pieces of diverse shapes and sizes, *whose numerical value were conventional*. Each piece *compacted* the information of a whole former set of simple pebbles––according to shape and size. The pieces of clay can be seen as embodiments of pre-mathematical symbols. This example shows clearly the cognitive impact that symbolic technology has had on human beings since early times.

As we become expert users of a symbolic system, we can work at the symbolic level without making a conscious effort; the system of symbols is transformed into a cognitive mirror in the sense that one's ideas about some field of knowledge can be externalized with the help of that system; then we can see our own thought reflected in the symbols and discover something new about our own thinking. From this perspective, it is crucial to understand how writing as a technology changed human cognition (Ong, 1999).

The former examples suggest the importance of a long term perspective to develop mathematical systems of representations. The study of these systems is instrumental to develop a proper epistemological and pragmatic perspective for mathematics education.

In modern educational systems, devices such as tables of functions, slide rules, and scientific calculators have been used, mainly, as devices to enhance computing power. However, more recently these devices have been used to help students graph functions, collect data and so on. However, these activities have been developed *inside* a curriculum explicitly designed as pre-computational (Moreno & Sriraman, 2006). The role of technology is conceived of as an *amplifier* of what could be done without that technology.

Nevertheless, the increasing availability of diverse tools and the now better understood nature of the symbiotic relation of a user-agent with a tool (Verillon & Rabardel, 1995), suggest that amplification is not enough as a unit of analysis to explain the students' use of technology. In a certain sense, this amplification process might be described as *been guided by the technology*. Let us first present some simple examples to make plausible the existence of a deeper layer of activity beyond amplification. Consider an artist, a violinist, for instance. The violin is like an extension of herself in the sense that while playing, the violin is *transparent*. The artist can feel the music *through* the violin. We will say that the violin has become an *instrument* not just a tool for the artist. There is a process by means of which the violin, that at the beginning is *opaque*, is transformed into an instrument, almost invisible, that allows the artist to display her art. That process is, in fact, a double process: the agent-user (in the present example, the violinist) explores the possibilities of the tool (the violin) and dialectically, modifies her own approach to the tool and to the knowledge (the music) generated by her activity. Her strategy to develop a playing technique is deeply linked to the workings of the tool. The cognition of the artist is transformed: her art is not the result of doing something better, something that she could do without the tool, but something intrinsically linked to the new activity that emerges from the new dialectical interactions agent-tool. When this finally happens, we say that the tool (violin) has been transformed into an instrument: it is one with the violinist. Instead of being guided by the tool (violin), the artist is *guiding the tool*. This is a multilevel process.

At the basis of the complex process by means of which a tool becomes an instrument, like the problem of explaining how a tool can *mediate* the cognitive processes of an agent-user. That any cognitive activity is a *mediated activity* has been aptly established (Wertsch, 1991). For research in math education, this thesis constitutes a starting point from which we would try, to understand the nature of the mediating role of digital tools in the learning and teaching of mathematics. Wertsch (1991), speaking of human action expressed his view with these words:

> The most central claim I wish to pursue is that human action typically employs mediational means such as tools and language and that these mediational means shape the action in essential ways. (p.12)

Let us try to make explicit a distinction between material tools and symbolic tools. In general terms, a material tool like a labor tool, affects the nature of the (mediated) human activity; it can modify the goal of that activity. On the other hand, a symbolic tool like the written language, affects the knowledge, the cognition of the reader. But what happens with a tool like a computer? It is, simultaneously, a tool that can affect the human activity (writing) and the cognition of the agent-user (reorganization of her ideas). In other words, the computer is externally oriented and, at the same time, internally oriented. The mastery of a technological tool like the microscope, for instance, affects the research activity of the researcher and, at the same time, modifies her knowledge. The conclusion seems inexorable: cognitive tools blend, dialectically, the activity of the agent-user and the transformation of her cognition.

Computing environments provide a cognitive mirror for studying the evolving conceptions of students and teachers. Their conceptions will be dynamically shaped during the process of going from one system of representation to another and so capturing different features, and behaviors of the mathematical objects under consideration. Graphing tools, for instance, produce a shift of attention from symbolic expressions to graphic representations. Representations are tools for understanding and mediating the way in which knowledge is constructed.

Let us insist that even if the new technologies are not yet fully integrated within the school mathematical universe, their presence will eventually erode or transform the mathematical way of thinking embedded in the system. In this context, rather than thinking of a sequence of themes or contents to be included in a curriculum proposal, it would seem relevant to

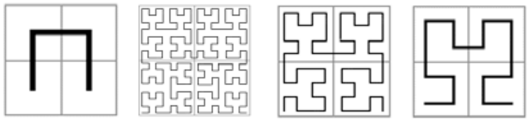


Figure 14.1a

structure and organize the curriculum in terms of fundamental mathematical ideas and concepts dynamically conceived. Students will need to develop mathematical resources, strategies and ways of thinking necessary to comprehend and apply mathematical ideas. It is evident and clear that as part of the scaffolding process to conceive of a technology oriented-curriculum, a new way to conceptualize mathematical objects and their study is needed. For instance, as formalization is relative to the medium in which it takes place, there is a need to reflect on the new ways students have to prove mathematical assertions in the classroom. Take for instance, the Hilbert space-filling curve (see Figure 14.1a), a continuous fractal space filling curve first described by Hilbert in 1891 (http://en.wikipedia.org/wiki/Hilbert_curve).

To prove it along classical lines is an intricate task, but what happens when one turns the result into a digital one? For instance, to argue in favor of the validity of the theorem, we can translate it thus:

> Given a (screen) resolution, there is a step in the recursive process that generates the curve that fills that screen.

This is a way to empower students in order for them to have access to deep results properly translated.

As Schoenfeld (1994) stated:

> Proof is not a thing separable from mathematics…is an essential component of doing, communicating and recording mathematics. (p. 76)

Nevertheless, the use of new digital media requires a fresh approach to this important mathematical and curricular topic. It is central for math education as a new applied epistemology. It is also important to recognize some limitations in using technology to solve particular problems. For example, Sriraman & Strzelecki (2004) illustrate examples that involve powers or exponential functions where students need to explore the problem using problem solving strategies (analyzing particular cases, looking for patterns, etc.) instead of trying to calculate results using hand calculators or computers. (Calculate the first or last digit of 2004^{2004}, for example).

MEDIATED ACTIONS: THE VOICE OF THE STUDENTS AND THE DEVELOPMENT OF MATHEMATICAL PRACTICES: RESEARCH STUDIES

In the previous sections of this chapter, we have sketched aspects of a framework in which we show some features of our educational system, the importance of thinking and organizing a curriculum proposal around fundamental mathematical ideas, and the relevance for students of using digital tools to access, comprehend, and apply powerful mathematical ideas. Curricular changes imply the need to constantly discuss the content and ways for students to learn the discipline. Romberg and Kaput (1999) stated that:

> The changes make it imperative that any answer to the question 'What mathematics is worth teaching'? Be periodically considered. … regardless of the specific content, the aims

> of mathematics teaching can be described in terms, as teaching students to use mathematics to build and communicate ideas, to use it as a powerful analytic and problem-solving tool, and to be fascinated by the patterns it embodies and exposes. (pp. 15–16)

It is also recognized that the mediation role of digital technology is important in helping students represent, identify, and explore behaviors of diverse mathematical relationships. An important goal during the learning process is that students develop an appreciation and disposition to practice genuine mathematical inquiry during their school learning experiences. The NCTM (2000) suggests that students in their problem solving approaches should pose questions, search for diverse types of representations, identify conjectures, and present different arguments to support and communicate their results. Here, the role of students goes further than viewing mathematics as a fixed, static body of knowledge; it includes that they need to conceptualize the study of mathematics as an activity in which they have to participate in order to identify, explore, and communicate ideas attached to mathematical situations.

> …Students themselves become reflective about the activities they engage in while learning or solving problems. They develop relationships that may give meaning to a new idea, and they critically examine their existing knowledge by looking for new and more productive relationships. They come to view learning as problem solving in which the goal is to extend their knowledge. (Carpenter & Lehrer, 1999, p. 23)

Based on the idea that students need to conceptualize their learning of mathematics as a problem solving activity, we have carried out several studies in which students have had the opportunity to employ different tools like dynamic software (Cabri), handheld calculators or Excel in their learning experiences (Moreno-Armella & Santos-Trigo, 2001; Moreno & Santos, 2004; Santos, 2004; Santos, Espinosa, & Reyes, 2006). Thus, we recognize that there are different ways in which students can use digital technology in their learning experiences. The idea that with the help of some software or handheld devices students can represent mathematical objects, explore different cases, and find loci or trajectories of points (segments or figures) was crucial in designing students' learning activities. What type of mathematical resources and strategies do students need in order to show an efficient or significant use of technology during their learning experiences? When does the use of technology become a powerful tool for students? What type of representations do students utilize during their problem solving approaches? These types of questions become relevant to explain the type of mathematical reasoning students develop as a result of using systematically computational tools in their learning activities. In what follows, we document features of students learning that show a process in students' adaptation in the use of technology. It involves going from utilizing the tool to carry out operations and static representations to showing distinct ways to solve and reason about the situation and problem solving approaches. At the latest stage, students not only search for different approaches to represent and solve problems, but also they explicit redesign or formulate their own questions or problems.

