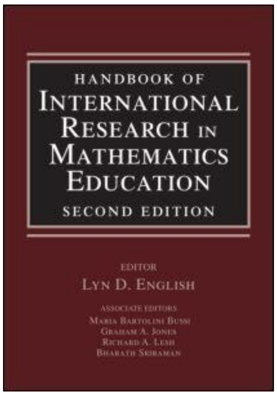

Handbook of International Research in Mathematics Education

Lyn D. English, Maria Bartolini Bussi, Graham A. Jones, Richard A. Lesh, Bharath Sriraman, Dina Tirosh

Teacher knowledge and understanding of students' mathematical learning and thinking

Publication details
https://www.routledgehandbooks.com/doi/10.4324/9780203930236.ch10
Ruhama Even, Dina Tirosh
Published online on: 19 Jun 2008

10 Teacher knowledge and understanding of students' mathematical learning and thinking

Ruhama Even
Weizmann Institute of Science

Dina Tirosh
Tel Aviv University

It is widely accepted today that teachers should be aware of, and knowledgeable about, students' mathematical learning. It is believed that such awareness and knowledge significantly contribute to various aspects of the practice of teaching. In this chapter, we critically examine this commonly held belief.

We begin this chapter by interpreting what one might mean by teacher knowledge and understanding of students' mathematical learning. Then we move to examining possible implications of such teacher knowledge on instruction. The third part of this chapter examines the assumption that teacher knowledge and understanding of students' mathematical learning is essential for good teaching in light of different theoretical perspectives. The fourth part discusses pre- and in-service teacher education that focus on different aspects of students' mathematical learning. Finally, we conclude this chapter by suggesting issues for further research.

WHAT IS ENTAILED BY STUDENTS' MATHEMATICAL LEARNING?

In coining the term "pedagogical content knowledge," Shulman (1986) contributed greatly to the initiation of the current discussion of what teachers need to know about students' mathematical learning. In this term, he referred mainly to "an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). In this part of the chapter, we re-examine this issue and further explore what might be implied by the phrase "*students' mathematical learning*." We focus on three aspects that have been at the center of researchers' attention during the last decades: (1) student conceptions, (2) different forms of knowledge, and (3) classroom culture.

Students' conceptions

In the last decades, many researchers have investigated students' mathematical ideas and conceptions as well as their development. Results of these studies show that learning mathematics is complex, takes time, and is often not straight forward (e.g., Bishop, Clements, Keitel, Kilpatrick, & Laborde, 1996; Bishop, Clements, Keitel, Kilpatrick, & Leung, 2003; Borasi, 1996; English, 2002; Grouws, 1992; Gutiérrez & Boero, 2006; Nesher & Kilpatrick, 1990;

Schoenfeld, Smith, & Arcavi, 1993; Smith, diSessa, & Roschelle, 1993). The findings indicate that students build their knowledge of mathematical concepts and ideas in ways which often differ from what is assumed by the professional community. In the following sections, we describe several lines of that research: theory building, misconceptions, and moving from misconceptions to knowledge.

Theory building

The attempt to develop a comprehensive theory that describes how students learn specific mathematical domains or concepts is rather rare in the field of mathematics education. A prominent example is the van Hiele theory, the most comprehensive theory yet formulated concerning geometry learning. It was developed by Pierre and Dina van Hiele half a century ago (Clements & Battista, 1992; Fuys, Geddes, & Tischler, 1988; Hershkowitz, 1990; Hoffer, 1983; Owens & Outhred, 2006; van Hiele & van Hiele-Geldof, 1958). The theory claims that when students learn geometry they progress from one discrete level of geometrical thinking to another. This process is discontinuous and the levels are sequential and hierarchical. The van Hiele theory also suggests phases of instruction that help students progress through the levels.

Several researchers have approached theory building differently from the van Hiele school. They have attempted to construct theories that are not specific to learning in a certain mathematical domain but rather, suggest general principles. One such approach relates to the acquisition of mathematical concepts (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Davis, 1975; Dubinsky, 1991; Sfard, 1991). This approach suggests that there is a chain of transitions from operational to structural conceptions. Some researchers (e.g., Sfard, 1991) further claim that operational conceptions are, for most people, the first stage in the acquisition of new mathematical concepts. A main, related claim is that processes performed on certain abstract objects turn into new objects that serve as inputs to higher level processes.

Misconceptions

A much more prominent line of research in the field of mathematics education is the study of errors. While the previous avenue of research focuses on general aspects of students' learning of mathematics, here, researchers usually focus on specific local concepts. Some researchers engage in describing in detail errors that students make in specific topics. Others explore additional dimensions. In this section, we briefly describe two such dimensions: sources of students' misconceptions and the evolution of misconceptions with age and instruction.

SOURCES OF STUDENTS' MISCONCEPTIONS

Many researchers find the study of students' errors fascinating. They devote their efforts to revealing possible sources of common students' errors. We illustrate this by using a widely documented error: the tendency to conjoin algebraic expressions (for example, to write the expression 2x+3 as 5x or 5). The literature suggests several different reasons for this tendency. One of them has to do with conventions related to not differentiating between conjoining and adding. For example, Stacey and MacGregor (1994) state that students may draw on previous learning from other fields to their work with algebraic symbols, e.g., in chemistry, adding oxygen to carbon produces CO_2. Tall and Thomas (1991) mention that due to similar meanings of "and" and "plus" in natural language, it is common for students to consider "ab" to mean the same as "a+b" because the symbol "ab" is read as "a and b" and interpreted as "a+b".

Another explanation that is often given for this error is that students face cognitive difficulties in accepting lack of closure and tend to perceive open expressions as incomplete (Booth, 1988; Collis, 1975; Davis, 1975). The latter explanation still leaves room for the question: Why do students feel this? A typical justification is that students expect the "behavior" of algebraic expressions to be similar to that of arithmetic expressions. Sometimes they expect a specific answer, that is, a final single-termed answer (e.g., Booth, 1988; Tall & Thomas, 1991); at other times, they interpret symbols such as "+" only in terms of actions to be performed, as is usually done in arithmetic, and thus conjoin the terms (e.g., Davis, 1975).

Another, somewhat broader explanation for the same behavior relates to the dual nature of mathematical notations: process and object (Davis, 1975; Sfard, 1991; Tall & Thomas, 1991). In algebra, the symbol 5x+8 stands both for the process "add five times x and eight" and also for an object. Often, students grasp 5x+8 only as a process to be performed and "add" 5x+8 in what seems to them a reasonable way, and obtain expressions such as 13x.