The role of teachers becomes relevant in providing a class environment that promotes students' experiences in reflecting, conjecturing, and persisting. In this context, the design and implementation of tasks, which favor the use of these experiences, continue to be a great challenge in problem solving instruction (Santos, 1998). Thus, an instructional goal is that the use of technology eventually becomes a powerful tool for students to make sense of information, to propose conjectures, and to examine different approaches to the problems.

Since curriculum proposals have recently made explicit the need and importance for students to use several digital tools in their learning mathematical experiences, it may give the impression that it is only in the last years that educators have come to consider the role of technology within our educational systems. But it is the understanding of the nature of the role of technology in the students' learning processes what has been changing in the last years.

It is relevant then, to have a long term perspective in order to gauge the role that digital technologies can play in 21st century education. Many researchers in Math Education have already taken a lead in this direction (see, for instance, Guin, Ruthven, & Trouche, 2005), opening a window to newer research and understanding. This is a direction we want to explore in the following pages. First, students will speak for themselves. Later, we will provide some faithful descriptions of mathematical activities developed in the classroom and finally we will explain, in general terms, a national project intended to provide a rich technological environment to probe the systemic impact of the digital technologies in educational systems. This is the purpose of the next sections.

STUDENTS' CONSTRUCTION OF MATHEMATICAL RELATIONS

First dialogue: The central angle theorem

We introduce a task aimed at examining the mediating role of the calculator (TI-92 plus, or Voyage200) during the students' construction of coherent oral expression of knowledge while dealing with geometric activities (Manouchehri et al., 1998, p. 437). Geometry software for calculators has usability limitations because the calculator's screen resolution and speed can make difficult the construction of objects in that environment. Nevertheless, it might enhance the cognitive activity of its users. We also have explored ways in which object manipulation and dragging help students discover the invariant properties of a geometric object. We have videotaped high school students' dialogues (15–17 olds), working with a graphic calculators (GC) TI-92 while interacting with their instructor. We now first present some of this work that involves students' understanding and development of the central angle theorem.

Looking at Figure 14.1, the teacher asks the students to do the following:

Teacher: Drag point B to the right and then to the left along the arc, but look carefully and try to answer these questions:
1) Does angle B change when you move point B along the arc?
Take point A and drag it to the left and to the right. Does angle B change?
2) Take point C and drag it to the left and to the right. Does angle B change?

Before the students started to complete the task, the instructor questioned them about their previous knowledge on the subject. Only one student knew that the angle B, in Figure 14.1, remains constant as long as one does not move points A or C. Interestingly, the students did not know how to explain this behavior. None of the participants knew the central angle theorem or that a triangle inscribed in a semicircle is always right. The following are some of the participants' answers taken from the task sessions:

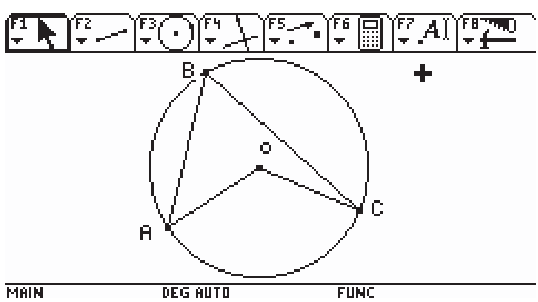


Figure 14.1 Exploring an angle behaviour.

Felipe: It looks as if the angle doesn't change even though point B is moving!
Manuel: Let me see, I can't see… maybe…
Felipe and Manuel are talking about Figure 14.1, while dragging point B to the left and to the right in their calculator screen.
Teacher: Do you think that the angle will change?

They both answered yes, and this is exactly what the rest of the group was expecting.

Teacher (addressing the entire group): Observe what is changing and what is not changing, and try to keep on doing the task. If you find something interesting for you don't hesitate to tell me.

The participants worked for 20 minutes on this task. Then, the teacher proposed the following construction and asked the corresponding questions:

Draw the segments from A and C to the center O of the circle (in Figure 14.1). Drag point A to the left and to the right; observe angle B as well as the angle formed by the segments that connect points A and C to the center O of the circle (central angle). Repeat the operation with point C.

Move point A or point C until they are collinear with O, the center of the circle. How is angle B changed?

Felipe and Manuel, moved point A until it was collinear with point C and with point O. Finally, they moved point B, showing the teacher what they were doing at every moment. The rest of the participants observed what Felipe and Manuel had found.

Felipe: It looks like when the angle in the middle is 180°, angle B is 90°!
Teacher: Why are you saying so?
Manuel: We have already tried it, and it seems that way. Look!

After 20 more minutes, nobody could further expand the argument about the rightness of the angle B. Then the teacher proposed the students to measure and label the central angle and angle B. After 15 minutes, the students called the teacher and showed him a table in a notebook, one column showing the values for angle B and the other showing the values for the central angle.

Felipe: One column is almost twice as large as the other!
Teacher: How can you express what you found?
Felipe: The center angle is two times greater than the other.
Teacher: Just like that?
Felipe: Ah ... Within a circle the center angle is two times greater than the other.

Two out of six teams continued to complete the tasks, but only one team (Felipe and Manuel) had made additional drawings and started labelling the angles. They argued that these additional drawings and labels were intended to "help them discover." The rest of the teams did not know what to do and did not propose any additional drawings.

What is the aim of these practical sessions? To document how students express and construct their arguments while trying to "prove" a theorem from the exploration of the links that exist between the different elements in the given figures. Of course, exploration and expression are possible, in enhanced ways, because of the dragging capability of the software. The central angle theorem is an *attraction pole*, a means to link circles, rays, radii, and tangents, to create a *local organization* (Moreno, 1996) from students' fragmented geometric knowledge.

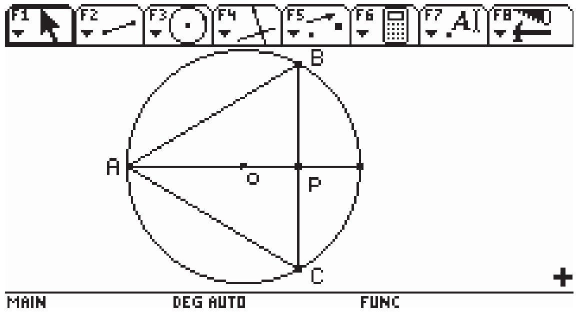


Figure 14.2 Exploring variation.

Second dialogue: Visualizing variation (1st part)

A student uses a marker and a whiteboard to make a drawing like the one shown in Figure 14.2. The instructor asks if the displacement of the chord BC, keeping it perpendicular to the diameter, changes the area of triangle ABC. The first answer is: "It does not change the area" and the argument is: "These sides (AB and AC) are shortened and this side (BC) is *thickened* (enlarged).".Students also mention that: "when the perpendicular BC gets longer, the sides AB and BC become smaller."

Subsequently, students use the graphic calculator (GC) and draw a triangle as the one shown in Figure 14.3.

Since point P is found below the center O, of the circumference, students begin their exploration dragging point P upwards, little by little. They maintain their area invariability hypothesis, stating the same compensating argument: what increases here, decreases there. Minutes later, to their surprise, they find the variation of the area. They had observed that by dragging point P upwards beyond point O, the "base" BC and the "height" AP decreases. Then, the instructor asks: When will triangle ABC reach its largest area? The students initiate a more systematized exploration and conjecture that the triangle with largest area has a diameter as a "base" and a radius as the "height," or "when point P is in the center."

As they continue the exploration and the teacher asks them to provide arguments to support their conjecture, some students ask whether they could measure the area with the calculator. But only a visual exploration without measuring the area was allowed at that moment. While one of the students uses the calculator, another observes with attention and says: "the area changes when point P moves upward from the center" but, moving downwards from the center, the area is constant." Later on, they are allowed to measure the area with the tools of the calculator. They quickly find that the largest area "seems to correspond to an equilateral triangle." Here, they were asked: "What makes you think it is equilateral?" One of them, referring to the sides of the triangle says: "At first sight, they seem to be equal." The teacher again asks for their reasons and both of them agree that the measure of the angles

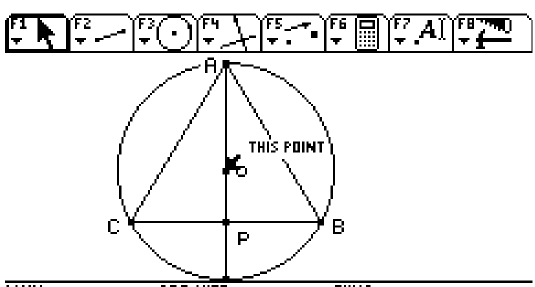


Figure 14.3 Looking for mathematical relations.

must be 60°. Using the calculator they confirm the measure of the angles and formulate their conjecture: "An equilateral triangle will have a maximum area whenever it is inscribed in a circumference."

While analyzing the corresponding video, we considered three moments:

1. When the problem was first posed and until students got the perception of the variation of the area.
2. When students' conjecture was that the largest area corresponds to a right triangle ("base equals diameter"), and
3. When they discovered that the equilateral triangle has the largest area.

Since the first moment, one of the students refuses to accept the area variation. Another has doubts, but once a partner tells him about the compensating argument "what increases here, decreases there," he becomes convinced. Because they have not measured the sides yet, they focus on the drawing of the whiteboard to wonder about the question. The orientation of the triangle they build up (Figure 14.2) does not help initially to respond quickly. Although they draw two triangles in the circle, they do not perceive its variation. This representation, the drawing on the whiteboard, shows its limitations. In order to grasp the variation, the orientation of the drawing was important (Figure 14.3) as well as the dragging, one of Cabri's capabilities. For quite a long time, one of the students was not able to perceive the variation, when point P is below point O. Nevertheless, he accepts that the area diminishes if point P moves upwards from the center but he maintains the invariability of the area when point P moves downwards from the center. His justification: "The height increases but the base decreases." When they drag downward point P, from the uppermost part of the construction, they observe that the "base" and the "height" increase at the same time and they easily accept that the area is changing. It is important to mention that since that moment, one of the students perceives the variation of the area whenever point P is over the diameter. However, both of them perceive, wrongly, that the triangle of larger area is "the one of longer base" (a diameter) and "height" (a radius), "when point P is above point O."

It was necessary, for these students, to use the measuring tools of Cabri to conjecture that the triangle of largest area is equilateral. For one of them, the measurement was strictly necessary to accept the variation of the area. In a later interview, one of the students said "of course, the calculator was decisive, without it I was not able to see that the equilateral triangle was the one of largest area," and besides, it was easier for him to perceive the variation of the area with the construction of Figure 14.2 because "I saw clearly how the 'base' and 'height' of the triangle changed. He was asked why he had built up the drawing with the calculator in that way (Figure 14.3) if they had it on the whiteboard, although with another orientation. His answer was: "With the base parallel to the horizontal it looks easier."

While trying to understand the students' perception of variation, we found that as a possible explanation that the compensating argument "what increases here, decreases there" is a concept deeply rooted in most high school students (aged 15–l7). How is student perception affected by the mediation of the calculator? Several students observed that working with the handheld calculator (GC) offered the chance to "drag the triangle." Continuously dragging the triangle creates the feeling of manipulation of the object (it is a kind of new realism) and of motion in real time. It is the succession of images taken as a continuous transformation that offers the student the chance to perceive the variation. Then, he/she offers the compensating argument. On the other hand, the action of the student is not limited to manipulating the object through the tool and seeing the screen. It seems that students "see" the variation as such because they generalizes the succession of images, and integrates them into a scheme. Perceiving variation is then a scheme of generalization. That is, the complex process involved in examining and quantifying variation properties embedded in mathematical objects seems to be facilitated with the use of technology (Mitchelmore, 1993; Sriraman, 2004).

Perception depends on movement. Afterwards, many participants improved their perception by visualizing some essential relationships in the drawings shown by the calculator. This is very likely to mean that the dragging effect is directly responsible for this improvement in the perceptive apprehension of the drawings.

When discovering a relationship in the drawings, some students declare it as a statement with certain formal structure, expressed in the language of the tool. The word "dragging" is common among students, and expressions such as "enter F2," "mark the angle," and others, reveal the influence of the tool.

What do calculator users see on the screen? The user sees what changes, sees the transformations of a drawing through time. That illusion of continuity is not only related to different images, but also to temporal space relationships that give a new meaning to the drawing. We can say that this kind of technology provides a new way of representation, executable representations, with clear potential for new learning.

It is a fact that students and teachers do not see the same in diagrams or drawings. It is very common that what students see in the drawings and diagrams, is not enough to understand what these drawings represent.

Dynamic geometry opens the access to the direct manipulation of the executable representations and provides the opportunity to displace these representations all over the screen. This capability can be used to visualize structural properties to the geometric figures. With the tool of dynamic geometry in the calculator, the users have the opportunity to carry out actions that are rather unfeasible with ruler and compass or other traditional tools. If the structural relationships of a figure are eventually visualized, then the student is empowered for formulating coherent statements such as definitions and theorems with the language provided by the calculator, at least. That is to say that we have two registers of representation (Duval, 2004), the drawing and verbal language, to articulate and interpret the results of geometric cognition as mediated by the calculator.

Third dialogue: Variation, variables and semiotic mediation in a dynamical environment

We continue our study of variation and variables in a dynamic geometry environment. As we have seen formerly, dragging is a tool well adapted to explore variation and change. It becomes important as well to develop a symbolic level that opens the window to a genuine understanding of these instrumental concepts.

Students will be able to deal with perceptual and cognitive difficulties as dragging allows to identify altogether the elements that change in a Cabri-figure as well as those that remain invariant while a figure is being dragged. In the context of geometry instruction, this facility has been used as a tool to distinguish between a drawing and the corresponding geometrical object—the deep geometric structure corresponding to what you see on the screen.

The immediacy of perception makes one believe that this is a universal and homogeneous ability that is installed ready-to-use to help us process information whatever the environment. Yet, this is not so: One can recognize a familiar face among others, but decoding a functional graphic, for instance, poses a higher cognitive demand on students (Duval, 2004). Recognizing variation is an even harder problem. A research reporting students' earlier difficulties with problems involving variation in a dynamic environment, is introduced in Moreno-Santillan (2002, 2004). We will now review new findings on these research themes. Excerpts from two interviews will exhibit some of the cognitive difficulties students face when dealing with these problems.

The interview

Some interesting cognitive behavior was made apparent in the interview with Estella, a 16-year- old-student. She is a bright student with a good working knowledge in mathematics.

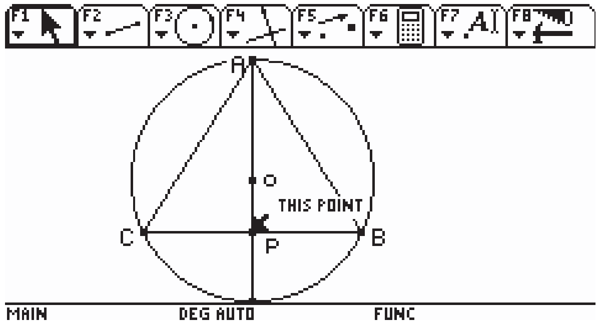


Figure 14.4 Searching for the largest area.

The instructor poses the (former) question concerning the largest area of the triangle in these terms (see Figure 14.4):

> An isosceles triangle is inscribed in a circle; one of the triangle's vertices A, coincides with an endpoint of a diameter and the remaining two vertices B, C, are the endpoints of a chord BC that is perpendicular to this diameter.

The instructor makes clear that the point P can be moved along the diameter. When the point P moves along this diameter, the chord BC also is moved and remains perpendicular to the diameter. In the following dialogue, we will keep close to the student's vocabulary during the interview.

Instructor: When I move point P (under the above constraints) what happens to the triangle? Does its area change? Does it remain the same?

Estela: The shape changes…it seems that the triangle remains as isosceles…but its area is the same.

Instructor: Can you explain?

Estela: Because here (pointing to the sides AB and AC) we are making the triangle smaller and here (indicating BC) we are making it larger…

The instructor continues, asking if there is a position of point P wherein the area changes, where the area is smaller or larger, and the student's response is:

Estela: No, the area remains the same…(Then she draws, on the slate, a triangle with BC very short)**…**for instance in this triangle, BC is smaller but AB and AC are much larger…(Figure 14.5)

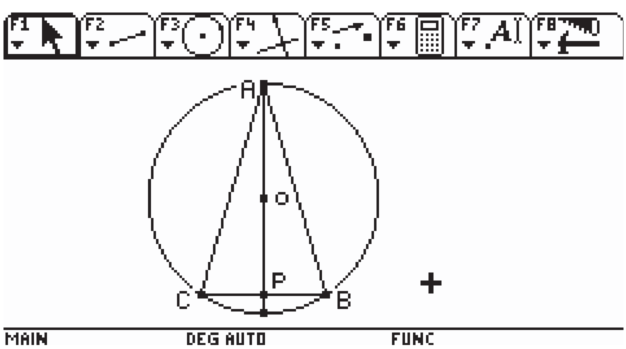


Figure 14.5 Exploring area changes.

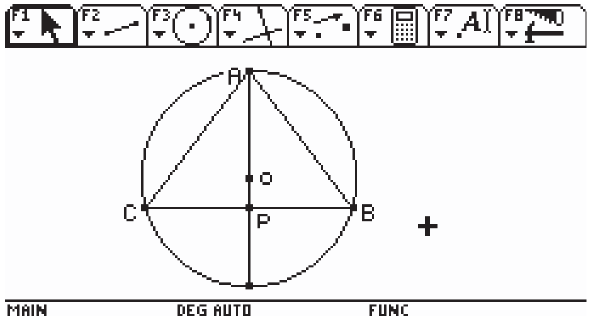


Figure 14.6 Looking for conditions of area changes.

The instructor asks Estella to construct the figure on the screen (students were using a TI-92 calculator). Afterwards, she begins displacing point P (upward) along the diameter without reaching the center O of the circle.

Instructor: Does the area change?
Estela: No…it does not.
Instructor: Why?
Before answering, Estela displaces P, very slowly, without reaching O.
Estela: If the side BC grows, the sides AC and AB are decreasing…(Figure 14.6).
Instructor: The area of the triangle…does it ever change?
Estela: No, because…

Then, Estela moves P towards O and, *for the first time during the interview*, P goes beyond O until almost touching A, and then drags P back beyond O. *At that very moment she discovers something new*, and says:

Estela: When I drag P beyond O, AB and AC gets shorter and BC also gets shorter…(Estela seems concerned and she drags P again. It is clear that something is disturbing her, attracting her attention; Figure 14.7.)
Instructor: What is going on?
Estela: I was looking at the area…it…it decreases, the area decreases!
Instructor: When does it decrease?
Looking at the slate she says:
Estela: If I drag P beyond O, the area is smaller.