We have stated previously that most of the research on students' errors aims for detailed descriptions of common mistakes in specific mathematical topics. Many instances of common errors, alternative conceptions and misconceptions are described in the research. On the basis of this volume of documented research, several theoretical frameworks attempt to describe general, underlying sources of students' incorrect responses. Here we briefly describe one theory, the Intuitive Rules Theory (Stavy & Tirosh, 2000). The essential claim of this theory is that irrelevant, external features of the tasks often determine human responses to mathematical and scientific tasks. For instance, students' responses to comparison tasks embedded in different topics are often of the type "More A - more B" (Stavy & Tirosh, 1996). One example relates to vertical angles. Studies have shown that when children in grades K–4 are presented with two vertical angels, drawn with the same length of arms, the equality of the angles appear to them as self-evident. However, when the same children are asked to compare two vertical angles, one drawn with longer arms than the other does, they claim that the angle with the longer arms is larger. This judgment exemplifies the effect of the rule "More A - more B" on students' responses. In this case the difference between the angles in quantity A (the perceived length of the arms) affects students' judgment about quantity B (the size of the angles). This and other rules bear the characteristics of intuitive thinking: They appear self-evident, are used with great confidence and perseverance, and alternative responses are excluded as unacceptable. The Intuitive Rules Theory explains numerous incorrect responses and has a strong predictive power.

EVOLUTION OF MISCONCEPTIONS WITH AGE AND INSTRUCTION

Another trend in research on error examination is the evolution of misconceptions with age and instruction. For example, Hershkowitz (1987) and Fischbein and Schnarch (1997) investigated the evolution with age and instruction of basic geometry concepts and probability, respectively. In the Hershkowitz study, subjects were students from grades 5, 6, 7, and 8, as well as preservice and inservice elementary school teachers. The tasks employed in the questionnaires were taken from the primary school geometry syllabus. In her analysis of errors, Hershkowitz identified several patterns of evolution of misconceptions with age and instruction. An expected pattern is that of errors which decrease with age and instruction. For instance, subjects were presented with several shapes and were asked to indicate those which were quadrilaterals. The findings show a great improvement with age in identifying the non-prototypical examples of quadrilaterals (e.g., concave). A deeper analysis reveals that some of these errors have the same pattern of overall incidence from one grade level to the next, as well as for students, preservice and inservice teachers. For example, when asked to identify right-angled triangles, students, preservice teachers and inservice teachers had difficulty in the identification of those triangles whose perpendicular sides are not in the vertical-horizontal (prototype) position. This difficulty decreases with age and experience, but the pattern of errors remains rather stable. A somewhat surprising pattern includes errors that increase with age and instruction. For example, subjects were asked to draw the altitude to one side of several given triangles including isosceles, unequal sided, obtuse-angled, and right-angled triangles. Contrary to what might be expected, the number of subjects who made the error of drawing all altitudes inside the triangle increased with age and instruction.

An example from a different domain is that of the intuitive use of heuristics in probability. In a comprehensive series of studies Kahneman and Tversky (Kahneman & Tversky, 1972, 1973; Tversky & Kahneman, 1982, 1983) found that when estimating the likelihood of events, people tend to use certain judgmental heuristics. When using the representativeness heuristic, for example, people estimate the likelihood of an event based on how similar it is to the process by which the outcomes are generated. For instance, many people believe that in a family of six children, the birth order sequence BGGBGB (B-boy, G-girl) is more likely to occur than either BBBBGB or BBBGGG. In the first case, the sequence BGGBGB may appear more representative of the expected 50-50 ratio of boys and girls in the population than the sequence BBBBGB. In the second case, the sequence BBBGGG does not appear representative of the random process of having children. When using another heuristic, the availability heuristic, people estimate the likelihood of events based on the ease with which instances of that event can be constructed or called to mind. For example, if a student is asked to estimate the probability of a car accident, the frequency of his/her personal contact with this event may influence his estimation. When studying the evolution with age of the use of these heuristics, Fischbein and Schnarch (1997) found that while the incorrect intuitive use of the representativeness heuristic decreases with age, the incorrect intuitive use of the availability heuristic gains greater influence.

Recently, Vosniadou and her colleagues suggest that the conceptual change approach, originally developed to explain students' difficulties in learning science, is a fruitful paradigm for examining the evolution of misconceptions with age and instruction, not only in science, but also in mathematics. In a paper entitled "Extending the Conceptual Change Approach to Mathematics Learning and Teaching" (Vosniadou & Verschaffel, 2004) these researchers argue that the conceptual change approach provides a powerful framework for explaining students' difficulties with certain mathematical concepts. They state that the term "conceptual change" is used to characterize the kind of learning required when an information to be learned is in conflict with the learners' prior knowledge, usually acquired on the basis of everyday experiences. Vosniadou and Verschaffel (2004) claim that in such situations a major reorganization of prior knowledge is required and that the use of additive mechanisms in situations requiring conceptual change is one of the major causes of misconceptions. Misconceptions are often caused when new information is added to an incompatible knowledge base, producing consecutive, synthetic models.

From misconceptions to knowledge

The early research on mathematics learning viewed students' errors as flaws that interfere with learning and need to be avoided, and as misconceptions that need to be replaced with correct knowledge. A newer trend in the field is the focus on what students know and can do, highlighting the useful and productive nature of students' limited knowledge, and the continuity in knowledge between novices and masters (e.g., Lamon, 2006; Mulligan, & Mitchelmore, 1997; Smith, diSessa, & Roschelle, 1993; Streefland, 1993; Urbanska, 1993). According to the older trend, researchers focused, for example, on how students unsuccessfully compare fractions such as 1/6 and 1/8, claiming that 1/8 is bigger because 8 is bigger than 6. In the newer trend, Mack (1990), for example, showed that the very same students solved problems involving comparison of fractions when the problems were meaningful to them and they were allowed to use their informal knowledge. Moreover, Smith, diSessa, and Roschelle (1993) showed fundamental similarities in characteristics of masters' and novices' knowledge about fractions. For example, both groups tended to construct strategies that were tailored to solving specific classes of problems instead of using the more general school-taught strategies.

Different forms of knowledge and kinds of understanding

Another aspect of students' mathematical learning and thinking that has been in the center of researchers' attention during the last decades is different forms of knowledge and kinds of understanding. The notions "knowledge" and "understanding" are multi-dimensional. Different forms of knowledge and various kinds of understanding are described in the mathematics education literature (e.g., instrumental, relational, conceptual, procedural, implicit, explicit, elementary, advanced, algorithmic, formal, intuitive, visual, situated, knowing that, knowing how, knowing why, knowing to). The following section presents a brief description of several of these forms, portraying the main themes.

Instrumental understanding and relational understanding: A dichotomy or a continuum?