After the interview, Estela remarked: "I discovered that the area would decrease because, if the base BC and the sides AB and AC are all smaller (than previously), then the area has to

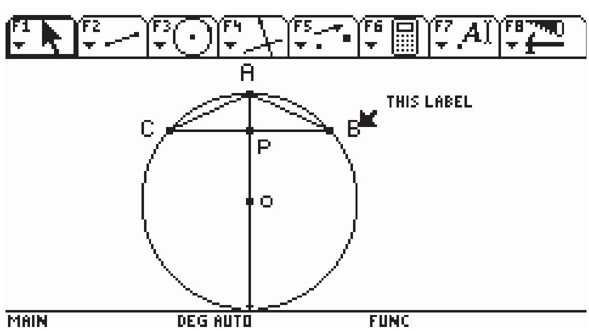


Figure 14.7 Dealing with a cognitive conflict.

be smaller. At the same time, if I drag P from A, without reaching O, the BC gets larger, AB and AC also gets larger, so the area grows." Estela has seen the simultaneous variation of three magnitudes when she displaces P along the diameter. Yet what she observes on the screen is not the variation of the area itself but an *index* (Deacon, 1997, p. 71) of this variation, a signal that engenders an inference: *the area becomes smaller if the sides become shorter*.

What is an index? Let us suggest an answer inspired by Deacon (1997, pp. 59–68): When a bird sings, an adult, by the side of a child, will indicate the bird and will utter: "bird." By pointing the bird out, the adult creates a level of reference––an *indexical* level of reference. Next time the child hears the song of the bird, she will utter 'bird" even *without seeing the bird*. She has established a connection between the song and the word "bird."

Previously, we said that Estella observes on the screen an *index of variation*. In other words, she observes a signal that engenders an inference. This inference is a first step to understanding the variation of area. Here is where a perceptual phenomenon becomes an *act of visualization*, that is, perception controlled by an interpretation.

We hope these students' dialogues provide some elements to establish that within a digital tool as the one selected here, Cabri-Geometre, there are elements that can be used to explore variation and variability with a specific intentionality to help pave the way towards their symbolic representation. In our study, the variable is an abstraction that generalizes a visual dynamic pattern and that finds its objectivity in the symbol.

In the following pages, we will present a more structural version of the work in the classroom. That is, we will describe classroom work from the viewpoint of *knowledge appropriation* by the students.

FORMULATING QUESTIONS AND THE USE OF TECHNOLOGY

We present students' problem-solving approaches that emerged from implementing learning activities in high school classes where students were encouraged to work as a learning community in which they valued not only personal contributions, but also their participation as a group. Students' engagement in processes of inquiring and explaining became the key ingredient while working with the tasks.

In this study, 16 Grade 12 students participated in a 4-week seminar that included two sessions per week (2.5 hours each session). The purpose was to ask students to employ dynamic software (Cabri) to solve mathematical tasks initially provided by the instructor. Later, the same students were encouraged to propose their own tasks or problems. During the first two sessions, the instructor gave a general introduction to the use of the software and illustrated the use of some commands to the whole class. In general, a student worked individually first, later in small groups of four members, and at the end of each session there was a general discussion with the whole class. Students could also exchange files and receive feedback from other participants. For the analysis of the students' work, we have chosen a task that was proposed by a small group. This task was solved during the last two sessions of the seminar. Throughout the analysis, we attached some comments or observations to describe particular students' behavior that appeared during the problem solving sessions; however, there is no attempt to show a detailed analysis of transcripts of their work. Instead, we identify a set of observations that illustrate mathematical relationships that emerged from students' interaction with the task. The teacher' participation played an important role in orienting (asking questions) the students' discussion that eventually led them to propose and examine those relationships.

Origin of the task

An important activity that appeared during the sessions was to ask students to formulate their own questions or problems. So, during the students' interaction with tasks or situations, they were free to explore connections or change original statements to examine and document the behavior of other relationships. A member of a small group mentioned that in order to formulate questions, it was important to identify basic properties embedded in representations of mathematical objects. For example, what do we know about rectangles? They have four sides (two pairs of parallel sides, perpendicular sides, four right angles, two diagonals, one center (diagonal intersection), and attributes such as areas perimeters and include pair of congruent right triangles (Pythagorean Theorem). Indeed, students agreed that in order to represent a task via the dynamic software, it was important to think of all figures in terms of properties and then select proper commands to represent that figure or mathematical object.

Can we construct a rectangle if we know only its perimeter and one of its diagonal?

This question was posed by one student to the whole class. Three different students' approaches emerged from the students' work in this task. Although in all of them the use of technology appeared to be relevant, we focused on identifying two approaches in which the software functioned as powerful tool not only in achieving the solution but also in exploring other geometric properties of the problem representations.

Solution process shown by three small groups

Students were aware that the process of solving a problem involves examining the statement from distinct perspectives. How can I represent the perimeter geometrically? What information does the perimeter provide about the sides of the rectangle? How the perimeter information is related to the diagonal? These were some of the initial questions discussed within a small group that eventually led students to represent basic information and use the software dynamic to connect such information. The important stages were:

1. Students represented the semi-perimeter as segment AB and chose point Q on it. That is, a + b is segment AB where a & b are sides of a rectangle. With this information they constructed the corresponding rectangle EHGF (figure 14.8). Here a = AQ = EH and b = QB = HG.
2. Students realized that by moving point Q along segment AB, a family of rectangles with a fixed perimeter was generated. They determined the locus of point G when point Q is moved along AB (Figure 14.9).
3. Students realized that the rectangle could be drawn in two different positions except when the rectangle became a square (Figure 14.9). That is, the dynamic representation of

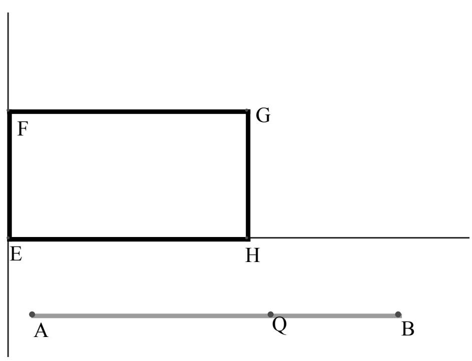


Figure 14.8 Constructing a dynamic representation of the rectangle.

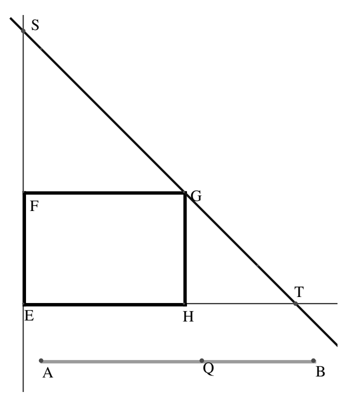


Figure 14.9 Identifying a family of rectangles.

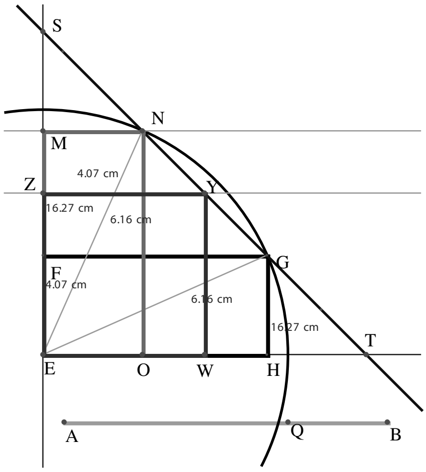
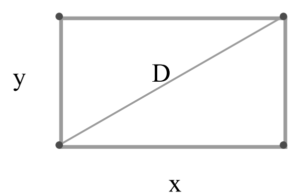


Figure 14.10 Drawing the rectangle.

the rectangle offered students the opportunity of exploring some properties of the figure directly.

Another approach shown by two small groups was to focus on the algebraic representation of the situation. That is, they decided to use x and y for sides of the possible rectangle and wrote the two corresponding equations:

$$y + x = \frac{p}{2}$$
$$x^2 + y^2 = D^2$$



Here, a student suggested to graph both equations, he mentioned that since P and D were given numbers, then the first equation represented a line and the second a circle (Figure 14.11).

Eventually, the graph became a referent for students to explain the existence of asked rectangle (the circle might intersect the line in one point, in the case of the square, in two points, and in the above figure, and no intersection points). The other small group that followed this approach gave an algebraic explanation regarding the solution of the system of equation.

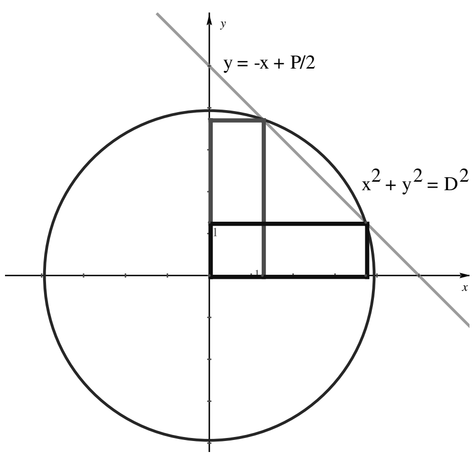


Figure 14.11 Graphing the algebraic representations.

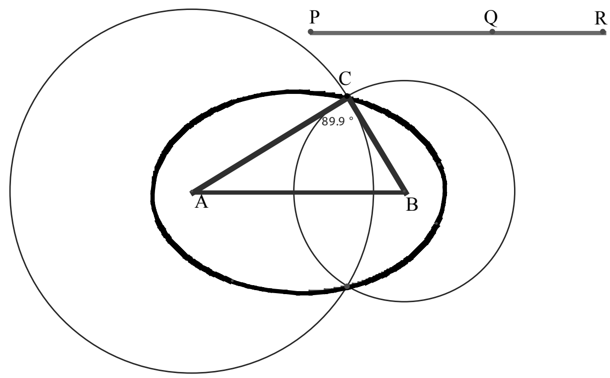


Figure 14.12 Relating the construction of the rectangle with an ellipse.

Yet another students' approach was to construct a family of triangles with perimeter equal to the sum of two sides of the rectangle plus the length of the diagonal. Here, they chose the given diagonal as a fixed side of the triangle and the other two sides of the triangle as the semi-perimeter of the rectangle. The software became a powerful tool to find the family of triangles with fixed perimeter (Figure 14.12).

On Figure 14.12, segment AB represents the given diagonal and segment PR is the semi-perimeter. Students drew two circles, one with center on A and radius PQ and other with center on B and radius QR. These two circles get intercepted in C. The locus of point C when Q is moved along PR is an ellipse (foci A & B and constant PR). Here, students focused on finding the triangle with angle ACB a right angle. To find it, they drew a circle with center, middle point of the diagonal AB, and radius half of the diagonal. The intersection of the ellipse and the circle will determine the vertex of the right triangle. Here, they also noted that there were cases in which angle ACB never became a right angle. In this case, it is observed that there is no intersection between the circle and the ellipse.

When these solutions were discussed within the whole class, it was evident that students realized that the use of the software provided a means to explore the task from diverse angles and perspectives. In particular, they were surprised that a variety of mathematical resources and ideas were present in each approach, which would have been impossible in the absence of technology. At the end of the session, a student asked: Can we construct a rectangle if we have its diagonal and its area instead of its perimeter? Here, students again were ready to explore this question through the use of the software.

Students showed during their interaction with the task different mathematical resources, representations, and strategies that allowed them to recognize strengths and limitations of their approaches to the problem. For example, the dynamic approach in which students focused on finding the locus of the fourth vertex provided enough information for them to identify all rectangles with fixed perimeter. Indeed, this representation helped them visualize that the locus of the fourth vertex cuts the axis at S and T to form a right triangle ETS and any rectangle inscribed in that triangle has the same perimeter (Figure 14.9). Here, they introduced the diagonal information to find the asked rectangle. The students' algebraic approach relied on a static representation in which they basically represented a particular case and discussed other possibilities of behavior of the two graphs in terms of the graphs intersection. The third approach in which students decided to construct a family of triangles with a fixed perimeter combines both a partial representation of the rectangle, that is a right triangle, and the power of technology to find all of them with a fixed base (the diagonal). When students moved points, found diverse loci, assigned measurements, and formulated and supported conjectures, it was clear that the software became a powerful mathematical tool for the students.

CONJECTURES AND THE USE OF DYNAMIC SOFTWARE

A second study focused on documenting high school students' use of dynamic software to identify and support mathematical conjectures or relationships. We are interested in showing students' approaches in which the use of the software helped them think of distinct arguments to support their conjectures.

> School mathematics should be viewed as a human activity that reflects the work of mathematicians –finding out why given techniques work, inventing new techniques, justifying assertions, and so forth. It should also reflect how users of mathematics investigate a problem situation, decide on variables, decide on ways to quantify and related the variables, carry out calculations, make predictions, and verify the utility of the predictions. (Romberg & Kaput, 1999, p. 5)

In this context, it is important to describe features of mathematical practice and instruction that framed the development of this study.

Learning mathematics involves a process in which students are encouraged to reflect on ways to connect or apply their knowledge. What does it mean to define something? Why definitions are important during the study of mathematics? How could I represent a particular phenomenon? What variables are important to identify or analyze particular relationship? How can I quantify variables? Is there any other way to solve the problem? How can the problem be extended? These are examples of questions that students should discuss when understanding mathematical ideas or solving mathematical problems. As Harel (in press) stated, "Problem solving is not just a goal but also the means—the only means—for learning mathematics. Learning grows only out of problems intrinsic to the students, those which pose an intellectual need to them."

A way to promote students mathematical reflection is to provide an instructional environment where they have the opportunity to participate in the construction of mathematical relationships. Thus, the instructional principles that shaped the development of the students' learning activities were based on what Hiebert et al (1997) called classroom dimensions. Here, the classroom is seen as a community of learners in which all the members share values and principles to understand mathematical ideas and solve distinct tasks or problems. In addition, the recognition that the student's use of tools mediates the ways how they think of the task or problem. Different tools offer distinct opportunities for students to work on problems.

> [Tools] can provide a convenient record of something already achieved…They can be used to communicate more effectively… and they can be used as an aid for thinking. (p. 10)

Thus, during the development of the sessions, the participants worked on the problems individually and in small groups. In general, the instructional activities were organized around a particular pedagogic approach in which the participants were encouraged to use an inquiry process to deal with the problems. Thus, the development of the sessions consistently showed the following structure:

1. Participants worked, initially, on the task individually and, later, in groups of two or three for some time. The role of the teacher was to monitor the participants' work and help them clarify (via questions) the statement of the task or encourage them to formulate questions while representing and exploring approaches to the problem. Each participant and the small groups handed in a written report showing their attempts or approaches to the task.
2. Small groups presented their work to the whole class. During each small group presentation, the rest of the group, including the teacher, asked questions or demanded explanations from students about what might not have clear or needed some elaboration

from the presenters. It was common that small groups' approaches showed distinct paths to represent a particular problem, and, during the presentation, it became important to discuss visible concepts that were relevant to solving the tasks.

3. The teacher identified strengths and limitations associated with each small group's presentation and discussed within the whole group mathematical ideas, strategies, concepts, and distinct representations that were relevant in the students' solution to the task. In addition, the teacher could have introduced a new concept, posed other questions, or analyzed extensions or possible connection of the original statement of the task or problem.
4. Students were asked to revise individually what it had been done during the process of dealing with the initial task. Here, each participant had the opportunity to incorporate new ideas, concepts, or strategies that he/she had judged to be relevant during the development of each session. As a result, students elaborated a written report in which they communicated systematically their ideas, resources, and strategies used to deal with the task.

In general, the instructional principle around this approach is the conceptualization of mathematical learning as an inquiry process that students need to engage in order to comprehend, reveal and contrast mathematical ideas, and present distinct arguments to communicate results.

Results

To describe what emerged or transpired during the development of the problem solving sessions, we decided to focus on presenting mathematical features that we identified as crucial during the students' interaction with the problems. Since we are not offering a detailed analysis of the students' performance, we decided to:

1. Identify distinct types of representations that students constructed and utilized during the solution of the activity;
2. Comment on mathematical properties that became transparent in using dynamic, numeric, and algebraic (including the use of derivative techniques) approaches to deal with the problem, and
3. Recognize the need and importance for the participants to examine those representations of the problems achieved through the use of technology and those that usually appear with the use of paper and pencil.

We document results about problem-solving approaches rather than problem solvers since our examples are taken from both students working individually, in small groups, and in class discussions.

Formulation of conjectures and their validation

With the use of dynamic software such as Cabri-Geometry or Sketchpad, students could construct dynamic representations of mathematical objects or problems. In this context, we have observed that students tend to use the software to first quantify attributes (lengths of segments, perimeters, angles, areas, etc.) and observe their behaviors or loci as a result of moving particular elements within the dynamic configuration. This process often leads the students to identify a particular conjecture of relationships that they need to support and communicate in terms of mathematical resources. As a result, students consistently examine the viability and pertinence of a particular conjecture or relationship in terms of using the software to (1) identify the conjecture visually, (2) examine whether the conjecture falls within a family of isomorphic objects (dragging test), (3) construct a macro that reproduces the construction and verify whether the conjecture was held in objects generated by the macro, (4) quantify and verify properties of mathematical objects to detect patterns, and (5) present formal arguments to prove the emerging conjecture. An example illustrates the way that students deal with a conjecture.

The problem: Cross' theorem

Squares are drawn on the three sides of a triangle. Show that the areas of the four shaded triangles are the same (Figure 14.13).

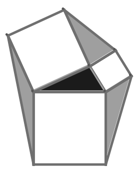


Figure 14.13 Are the triangles' shaded areas the same?

Students, with the help of the software, represented this problem dynamically. This representation was useful not only to "verify" the triangles' areas relationship, but also to prove the theorem and to formulate a related conjecture.

What if we draw rectangles instead of squares on each side of the given triangle, how are the shaded areas? This was the initial question that students agreed to explore. However, they noticed that they could draw many rectangles taken as the base one side on the given triangle and as a consequence it was difficult to trace the area behavior of the shaded triangles. How can we relate the construction of the three rectangles based on the sides of the given triangle? Discussing this question led them to realize that it was important to introduce another condition to draw the corresponding rectangles. The condition that they decided to take into account was that the corresponding sides of the rectangles would share the same proportion. That is, they decided that

$$\frac{EC}{CB} = \frac{GA}{AC} = \frac{IB}{BA} = \frac{1}{2}.$$

They drew the rectangles holding this condition and observed that the areas of triangles CEF, AGH and BDI were the same (Figure 14.14) and wondering if this relationship would hold for any triangle.