In an extremely influential article, Skemp (1978) presented his view on the distinction between two kinds of understanding in mathematics: relational and instrumental. Relational understanding is described as knowing both what to do and why, while instrumental understanding entails "rules without reasons" (Skemp, 1978, p. 9). Skemp argued that although instrumental mathematics is easier to understand within its own context, its rewards are more immediate and apparent, and one can often obtain the right answer more quickly and reliably, relational mathematics has the advantages of being more adaptable to new tasks, being easier to remember and capable of serving as a goal in itself. Skemp further asserted that the kind of learning which leads to instrumental mathematics includes the learning of an increasing number of fixed plans by which pupils can find their way from particular starting points to required finishing points. These plans tell them what to do at each choice junction, but there is no awareness of the overall relationship between successive stages and the final goal, and the learner is dependent on an outside guidance for learning each new plan. In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can produce an unlimited number of plans for getting from any starting point to any finishing point within the schema. The more complete a pupil's schema is, the greater his/her feeling of confidence in his/her own ability to find new ways of "getting there" without outside help. These schemas, however, are never completed and the process of constructing them is self-continuing, independent of particular ends to be reached, and a self-rewarding, intrinsically satisfying goal in itself.

Skemp argued that these two kinds of knowledge are so different that there is a strong case for regarding them as different kinds of mathematics. He opposed to instrumental mathematics, hinting that the term "mathematics" ought to be used for relational mathematics only, and raised several, severe problems that could occur when pupils whose goal is to understand instrumentally are taught by a teacher who wants them to understand relationally, or vice versa.

Skemp's work contributes significantly to the long-standing debate on the relative importance of computational skill versus mathematical understanding and to further investigations and discussions on this issue. For example, Nesher (1986) asserts that the dichotomy between learning algorithms and understanding is both superficial and misleading, arguing that research on mathematical performance does not inform us about the relationship between success in algorithmic performance versus success in understanding, nor does it give evidence about the contribution of understanding to algorithmic performance. She also contends that the possibility of teaching for understanding in mathematics without attending to the algorithmic and procedural aspects is questionable. In a similar vein, Resnick and Ford (1981) suggest that memorization of certain facts and procedures is important not so much as an end in itself but as a way to extend the capacity of the working memory by developing automaticity of response. They argue that when certain processes can be carried out automatically, without need for direct attention, more space becomes available in the working memory for processes that do require attention.

Other researchers in mathematics education also question the usefulness of instrumental-relational dichotomy and raise various, related issues. Hiebert and his colleagues (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986), for instance, suggest that both conceptual and procedural knowledge are required for mathematical expertise. They define conceptual knowledge as knowledge which is rich in relationships. The learning of a new concept or a relationship implies the addition of a node or link to the existing cognitive structure, thus making the whole more stable than before. Procedural knowledge, on the other hand, is a sequence of actions that can be learned with or without meaning. Hiebert and Carpenter (1992) suggest that the relationships between conceptual and procedural knowledge may range from no relationship to a relationship so close that it becomes difficult to distinguish between them.

We have shown that different researchers in mathematics education take different points of view on the dichotomy/continuum issue. While Skemp assumes a dichotomy between instrumental and relational knowledge, and Nesher (1986) and Resnick and Ford (1981) question its usefulness, Hiebert and Carpenter and other researchers suggest that absolute classifications are impossible.

Algorithmic, formal and intuitive dimensions of mathematics: Interactions and inconsistencies

In several of his numerous writings, Fischbein suggested that any mathematical activity requires the use of three basic dimensions of mathematic knowledge: algorithmic, formal and intuitive (Fischbein 1983, 1993). These three types of knowledge are essentially different from the types of knowledge described in the previous section. The algorithmic dimension consists of rules, procedures for solving *and* their theoretical justifications. The formal dimension includes axioms, definitions, theorems and proofs. The intuitive dimension is a kind of cognition that comprises the ideas and beliefs about mathematical entities and the mental models that are used for representing mathematical concepts and operations. Intuitive knowledge is characterized as the type of knowledge that we tend to accept directly and confidently as being obvious, with a feeling that it needs no proof. This type of knowledge has an imperative power; that is, it tends to eliminate alternative representations, interpretations or solutions.

Fischbein argues that these three dimensions of knowledge are not discrete; they overlap considerably. Ideally, these dimensions should cooperate in the processes of concept acquisition, understanding and problem solving. In reality, though, this is not always the case. Both the formal and the algorithmic dimensions of knowledge can become rote for the students. Often there are serious inconsistencies among students' algorithmic, intuitive and formal knowledge. Such inconsistencies could be the source of common difficulties learners encounter in their mathematical activities, such as misconceptions, cognitive obstacles and inadequate usage of algorithms.

Knowing-about and knowing-to: Knowing facts versus knowing to act

A rather frustrating phenomenon, often described by both researchers and teachers, is that students who are known to have all the knowledge that is needed to solve a problem, are unable to employ this knowledge to solve unfamiliar problems (see, for instance, Schoenfeld, 1988). Attempting to explain this phenomenon, Mason and Spence (1999) define a special form of knowing: Knowing-to act in the moment. Mason and Spence describe and discuss some traditional epistemological distinctions between sorts and degrees of knowledge of mathematics, including knowing that (something is true), knowing how (to carry out some procedure) and knowing why (having some stories to account for something). They argue that education driven by these three types of knowledge, which constitute knowing-about mathematics, sees knowledge as a static object that can be passed on from generation to generation as a collection of facts, techniques, skills and theories.

Mason and Spence contend that knowing-about is a distant, detached form of knowledge, exhibited rather than used, and that such knowledge does not automatically develop the awareness that enables students to use this knowledge in new situations. They suggest that a fourth form of knowledge, knowing-to act in the moment, is the type of knowledge that enables people to act creatively rather than merely react to stimuli with trained or habituated behavior. Mason and Spence claim that knowing-to requires sensitivity to situational features and some degree of awareness of the moment, so that relevant knowledge is accessed when appropriate. They describe the interactions among these four types of knowledge, suggesting that knowing-to is the critical form of knowing, the type of knowing students need in order to engage in problem solving where context is novel and resolution non-routine.

Classroom culture

An important issue that has received the attention of the mathematics education community in recent years is classroom culture (Cobb & Bauersfeld, 1995; Cobb, Stephan, McClain, & Gravemeijer, 2001; Even & Schwarz, 2003; Lerman, 2006). This new focus signals a shift from examining human mental functioning in isolation (a characteristic of most of the research described in the previous two sections) to considering cultural, institutional and historical factors. The mathematics education community increasingly embraces the view that cultural and social processes are integral to mathematical activity.

Pimm (1987), for instance, in his examination of the types of interaction commonly found in mathematics classrooms, demonstrates how, in many cases, teacher questioning is aimed at breaking up teacher monologue, making sure students are listening, and ascribing if the particular student questioned has grasped what is being explained. Correspondingly, Pimm reveals how what might seem at first glance as students answering mathematical questions asked by the teacher, actually covers a particular type of classroom communication where students aim at guessing what the teacher has in mind.

To illustrate how such classroom norms are supported, we present an episode observed in an algebra lesson (Robinson, 1993). On the board, the teacher (T) wrote two expressions, one simple and the other complex: $4a+3$ and $\frac{3a+6+5a}{2}$. Then, he asked the students (S's) to substitute a fraction in both expressions:

T: Substitute a = ½.
S1: You get the same result.

Then the teacher asked:

T: Are the algebraic expressions equivalent?

The students initiated a debate of this issue among themselves:
S2: No, because we substituted only one number.
S1: Yes.
S3: It is impossible to know. We need all the numbers.
S4: One example is not enough.

Clearly the students were engaged, on their own initiative, in a genuine and important mathematical discussion. But the teacher ignored the students' discussion completely and stated:

T: We can conclude—it is difficult to substitute numbers in a complicated expression and therefore we should find a simpler equivalent expression.

While the substitution of a = ½ in the two given expressions might lead naturally to the conclusion that we should find a simpler equivalent expression (as was originally planned by the teacher), this was, by no means, the response appropriate to the discussion taking place in that classroom at that moment. Several negative lessons students may easily learn from such experiences are: that their mathematical thinking is not valued; only what the teacher has in mind is important; that mathematics does not necessarily make sense; that the teacher is the sole authority for determining the correctness of answers.

Several mathematics educators (e.g., Ball, 1991a; Chazan, 2000; Cobb, Stephan, McClain, & Gravemeijer, 2001; Cobb, Yackel, & Wood, 1989; Hershkowitz & Schwarz, 1999; Lampert, 1990, 2001; Schoenfeld, 1994; Yackel, 2001) have attempted in recent years to support the development of a different mathematical culture in the classroom. One of the main characteristics of the revised culture is the alteration of traditional roles and responsibilities of teacher and students in classroom discourse. These researchers and others (e.g., Arcavi, Kessel, Meira, & Smith, 1998; Goos, 2004; Wood, Williams, & McNeal, 2006) investigate mathematics learning and knowing in these classrooms. They document and examine, either explicitly or implicitly, the evolution of behaviors that sustain classroom cultures characterized by social norms, such as explanation, justification, argumentation, and intellectual autonomy, as well as sociomathematical norms (a term coined by Yackel and Cobb, 1996), such as what counts as mathematical explanation and justification, and what are mathematically different solutions. Examples of the latter are given in the next part.

WHAT CAN HAPPEN IN THE CLASSROOM?

It is not reasonable to assume that there is a simple connection between teachers' knowledge and understanding about students' mathematical learning and the process of instruction. Rather, when applied in practice, such knowledge interacts with a combination of many factors, for example: knowledge about mathematics and about didactics; self-confidence in knowing mathematics and in knowing to teach; personal theories and beliefs about mathematics, teaching, learning, and students; the nature of student assessment (e.g., external/internal, traditional/alternative); the character of the educational system (e.g., centralized/discentralized, goals for teaching mathematics at school); participating parties (e.g., principal, supervisor, parents, colleagues). Still, the contribution of teachers' knowledge and understanding about student mathematical learning to their instructional practice cannot be ignored. This is illustrated in the following cases.

Knowing and not knowing about "finishing" open expressions

Benny, Gilah, and Batia were seventh-grade teachers, participating in research on teaching algebra (Tirosh, Even, & Robinson, 1998). They were teaching algebraic expressions from the same textbook. Benny's behavior suggests that he is unaware of students' tendency to conjoin or finish open expressions. He does not mention this issue in an interview when asked to describe students' difficulties related to learning algebraic expressions, nor does he address it in his lesson plans.

When designing the teaching of simplifying algebraic expressions, Benny plans to provide students with a rule of "adding numbers separately and adding letters separately." During the lesson, he states the rule and keeps repeating it. When an incorrect response is given, he often states that this is wrong, and repeats the rule. The following fragment describes what happened in his class when he tried to apply his plan.

Benny writes the expression 3m+2+2m on the board and asks: "What does this equal?" He immediately follows with the rule: "Add the numbers separately and add the letters separately." Then he suggests coloring the "numbers": 3m+2+2m (as if 3 and the other 2 are not numbers), and writes 5m+2. A student asks: "And what now?" Another student suggests: "7m." The teacher (rather surprised by this answer) says: "No! 5m+2 does *not* equal 7m," and he repeats the rule again: "The rule is: add the numbers separately and add the letters separately" (note that this rule may actually lead to 7m). Then he gives the students another example and colors the (free) numbers: 4a+5-2a+7. The teacher emphasizes the rule by dictating it to the students and asking them to repeat it out loud. The rest of the lesson is devoted to work on similar exercises. The students continue to experience difficulties.

In contrast to Benny, Gilah is aware of students' tendency to "finish" open sentences. When asked during an interview to mention various difficulties related to the learning of algebra, she specifies, among other things, students' tendency to "simplify" expressions such as 3x+4 to 7x. She further explains: "Students tend to make the expression as simple as possible. They tend to 'finish' it [the expression]." In her opinion, this is *the* main obstacle in teaching how to simplify algebraic expressions. Therefore, she planned a comprehensive activity, devoted to acquaintance the students with the notions of like and unlike terms, to be taught before the lessons on the simplification of algebraic expressions. She spent time and effort on teaching and directing the students towards the use of this one specific method. In an interview she claims:

> I think that differentiating between like and unlike terms should precede the issue of simplifying algebraic expressions. There is a need to work extensively on the topic of like and unlike terms.

Her introductory activity consists of two main parts. In the first one, "Identifying like terms," students are told that "like terms are terms that have an identical combination of variables" and they receive a variety of examples of like and unlike terms (e.g., $2x^2$ and $4x^2$, 3ab and 6ab, 5a and $6a^2$, 2bc and 3ac, 3ab and –2ba). Then, they practice and discuss identifying like and unlike terms. In the second part of this activity, "Collecting like terms," students are told that "in order to simplify algebraic expressions, one can collect like terms." The students then receive several examples that illustrate how to collect like terms, starting with 4a+2a = 6a and gradually reaching more complicated expressions such as 2xy+4x+1.5y+6xy+y = 8xy+2.5y+4x. The examples are accompanied by written descriptions, which highlight the like terms and the result of their collection. After discussing the examples, the students practice simplifying algebraic expressions by collecting like terms. As the class progresses, Gilah and her students keep referring to the notions of like and unlike terms. They use them to determine if and how a given algebraic expression can be simplified.