At this stage, the goal for students was to first investigate if for other triangles with the same construction, the area relationship was maintained and, second, to look for arguments to support the conjecture. The software became a powerful tool to explore both the plausibility of the conjecture and the search for ways to validate it. We illustrate the ways students showed to deal with this conjecture.

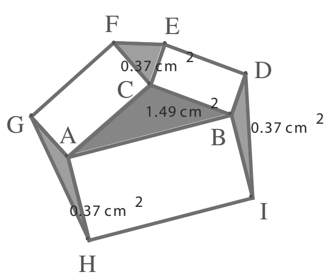


Figure 14.14 Drawing rectangles with proportional sides.

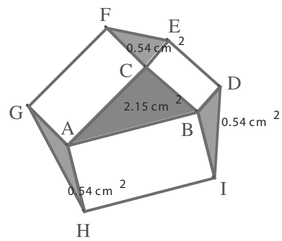


Figure 14.15 Visual recognition of a relationship in triangles CEF, AGH, and BDI.

Visual recognition

An important feature in using dynamic software is that mathematical figures can be drawn accurately. In this case, students drew triangle ABC and the corresponding rectangles (with the same constant of proportionality among their sides) and visualize that for this case, the area of triangles CEF, AGH, and BDI were the same. The initial visual conjecture is supported with the corresponding areas calculation (see Figure 14.15).

The dragging test

Here students explored the validity of the conjecture for a family of triangles. With the use of the software, they moved the position of the vertices of the given triangle ABC to generate a family of triangles with the same construction. They observed that when one vertex is moved, the family of triangles generated held that the area of triangles CEF, AGH, and BDI was the same (see Figure 14.16).

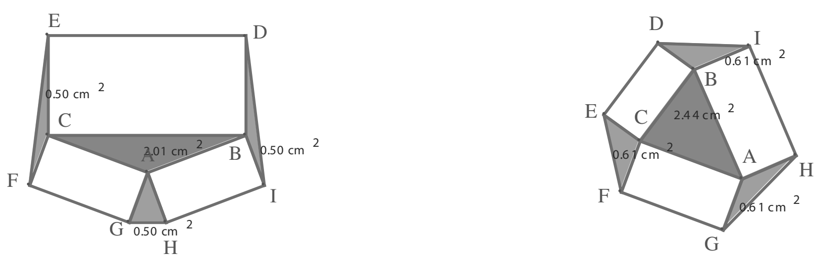


Figure 14.16 Verifying the conjecture for different positions of triangle ABC.

Constructing a macro

Another way to verify the conjecture was that students built a macro to reproduce the construction for any given triangle. That is, students identified initial objects (triangle ABC and ratio R of rectangle sides) and final objects triangles CEF, AGH, and BDI in order to reproduce the construction for any given triangle. By applying this macro to different triangles, students confirmed the conjecture, that is, in all triangles they applied that macro, they observed that triangles CEF, AGH, and BDI all have the same area. Figure 14.17 shows two of those triangles.

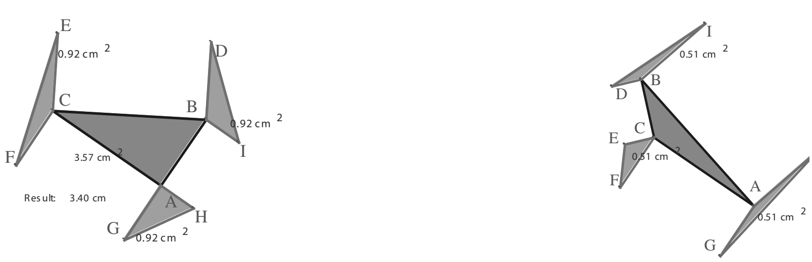


Figure 14.17 Applying a Macro to verify the conjecture.

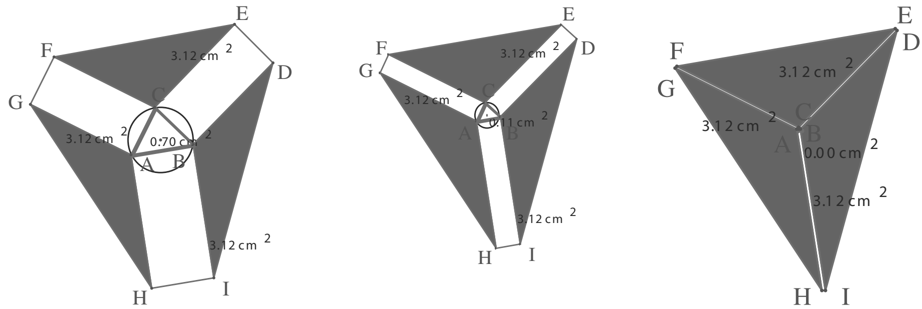


Figure 14.18 Moving vertices A, B, and C to the center of triangle ABC.

Rigid motion

Yet another students' approach to show that areas of triangles CEF, AGH, and BDI were congruent, involved moving vertices of triangle ABC to center of the triangle (intersection of the medians). Students observed that when vertices A, B, and C became the center of the triangle, then triangles CEF, AGH, and BDI shared the same area (see Figure 14.18).

Quantifying attributes and patterns

It was easy with the use of the software to quantify attributes (lengths, areas, angles, etc.) of the figure and observed their behaviors. For example, in addition to observing the behavior of the triangles areas, students focused on comparing (ration) areas of triangle CEF and triangle ABC for distinct values of the proportionality coefficient of the sides of the rectangles (see Figure 14.19). Based on this information, they notice that

$$\frac{\textit{area} \text{ of } \Delta CEF}{\textit{area} \text{ of } \Delta ABC} = r^2$$

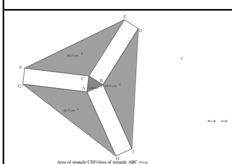
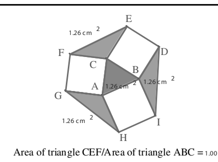
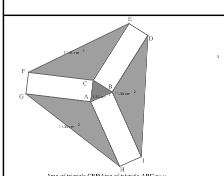
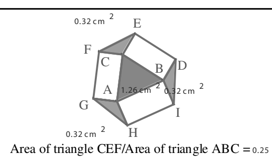
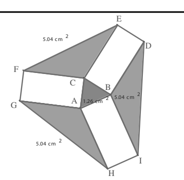
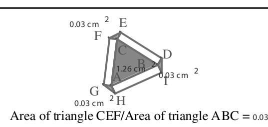
Figure	$\frac{Area\ \ CEF}{Area\ \ ABC}$	r	Figure	$\frac{Area\ \ CEF}{Area\ \ ABC}$	r
	16	4		1	1
	9	3		.25	.5
	4	2		.16	.4

Figure 14.19 Looking for patterns.

There is evidence that the use of dynamic software shaped and determined the way that students thought about the process of formulating conjectures and methods to examine and validate the conjecture. In particular, the dynamic representations of the problems and mathematical objects that students generated with the help of the software became an experience-enhancing platform to identify and explore mathematical relationships. The easiness to quantify mathematical attributes and the exploration of visual representations of problems were fundamental activities that permeated the students' process of formulation of conjectures. To assess the validity of the conjecture, students relied on the use of the software to evaluate particular cases visually, to examine family of cases by dragging particular elements within the representation, to construct a macro to also examine a family of cases, and to observe patterns that emerge as a result of exploring invariance in the behavior of particular data (changing the ratio coefficient of rectangles' sides, for example). In this context, the use of an analytical method to prove the conjecture came out naturally as a way to confirm the validity of the conjecture.

A NATIONAL TECHNOLOGY PROJECT

The burst of new technologies in education has frequently produced uncritical optimism about the possibility of transforming the foundation of educational systems. For this reason, it became necessary to challenge paradigms supporting the belief that it is through the mechanical use of these new tools that the great majority of individuals would be able to access complex and powerful mathematical notions. The access to powerful ideas has to take into account, from the start, the mediation between the technological tools and the sociocultural environment surrounding individuals and schools. The different strategies that students, teachers, and schools establish to incorporate technology depend to a great extent on the interpretative resources developed within this environment. Although we do not study in depth the relationship between culture and epistemology, it is an essential issue in education. According to Balacheff and Kaput (1996), the main impact of information technology on educational systems is epistemological and cognitive, because it has contributed to the production of a new form of realism in mathematical objects. This new form of realism depends on the interpretative resources provided by the sociocultural environment. At the same time, however, the existence and use of this technology can transform the initial interpretations derived from the sociocultural environment. Thus, technology has the power to become a sociocultural and educational agent for change but this process of change is complex.

In the proceedings of a world conference on higher education, *Higher Education in the XXI Century*, held in Paris in October 1998, attendees concluded that it is necessary to encourage research through the construction of networks that allow democratic access to knowledge and through the adaptation of new communication technologies to national requirements. Resources of productivity lie in the technology of knowledge generation, information processing, and symbol communication (Castells, 1996).

National project

In Mexico, as in many other countries, the incorporation of technology into the educational system is driven by a policy of primary importance. A number of educational plans exist that incorporate technology into classrooms. One of these is our national project, Incorporating New Technologies into School Culture: The Teaching of Mathematics in Secondary School, funded by the Ministry of Education and the National Council for Science and Technology in Mexico (Conacyt, Project 526338S). This project is aimed at:

1. Gradually incorporating various pieces of technology into the mathematics and science curricula at the secondary school level
2. Implementing the use of technology supported by a pedagogic model that allows the construction of learning environments oriented toward the improvement of mathematical education
3. Encouraging the design and use of computing environments that can improve the traditional teaching and learning methods (i.e., with paper and pencil).