Like Gilah, Batia's lesson planning, her teaching, as well as her interviews, all make it obvious that she is aware of students' tendency to "finish" open expressions. For example, in her written lesson plan she writes: "I expect difficulties in problematic cases [such as] 2x+3=5x." Also, in an interview, when asked about difficulties that students commonly encounter when studying algebraic expressions, she mentions, among other things, the tendency to add 2a+3 and get 5a, stating: "They need to get an answer, it does not seem finished to them." The following interchange illustrates how Batia uses her knowledge about this common mistake in instruction:

Teacher: What is 3+4x?
Student: 7x.
Teacher: How about 7?
Student: Maybe?!
Teacher: Well, let's see again. 3+4x. What is the operation between 4 and x?
Student: Multiplication.

Teacher: So, first we have to determine what 4·x could be. Can we know that?
Student: No!
Teacher: So, can I first add the numbers?
Student: No! OK, I got it.

A main difference between Benny, and Gilah and Batia was that Benny was unaware of the students' tendency to finish open expressions, while Gilah and Batia were. Consequently, Benny was surprised when his students encountered so many difficulties and his teaching decisions were not related to his students' problems. In his reflection on the lesson, Benny expressed his dissatisfaction and frustration. He explained that he sensed there was a problem but he did not understand its sources. Gilah and Batia's students, on the other hand, seemed comfortable with this notion and way of work and rarely made mistakes.

While coming from different starting points regarding understanding of their students' mathematical learning, both Benny and Gilah chose to provide the students with a rule. Both teachers used some version of the "collecting like terms" approach, which is commonly used when teaching simplifying algebraic expressions. Benny started to use this method without taking into account the students' specific mistake. In his class students seemed unwilling to accept expressions including a "+" sign (such as 3 + 2a) as final answers. Gilah, on the other hand, as a way to address the specific students' mistake devoted an extensive period of time to practicing "collecting like terms" before dealing with simplifying algebraic expressions. Indeed, in her class students seemed to have mastered this skill.

At first sight, Gilah's awareness of her students' mathematical learning led to successful instruction. It enabled her to plan her teaching accordingly and to navigate the instruction so that students learned what she intended them to. However, the long-term implications of such a method on students' general knowledge and conceptions of mathematics is questionable. Gilah's teaching approach consisted of what Davis (1989) refers to as a course in which the student is asked to perform some fragmentary, individual, small rituals. These skills are presented to students as "rituals to be practiced until they can be executed in the proper, orthodox fashion" (p. 117). We would like to join Davis in his claim that when using such an approach, the student sees no purpose or goal in the activity. "Consequently, the student sees no reason why the ritual is performed in one way and not another." Davis mocks the theory underlying such didactive approaches which assume that "if the students spend enough time practicing dull, meaningless, incomprehensible little rituals, sooner or later something WONDERFUL will happen" (p. 118). Gilah seems to emphasize procedural knowledge only, with no explicit consideration of other kinds of knowledge nor of classroom culture.

Batia, who like Gilah, was ready to face classroom situations where students "finish" algebraic expressions, did not choose to use one specific approach. Rather she used her rich repertoire of strategies (of which we presented only one), all of which are characterized by short and quick teacher-student interchanges. In such situations, students rarely interact with each other or discuss each other ideas. Batia's understanding of students' mathematical learning enabled her to make quick relevant responses to students that took their understanding into consideration. However, the nature of the discourse in her class, and her exclusive focus during her interviews on the cognitive development of her students, signal that she did not pay explicit attention to classroom culture.

Attention to student ways of learning and knowing

The two teachers, Magdalene Lampert and Deborah Ball, to whose work we refer in this section are not ordinary teachers. Both are university professors and experienced schoolteachers, whose theoretical and practical knowledge (about mathematics, teaching mathematics, students, the educational system, and related factors) is much deeper and broader than that of a regular schoolteacher. The classroom culture in their classes is very different from those described in the previous section. They (Ball, 1991b; Lampert, 1990) explicitly explain what classroom culture they are aiming for, and consciously encourage their students' intellectual autonomy and their development of specific social and sociomathematical norms. That is, they pay attention not only to students' individual learning and cognitive development, but also to the development of the classroom culture. For example, in one of the lessons cited (Lampert, 1990), the teacher presented her fifth graders with the problem of finding the last digit of 7 to the fifth power. The students offered three conjectures: 1, 9, and 7. The following excerpt illustrates how the teacher navigated the class discussion and how she encouraged the development of norms such as: students are to make conjectures, explain their reasoning, validate their assertions, discuss and question their own thinking and the thinking of others, and argue about what is mathematically true.

Teacher: Arthur, why do you think it's 1?
Arthur: Because 7^4 ends in 1, then it's times 1 again.
Gar: The answer to 7^4 is 2,401. You multiply that by 7 to get the answer, so it's 7 x 1.
Teacher: Why 9, Sarah?
Theresa: I think Sarah thought the number should be 49.
Gar: Maybe they think it goes 9, 1, 9, 1, 9, 1.
Molly: I know it's 7, 'cause 7 ...
Abdul: Because 7^4 ends in 1, so if you times it by 7, it'll end in 7.
Martha: I think it's 7. No, I think it's 8.
Sam: I don't think it's 8 because, it's odd number times odd number and that's always an odd number.
Carl: It's 7 because it's like saying 49 × 49 × 7.
Arthur: I still think it's 1 because you do 7 × 7 to get 49 and then for 7^4 you do 49 × 49 and for 7^5, I think you'll do 7^4 times itself and that will end in 1.
Teacher: What's 49^2?
Soo Wo: 2,401.
Teacher: Arthur's theory is that 7^5 should be 2401 ×2401 and since there's a 1 here and a 1 here ...
Soo Wo: It's 2,401 × 7.
Gar: I have a proof that it won't be a 9. It can't be 9, 1, 9, 1, because 7^3 ends in a 3.
Martha: I think it goes 1, 7, 9, 1, 7, 9, 1, 7, 9.
Teacher: What about 7^3 ending in 3? The last number ends in ... 9 × 7 is 63.
Martha: Oh ...
Carl: Abdul's thing isn't wrong, 'cause it works. He said times the last digit by 7 and the last digit is 9, so the last one will be 3. It's 1, 7, 9, 3, 1, 7, 9, 3.
Arthur: I want to revise my thinking. It would be 7 × 7 × 7 × 7 × 7. I was thinking it would be 7 × 7 × 7 × 7 × 7 × 7 × 7 × 7. (Lampert, 1990, pp. 50–51)

Although the teacher does not respond immediately to every student's question, statement or conjecture, as the previous teachers did, she seems extremely attentive to her students' mathematical thinking. Occasionally she interjects a clarifying question or remark that propels the mathematical discussion forward while allowing enough room for her students to take a principal role in the discussion.

The episode above might create the impression that such extraordinary teachers always understand their students' ways of thinking. However, there are features that are inherent in the task of hearing and assessing students' thinking and learning that make this task very difficult (Ball, 1997). The "Shea numbers" episode (Ball, 1991b) is illuminating in highlighting how complicated and challenging it might be for the teacher to understand students' mathematical thinking. Ball's third-grade class talked about even and odd numbers. A student whose name is Benny made the observation that even numbers can be "made" from two other even numbers, e.g., 4+4 and 6+6. Following this, another student, Shea, commented that he had noticed something special about the number six. He claimed that six could be an odd *and* even number. He further explained that,

> I'm just thinking that it can be an odd number, too, 'cause there could be two, four, six, and two, three twos, that'd make six… And two threes, that it could be an odd and an even number. Both! Three things to make it and there could be two things to make it. (Ball, 1991b)

Ball, who interpreted Shea's claim as connected to Benny's observation, thought that Shea's point was that two odd numbers could also make an even number. She then explained to Shea that Benny's observation was not that all even numbers are made up of two even numbers. Rather, as Shea just suggested, some of the even numbers, like six, are made up of two odd numbers. However, this was not what Shea suggested. As later became apparent, he claimed that if splitting up fairly into two groups (i.e., an even number) makes an even number, then splitting up fairly into three groups (i.e., an odd number) makes an odd number. According to Shea's definition, six is indeed both an even and odd number. Viewing sensitivity and attention to students' thinking as critical attributes of a teacher's role, and caring about the development of a classroom culture where explanation, justification, argumentation, and intellectual autonomy are norms, Ball eventually, with the help of other students, came to understand Shea's mathematical thinking.

DIFFERENT PERSPECTIVES ON TEACHERS' KNOWLEDGE ABOUT STUDENT LEARNING

Earlier, we presented three main aspects of student mathematical learning: student conceptions, different forms of knowledge and classroom culture. It is generally agreed that it is important for teachers to be knowledgeable about these three aspects. This section examines how this assumption fits with three main learning perspectives. Following Greeno, Collins, and Resnick (1996), we focus on behaviorism, constructivism, and situationism perspectives.

Knowing about student conceptions

Behaviorism views learning as the process in which associations and skills are acquired. A basic assumption is that any use of a wrong association tends to strengthen it. Therefore, it is essential to prevent students from making mistakes or from being exposed to errors made by their peers. Behaviorists state explicitly that it is impossible for anyone (including teachers) to know what goes on in the students' mind. They direct teachers toward determining the correctness of the students' responses, not the students' conceptions.

According to constructivism, children's knowledge differs not only quantitatively but also qualitatively from that of the adult. A basic assumption of constructivism is that knowledge is not communicated but constructed by unique individuals. Constructivists claim that when teaching mathematics, the teacher should attend to students' thinking, form an adequate model of students' ways of viewing an idea, and then construct a tentative path along which students may move to construct a mathematical idea. Accordingly, the very essence of constructivism is to know and understand student conceptions.

The situationist perspective focuses on the kinds of social engagements that provide the proper context for learning to take place. Learning is perceived as a process that takes place in a participative framework, not in an isolated individual mind. The learner does not gain a discrete body of abstract knowledge, which s/he will then apply in other contexts. Rather, knowing is viewed as the practices of a community and the abilities of individuals to participate in those practices; learning is the strengthening of those practices and participatory abilities. Thus, the situationist perspective attends to students' ability to participate in shared mathematical activities.

Knowing about forms of knowledge

Behaviorists view knowledge as an organized accumulation of facts, skills and procedures. Consequently, these are the types of knowledge that teachers are apt to emphasize in instruction. The constructivist perspective emphasizes the development of different forms of knowledge such as conceptual knowledge, problem-solving strategies, and meta-cognitive abilities. Consequently, teachers should be knowledgeable about different forms of knowledge. Knowing-to is a central feature of participation. However, since the situationist perspective does not concentrate on knowledge per se, knowing about different forms of knowledge may well be considered irrelevant for teachers according to this approach. Instead, what might be important is attention to participation in complex activities, which involves the use of different forms of knowledge.

Knowing about classroom culture

Learning environments designed according to behaviorist principles are organized so that teachers efficiently transmit facts and procedural knowledge. Usually, the teacher presents correct procedures and provides opportunities for practice. The focus is on the individual student and the classroom is viewed as a collection of individual students. Constructivist learning environments are designed to provide students with opportunities to construct conceptual understanding and to foster problem-solving and reasoning abilities. Constructivism[1] also focuses on the individual student, not on building a community of learners. In the situationist perspective, by contrast, the classroom culture is the essence. Teachers represent the community of practice, exemplify valued practices, encourage the development of desired norms, and guide students as they become increasingly competent practitioners.

Navigating between perspectives

It is clear that each learning perspective approaches teacher knowing about student learning differently. We join Sfard (1998) in arguing that choosing and being completely loyal to one learning perspective is counter-productive in educational practice. Adherence to one theoretical perspective might seem an advantage as it eliminates confusion and contradictions. But the task of teaching is much too complex to be reduced to clear-cut global principles, to be applied in all circumstances. We believe that understanding student conceptions, both those documented in the research literature and those known from experience, would assist teachers to adjust instruction to where their students are in their mathematical understanding. Also, it is important for teachers to be aware that knowing mathematics cannot be reduced to one simple form of knowledge. Furthermore, teachers should be aware that classroom culture is inseparable from learning mathematics, as learning always occurs in a specific sociocultural environment. Teacher understanding of the interrelations between classroom norms and mathematics learning is essential for designing an appropriate learning environment.

TEACHER EDUCATION: WHAT AND HOW

Research and professional rhetoric (e.g., Barnett, 1991; Cobb & McClain, 1999; Even, 1999a, 1999b; Even & Markovits, 1993; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Llinares & Krainer, 2006; National Council of Teachers of Mathematics, 1991; Rhine, 1998; Scherer & Steinbring, 2006; Simon & Schifter, 1991; Tirosh & Graeber, 2003) recommend that attention be paid to students' mathematics learning and thinking in teacher education and professional development programs. This recommendation is based on the view that awareness to, and understanding of, students' mathematics learning and thinking are central to good teaching and that such awareness and understanding would not happen automatically. Consequently, the development of such awareness and understanding needs to be part of (preservice and inservice) teacher education curriculum. We do not attempt to provide here a survey of programs that adopt such practice. Rather, we limit ourselves to discussion of what it might mean for teacher education to focus on student mathematical thinking and learning. We organize our discussion around the three aspects of student mathematical learning which serve as our foci points throughout this chapter: student conceptions, different forms of knowledge, and classroom culture.