The general objectives of this program are to raise education standards, to train teachers in the use of technology, and to broaden the students' opportunities of education.

Initially, the program covered 15 states (out of 32) including both rural and urban sites. The software selected for the project includes Cabri-Géomètre, spreadsheets, and algebraic calculators (TI-92 Plus). To achieve the aims mentioned above, we have had to investigate the impact of technology on the teaching and learning processes. This has several implications for the assessment and implementation of the program. For instance, *usability* problems can affect the student's achievement of educational goals—the raising of educational standards or the advantages that students gain from introducing technology into the classrooms. The reported results of this project take into account the progress made concerning the global goals we set for ourselves at the start of the project. Among the proposed goals is that of exploring the effects on the cognition of the students of the use of computational instruments into the teaching model. We will report on those results mainly from the perspective of geometry and algebra because in the development of the project, it has become apparent that these disciplines are the most promising in the context of our work, and may allow students to develop powerful ideas in the mathematical education field.

Some results from fieldwork

During the development of the project, we conducted interviews and written tests to evaluate the students mathematical learning as mediated by computing tools. These are resources to obtain information that can be used as feedback for the general management and assessment of the entire project. Following are the results obtained in interviews and written tests while working with Cabri-Géomètre. This work is done in secondary schools (12- to 16-year–olds). One of the problems that influenced the students' completion of tasks in Cabri-Géomètre was *learning to draw with the mouse*. When the teacher asked the participants to draw geometric figures, they tended to use the mouse *as a pencil metaphor*. For instance, they experienced difficulties drawing segments because they moved the mouse from left to right as if they were using a pencil. Similarly, when asked to construct a triangle they proceeded by drawing three different segments instead of selecting the *triangle* option from the menu bar. The figure they drew looked like a triangle but did not have its properties. For instance, students were amazed when they tried to assign an area to those triangles and the environment was not responding in this respect. As a result, the users experienced difficulties in understanding relationships between the triangles they had drawn and the ones they could draw using the corresponding Cabri-Géomètre menu.

We also became aware of other problems, such as the difficulty in measuring angles in the Cabri-Géomètre environment. Some children were not able to see an 89.98° angle as a 90° angle. Our school mathematical culture is one that still demands "exactness." Indeed, the use of calculators and computers offers students the possibility of gradually recognizing the importance of approximating and estimating quantities or mathematical operations or when dealing with numerical domains. However, we have to be careful with considerations of *usability* especially when the technology was not originally designed for the cultural context in which it is being incorporated.

The above findings becomes central in evaluating implementation outcomes. If usability problems, as culturally determined, are not adequately taken into account, the introduction of the technology into schools might fail to achieve its original goals. In other words, although technological tools might be adequately designed to meet specific educational goals, if students cannot use them because it is culturally inadequate, the implementation will fail. If assessments do not take into account such usability issues and contemplate exclusively educational indicators, such as student achievement, then these assessments will probably end up recommending inadequate prescriptions in many cases.

Other problems that influenced the user–task interaction was that the technology easily shifted from being an educational tool to being an educational goal. At the end of the sessions, we asked students what they have learned, and the majority answered, "to use the computer." The latter can be due to both usability and to the novelty of using computers. Nonetheless, this is an important issue because if the software is not being used as expected, then the initial educational goal will not be achieved.

According to Balacheff and Kaput (1996), design can aid the development of fluency between diverse mathematical representations, but it can also lead to the construction of misconceptions and misunderstandings. Therefore, the interface can no longer be considered as a mere superficial layer because what is involved is not mere perception but interpretation (Balacheff & Kaput, 1996, p. 475). They provide a well-documented review of existing computational technology in mathematics and describe how differences in design can affect the student's mathematical experience.

Final remarks on the projects

We have described different projects developed within the Mexican educational system at different levels of implementation. We have also described the implementation of technology that today and in the near future might impact school practices and lead to new educational developments. Finally, we have tried to exhibit the type of mathematical understanding achieved by students during the implementation of these new approaches, for instance, those mediated by algebraic calculators and computers. Let us recall our description of the idea of *access to powerful mathematics* through school culture. It mainly has to do with providing students with the opportunity of experiencing the construction of mathematical knowledge at school according to their level of development; with developing students' creativity through exploration and discussion of the different approaches that emerge from discussing questions posed within the classroom, and with developing students' individual computing techniques or procedures.

It must be emphasized that these projects have been conceived to respond to the real educational problems that are seen in our particular society.

We believe that in the decades ahead these problems, and the whole sociocultural environment from which they result, will continue in our country and in Latin American countries in general. In particular, we will continue to face the need to take care of considerable populations with high levels of dispersion, and, at the same time, we will have to respond to the problems created by the addition of new technologies to our school systems.

The characteristics of sociocultural and economic development, which we have sketched in this chapter, are widely shared among Latin America. It is from this perspective that we see great potential in the educational proposals found in our projects. These are viable projects that go beyond the particular conditions of our country. Of course, the incorporation of information technology into school systems must be gradual, and adopting a systematic approach. By this we mean that it is not just a matter of installing equipment in the absence of an educational and social project, which may lend importance to the social acceptance of these technologies. Broad social support is indispensable, and may better the quality of education and generate conditions in which new conceptual frameworks (powerful ideas) may spread to the largest possible number of schools within a country. We now consider issues related to the development of the technology project from the viewpoint of the community and school culture.

Researchers are aware that educational software is not culturally neutral (Crawford, 1990). For instance, the design of educational software incorporates the values and priorities of the designer. The designer's sociocultural environment will play a role––which could be an implicit role—when producing a piece of educational software. This is closely related to *usability issues* that we have discussed in a previous section of this chapter, and this issue is central to our project because we are using several software environments that underlie a series of powerful mathematical ideas. Many of these ideas are closely bonded to the curriculum, but others are not. The latter convey an opportunity to explore future changes that might be incorporated into the already-mentioned curriculum. Computational environments enhance students' access to powerful ideas. The feasibility of dealing with general mathematical ideas within a computational environment highlights an important feature of these tools and environments: the access to *systematization*, a true powerful idea.

The technology project has demanded a global and local level of assessment. The global level focuses on understanding the educational system as a complex one: the interactions of students, teachers, parents, and administrators all within an educational environment. The goal of this level of assessment is to regulate the educational processes taking place at school. This includes taking care of teachers' evolving conceptions, administrators and parents' new attitudes toward technology. On the other hand, the local level concentrates mainly on case studies. The latter is sought to provide useful feedback for improving dissemination and implementation, as well as to produce auditable trails of documentation that can reveal the nature of achievements.

Data from the local level of assessment such as filmed interviews with students were used to analyze the evolution of skills and specific knowledge according to the mathematics curriculum. Tasks were designed with a model of collaborative work in the classroom in mind. These tasks were implemented according to evolving lines in the different curriculum contents- for instance, from intuitive to exploratory dynamic geometry.

Different pieces of software (Excel, Cabri-Géomètre, SimCalc) have been used at different sites, and the calculator is being used at every site. Pupils collaborate in pairs and small groups of three when working in front of the computer. When using the calculator, they work individually but also in groups. Teachers have noted that when students share a calculator, they often seemed to have an advantage. In addition, when two students each have a calculator but work together, they generate a variety of approaches and also discuss their work. The teacher's role consisted of (1) providing support to students as they worked out the activities described in the worksheets, and (2) organizing collective discussions to enhance individual experiences and problem-solving abilities. In addition to being a mediator during the classroom activities, the teacher is also a mediator between students and the tools as the students learn to use them.

The focus of this work is to cast light on the role of computing tools as shapers of school mathematical culture. The results discussed here provide evidence of the impact of learning environments on the ways in which children express their mathematical thinking. This is in part due to the close interaction occurring between the students *and* the tools. For instance, while working with the calculator, students can enter a formula and observe results of the calculations carried out with that formula. The student becomes aware of a broad generality expressed by the formula instead of looking only at the symbolic manipulation. In this fashion, he or she is introduced to a powerful mathematical idea. Many researchers participating in this project have observed this trend during the development and implementation of diverse activities.

We can add some remarks from the global assessment perspective:

1. Parents value technology because it brings better career opportunities to their children.
2. Teachers point out that technology helps build a new learning milieu within the classroom in which new strategies for problem solving and new ways of introducing teaching materials can emerge.

Also from a global perspective, the project tries to answer questions such as the following:

1. What new insights are productive teachers developing?
2. Are the teachers and parents' expectations evolving together with the project?
3. Is the evolution of values manifested accordingly to regional cultures?

Teachers clearly do not want their involvement in the project to be determined by the whims of elected political representatives; they want to ensure a continuing participation in it.

When the teachers were asked about their perceptions of the quality of students' learning, they said that they were pleased that their students were more interested in mathematics. Students were learning to reason and had become more sensitive to the introduction of mathematical ideas before they dealt with them in the normal classroom.

Teachers play a central role in helping students assimilate what they know. Teachers seem very comfortable with technology now and seem to be more worried about other issues within the project, such as student assessment and student commitment (Lesh, personal communication). *Now is the time to provide teachers with the tools to consider and promote new ways of learning, not only as an internal process but also as a social event.*

REFLECTIONS ON DIGITAL TOOLS AND ENVIRONMENTS

Our field is at the crossroad of science, mathematics, and educational practices. The interests of the corresponding community include working with people whose learning takes place at schools and studying the corresponding institutions that provide the intellectual conditions to learn. The urgency of teaching and learning, so distant from the research activities, has resulted in *practices without corresponding theories.* We must make clear that we are not dismissing the considerable and important results the community has produced. We simply want to underline that institutional pressures can result, more frequently than one would like, in the derailing of research goals.