Educating about student conceptions

Many teacher education programs center on developing teachers' knowledge about students' mathematical conceptions. Some concentrate on teaching specific theories and models of students' mathematical thinking. Others aim at developing awareness that students often think differently about mathematical concepts than what might be expected. A pioneering project entitled Cognitively Guided Instruction (CGI) has focused on enabling inservice elementary school teachers to understand their students' thinking by using a specific research-based model of children's mathematical thinking (Fennema et al., 1996). The researchers presented teachers with a model of children's thinking about basic addition, subtraction, multiplication and division word problems. The model distinguishes between several problem types and identifies the relative difficulty of each category. During workshops, teachers learned to recognize differences among word problems, to identify the solution strategies that children might use to solve different problems, and to organize these strategies into hierarchical levels of thinking. The findings indicate fundamental changes in the beliefs and instruction of the participating teachers. The teachers' role evolved from demonstrating procedures to helping children build on their mathematical thinking by engaging them in a variety of problems, and in encouraging them to talk about their mathematical thinking. Such changes in instruction were later directly related to changes in students' achievements.

While the CGI Project aims at professional development of elementary school teachers, the Manor Project focuses on the development of a professional group of secondary school mathematics teacher-leaders and inservice teacher educators (Even, 1999a, 2005a). Part of the Manor Program centers on deepening and expanding the participants' understanding about students' conceptions and ways of learning different topics in mathematics. The aim is to assist participants to look at mathematics learning "from the student's point of view," to examine what might be the meaning of the widespread constructivist claim that students' ideas are not necessarily identical to the structure of the discipline nor to what was intended by instruction; and that students construct and develop their own knowledge and ideas about the mathematics they learn.

In contrast with the approach of the CGI, the Manor Program participants are not provided with explicit research-based models of children's thinking in specific mathematical topics. Research on student thinking at the level of junior and senior high school mathematics does not seem to support the existence of such models. Rather, similar to the Integrating Mathematics Assessment (Rhine, 1998) and the Mathematics Classroom Situations (Even & Markovits, 1993; Markovits & Even, 1999) approaches, the aim is for the participants to become acquainted with research-based key features of student thinking in different mathematical topics (i.e., cognitive development and aspects of mathematical thinking in algebra, analysis, geometry and probability). The purpose is to challenge and expand the participants' understanding of students' ways of making sense of the subject matter and the instruction.

The Manor Program focuses on deepening the academic background of the participants and in line with the model proposed by Leinhardt, Young, and Merriman (1995), it emphasizes the synthesis of theoretical and practical sources of knowledge. To help the participants become familiar with relevant research literature, a large part of the program includes reading, presentations and discussions of research articles on students' mathematical conceptions and ways of thinking, and on classroom cultures that support and promote the development of mathematical reasoning.

Participants then are directed to examine the theoretical knowledge acquired from reading and discussing research in the light of their practical knowledge. Vice versa, the participants are guided to build upon and interpret their experience-based knowledge using research-based knowledge. To do so, the participants are asked to choose one of the studies presented in the course and replicate it (or a variation of it) with their own students. Intellectual restructuring depends on deep processing of experiences (Desforges, 1995), which is more likely to occur if the activity requires personal involvement and presenting the ideas and reasoning to others (Chinn & Brewer, 1993). Therefore, the participants are required to write a report that describes the students' ways of thinking and difficulties, and to compare the results with those of the original study.

It appears (Even, 1999b) that for the participants, acquaintance with research in mathematics education via discussion of research articles supports the development of what were initially intuitive, naive and implicit ideas about student mathematics learning, into more formal, deliberated and explicit knowledge. Replicating a study further expands theoretical knowledge, and helps to develop better understanding of the issues raised and discussed in the articles they read. Redoing a mini-study with real students provides opportunities for examining theoretical matters by particularizing them in a specific context. For example, reading and discussing research contributed to learning in general about how students construct their own knowledge. The mini-study made general theoretical ideas more specific, concrete and relevant, illustrating what the constructivist view might mean in a practical context. By conducting a mini-study with real students the participants learned that what they thought they knew about their students was not necessarily a good representation of the students' knowledge and abilities (similar results are reported by Lerman, 1990 and by D'Ambrosio and Campos, 1992). Depending on their background and the specific project they chose to work on, some participants learned that, contrary to expectations, sophisticated mathematical ideas that seemed too difficult, can actually be dealt with successfully by their students. Others found that even well planned teaching might not produce the kind of learning they expected.

Educating about different forms of knowledge

Learning about various forms of mathematics knowledge is not listed as an explicit aim of most teacher education programs. However, highlighting instrumental and relational knowledge, and procedural and conceptual understanding are implicit goals of many of them. Here, we shall briefly describe a 1-year preservice elementary school teacher program, Students' Thinking About Rationals (STAR), which concentrates on participants' subject matter knowledge and pedagogical content knowledge of rational numbers (Tirosh, 2000). One aim of this program is to familiarize prospective teachers with Fischbein's framework of the three basic dimensions of mathematics knowledge (described in the first part of this chapter). It was believed that this framework could support teachers in their attempts to foresee, interpret, explain and make sense of students' mathematics learning. More specifically, this framework was introduced, discussed, and used as a means that could assist teachers in their attempts to predict possible students' mistakes in various rational-numbers-tasks and to hypothesize about possible sources of given mistakes.

Fischbein's framework was used on many occasions in the course. We present here one example relating to division of fractions. Participants were presented with four division expressions and were requested to (a) calculate each of these expressions, (b) list common mistakes seventh-grade students may make after completing their studies on fractions, and (c) describe possible sources for each of these mistakes. One of the expressions was ¼ : ½. At the beginning of the course all participants calculated this expression correctly. Most of them argued that the (only) common mistake students would make is ¼ ÷ ½ =2 and that this mistake will originate from a bug in the *algorithm* (e.g., ¼ : ½ = 4/1 · ½ = 2). During the course, the instructor uses Fischbein's framework to exemplify that the same error may have other sources. She demonstrates that such a response could derive from the commonly held *intuitive* belief that in division, the dividend should always be greater than the divisor (and therefore ¼ : ½ = ½ : ¼ = 2), from inadequate *formal* knowledge (e.g., division is commutative and therefore ¼ : ½ = ½ : ¼ = 2) or from other sources. By the end of the course, most participants were acquainted with Fischbein's framework and used it to guide their attempts to describe common incorrect responses.

Educating about classroom culture

As the focus on socio-cultural aspects is relatively new among mathematics educators, it is only natural that most of the emphasis is currently centered on examining socio-cultural aspects of student learning and not yet on educating teachers about it. Below we describe some pioneering work in this direction.