The presence and use of digital technologies have introduced new ways of looking at mathematical cognition and, as a consequence, the tools used offer the potentiality to reshape the goals of our entire research field. Nevertheless, the tension between the local and the global mathematical approaches emerges again, and we have come to think that presently, only local explanations are possible in our field. *Local theories* might be the answer to the plethora of explanations we encounter and deal with. But even if local, a mathematics education theory must be developed from a scaffolding (perhaps a field of pragmatic evidences) that eventually crystallizes in the theory under construction.

With the new digital tools at hand, students are capable of producing a formulation associated with their explorations and express it in the language of the digital medium in which they are working. In this sense, the computing environment becomes an *abstraction domain* (see, for example, Moreno & Sriraman, 2006), which can be understood as a scenario in which students can make it possible for their informal ideas to begin coordinating with their more formalized fragments of knowledge on a particular subject. Let us give a simple example to illustrate the meaning of this idea. A father, for his son's 15th birthday, presents him with a beautiful table and explains:

> Whenever you act in an incorrect way, you should introduce a nail in your table. And when you correct your mistake you should extract the nail. Following this rule, you will have a precise idea of your ethical behavior, late in your life.

Knowledge is *situated* knowledge. The father has used this story to explain to his son the meaning of the abstract concept of ethical behavior. The story becomes an abstraction domain wherein *the general lives in the particular*.

This happens, for instance, when students are working within a dynamic geometry activity and one of them wants to explain to her fellows how to build the perpendicular bisector of a given segment. In our work, we have witnessed how students use expressions like "open F4 then press 4...then...." or "use compass" or "use perpendicular bisector." These expressions, adopted from the various software menus, become part of a "language" that refers to practical activities within the dynamic environment. The tool and the geometry environment become an abstraction domain, an example of material intelligence that helps students to understand generality and systematization from a concrete form of embodiment. In the case of the study of variation we have previously introduced, the dragging action worked as a situated abstraction device for students to shape their ideas by means of an exploration (made feasible by dragging) *on* geometrical figures. The abstract concept, variation, arising out of such exploration is still *linked* to the environment: The student can talk of its general properties but only (at an early stage of conceptual development, at least) by means of the language supplied by the environment. This is part, as well, of the process of *socialization* of the tool. Needless to say, this is crucial in the classroom.

These ideas are closely related to the idea of *situated theorem* (Moreno & Sriraman, 2005). For instance, students might produce some observations *situated* within the computational environment they are exploring, and they could be able to systematize their observations by means of tools and activities proper to the digital environment at hand. That happens, for instance, when the students try to invalidate by dragging a property of a geometric figure but they cannot. That property, then, becomes a (situated) theorem expressed via the tools and whose meaning is intimately linked with the dynamical environment. Goldin and Kaput (1996) had already commented on the important distinction between *abstract mathematical reasoning* using formal notations (that interact meaning-fully with other kinds of cognitive representation), and symbol manipulation that is merely *de-contextualized*.

Today, there is substantial evidence that the encounter of the mind with *distributed cultural systems* has altered human cognition and has changed the tools with which we think. The origins of writing and how writing, as a technology, changed human cognition is key from this perspective (Ong, 1999). This suggests the importance of studying the evolution of mathematical systems of representation as a vehicle to develop a proper epistemological perspective for mathematics education. For instance, Goldin (1998, 2003) has proposed a model for mathematical learning that is based on the different types of representations one invokes when engaged with a problem in mathematics.

There is an attribute of *executable representations* on which we want to focus: They serve to externalize certain cognitive functions that formerly were executed only by people. That is the case, for instance, with the exploration of a geometrical object made feasible by dragging. Another significant example is provided by zooming into the graph of a function. These immediate manipulation on mathematical objects have been generating a new mathematical realism that lowers the cognitive burden on the student making explicit its educational value. Let us illustrate this point with an experience in the classroom that speaks for itself. Zooming in on the graph of a function results in straightening the graph in a small interval. In other words, applying the zoom can be seen as *taking the derivative* in the graphical register of the function (Duval 2004; Tall 1996, p. 310).

At this point, we considered the possibility that the didactic virtues of a cognitive conflict could promote the students' levels of conceptualization (i.e., they could generate a powerful

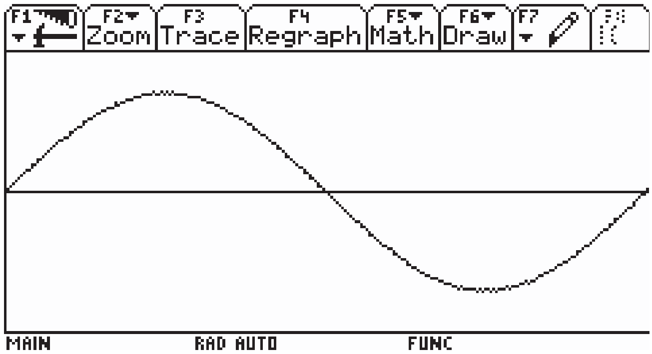


Figure 14.20 Graphic representation.

idea) with regard to the graphing of functions through the resources of the TI-92 calculator, for example. This manifested itself when we presented the students with the function

$$Y = \sin(x) + (\sin(100x) / 100).$$

Zooming in on any point of the graph of this function (see Figure 14.20) causes unexpected behavior: the new function graph reveals an oscillatory behavior that was hidden in the initial graph.

There are many things that become clear through this exploration (see Figure 14.21). First, this is not possible without the help of the computational resources at our disposal. Second, it allows the relationships between the graphing and the screen's resolution to be systematized, in this case, for the TI-92 calculator. This is equal to the achievement of a powerful idea, which includes the understanding of the screen as a representational space.

Weierstrass function,

$$\Sigma(2/3)^n \cos(9^n \pi x),\ n \geq 0$$

marked a milestone in the development of mathematical analysis, because it is a continuous but nondifferentiable function. We do not present our students with formal proof of the theorem. Nevertheless, we can use Von Koch construction (Moreno & Sriraman, 2005) rather, as a domain of abstraction. The digital environment is particularly well adapted to the purpose of providing meaning to Von Koch construction of the new object that appeared pathological in the mathematical landscape of his time. We were interested in seeing kinds of statements presented by the students in discussing the process of graphing.

The idea of considering the effects of the screen resolution on the graph turned out to be powerful (see Figure 14.22). Through the instrumentalization of this idea, students could

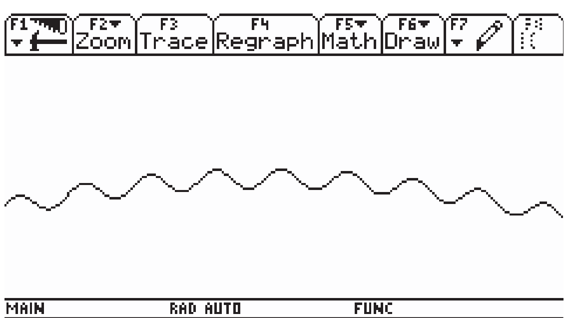


Figure 14.21 Zooming around the maximum.

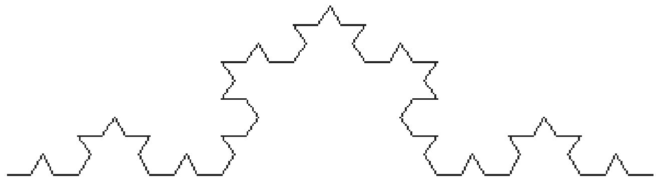


Figure 14.22 Von Koch construction.

discover the degree of complexity of the function, perhaps only from a visual-dynamical point of view, but even this objective is worthwhile because it opens a window into a mathematical world with the potential to enhance understanding beyond the curriculum. The object built on the screen is very manipulable opening the door to what Balacheff and Kaput (1996) called a "new mathematical realism," as well as connecting mathematical objects with actions performed on the objects. The computer adds a new system of representation that has an additional virtue: being executable.

Today, mathematical culture has evolved and those curves that were seen as nonobjects, are emblematic in the world of fractals. There are different cognitive demands according to the tools we use to accomplish a task. If we draw a circle using the border of a circular object as a guide, we obtain some valuable information on the control we have to practice with our hand. So obtained, the information can be understood to result from the mediational role of the tool (the border of the circular object we used). Drawing within a computational environment set forth a different cognitive demand for students. The nature of the mediational tools applied in each case support this assertion. There is a considerable amount of research on this topic. Recently, Chassapis (1999) discussed the mediation of tools in the development of the concept of a circle. He suggested that human actions and thought are different when students work with a compass than when they work with tracers and templates. From the first moment students access the tools as instruments to enhance their expressive power, after considerable work with the mediating help from teachers, they might enter the higher level of reconceptualization.

With computer explorations, we can associate the notion of a *situated theorem* when the tools employed become visible as part of the expression (Moreno & Sriraman, 2005, Moreno & Sacristán, 1995, 2003). This means that students can develop an ability to state general propositions in the language of the environment. A powerful idea for designing any educational environment for the 21st century comes from considering that digital environments derive their educational power from their ability to manipulate and externalize abstract ideas.

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