In their work with inservice elementary school teachers, Cobb and McClain (1999) emphasize that one of their goals is to help teachers to locate "students' mathematical activity in social context by attending to the nature of the social events in which they participate in the classroom" (p. 29). These researchers and their colleagues (Cobb & McClain, 1999; Cobb, Stephan, McClain, & Gravemeijer, 2001) acknowledge the need for teachers to learn about the social aspects of mathematics learning, and use episodes from classrooms to serve as a basis for conversations with teachers about the role of the teacher in supporting the development of sociomathematical norms.

Lampert and Ball (1998) work for several years with preservice elementary school teachers towards pedagogical inquiry. Among the various aspects of teaching they attend to, they design tasks to help prospective teachers consider classroom culture, stating explicitly that classroom culture is "one of the core dimensions of practice and hence an important idea for prospective teachers to learn" (p. 111). Lampert and Ball created multimedia records of practice; a comprehensive record of information of various kinds (video and text) about what occurred in the third- and fifth-grade mathematics classes they were teaching during the 1989–90 school year. Preservice teachers explore the records of practice in the multimedia environment aiming at identifying items that exemplify key elements of the culture of the classroom, formulating conjectures and explanations about the teacher's role in establishing and maintaining these elements of classroom culture. In doing so, Lampert and Ball treat what constitute classroom culture and how it can be developed in a classroom as content to be learned by prospective teachers. They design opportunities for the prospective teachers to engage in learning this content and to organize their ideas conceptually. Lampert and Ball (1998) suggest that the course they designed influenced students' conceptions of mathematics and what it means to know mathematics to include discussion, debate, and collaboration.

LOOKING TO THE FUTURE

In this chapter, we discussed teachers' knowledge and understanding of students' mathematical learning and thinking. Three main relevant issues are:

- *What* should teachers know and understand?
- *How* may they learn?
- *When* may they learn?

In the preceding sections, we focused on the first two questions ("What" and "How") in light of the information provided by the research literature. Much less is known about the third question, "When" (e.g., during preservice education? during inservice professional development?). We approached the first two questions by referring to three aspects: (1) student conceptions, (2) different forms of knowledge, and (3) classroom culture.

There are other issues that need to be examined: what do teachers need to know about these aspects and what are promising ways for teacher learning about them? For example, regarding student conceptions, a spontaneous solution may be to choose the most salient ones. However, students' conceptions may differ according to the curricula they study, the classroom practices they experience, and other factors. The extent to which mathematical ways of thinking and difficulties are embedded in a particular approach to learning and teaching still needs to be studied. For instance, it is possible that the tendency to conjoin open expressions will be found only in classes that use the traditional approach to teaching algebra. It might not be found in classes that use curricula that attempt to provide students with a broader context, one in which not completing the expression makes sense, offers some advantage, and does not simply remain another formal exercise.

A similar issue emerges in relation to educating teachers about forms of knowledge. Currently, there is no one single theoretical framework that is widely accepted by the mathematics education community. Should such consensus be reached? Should we wait until this line of research is more advanced before we make decisions regarding its inclusion in teacher education? If one feels that we should not wait for more information, decisions should be made regarding which and how many frameworks will be used. In the meantime, we need to obtain more information about the impact of focusing on different forms of knowledge in teacher education.

With respect to research on classroom culture, we feel that the literature does not provide enough critical analyses of problematic aspects, of advantages and disadvantages of adapting the current advocated classroom culture. Missing are analyses that take into account the complexity of actual mathematics instruction that needs to consider various (and sometimes conflicting) factors, facets and circumstances. Even if we adopt the vision of a desired classroom culture as advocated today in reform documents (e.g., National Council of Teachers of Mathematics, 1991, 2000; Australian Education Council, 1990), we are still faced with questions concerning "How?": Is it necessary for teachers to experience a desired classroom culture as learners? Is it sufficient? Do they need to observe such classrooms? Is it enough? Do they need to actually experience teaching in such classrooms as student teachers?

Understanding students' thinking is a problematic notion, as pointed out by Wood (2004). In this chapter, we aimed to contribute to unpacking the meaning of understanding students thinking by focusing on three aspects: student conceptions, different forms of knowledge, and classroom culture. But, of course, more work is still needed to develop common meanings.

Still, recent studies (Even, 2005b; Even & Wallach, 2004; Wallach & Even, 2005) suggest that the problem is not only in the vague meaning of understanding students thinking. These studies show that there are often discrepancies between what students say and do, and what teachers understand, suggesting that teacher interpretation of students' understanding, knowledge and learning of mathematics draws on rich knowledge base of understandings, beliefs, and attitudes. Consequently, the process of teacher sense-making of students' understanding involves ambiguity and difficulties. As we saw earlier, current teacher education programs aim to raise teachers' *awareness* of the importance of understanding their students' mathematics conceptions and ways of thinking, and to develop teachers' *knowledge* about different ways in which students think and reason mathematically. The recent findings (Even, 2005b; Even & Wallach, 2004; Wallach & Even, 2005) suggest that in teacher education it is also important to address the problem that what the teacher understands could be different from what students are saying or doing.

Finally, although we raised many issues in this chapter regarding teacher knowledge and understanding of students' mathematical learning and thinking that still need to be explored, we would like to stress that our research community has made a considerable progress with respect to this issue in the last decade. This research has advanced our understanding of the complex nature of teacher knowledge in general, of teacher knowledge and understanding about student mathematical learning and thinking, in particular, and of the interrelations of this kind of knowledge with instructional practice. We look forward to seeing what exciting research the next decades will bring.

NOTE

1. Social constructivism does take account of the social aspect of learning. Yet, it centers on the individual learner in a social context and not on the class as a community.

REFERENCES

Arcavi, A., Kessel, C., Meira, L., & Smith, J. P. III (1998). Teaching mathematical problem solving: An analysis of an emergent classroom community. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *CBMS Issues in Mathematics Education, Volume 7: Research in collegiate mathematics education* (pp. 1–70). Providence, RI: American Mathematical Society.

Australian Education Council (1990). *A national statement on mathematics for Australian Schools*. Melbourne: Curriculum Corporation.

Ball, D. L. (1991a). What's all this talk about "discourse"? *Arithmetic Teacher*, *39*(3), 44–48.

Ball, D. L. (1991b). *Shea numbers* (A video package). East Lansing, MI: Math Project, Michigan State University.

Ball, D. L. (1997). What do students know? Facing challenges of distance, context, and desire in trying to hear children. In B. J. Biddle, T. L. Good, & I. F. Goodson (Eds.), *International handbook of teachers and teaching* (pp. 769–818). Dordrecht, The Netherlands: Kluwer.

Barnett, C. (1991). Building a case-based curriculum to enhance the pedagogical content knowledge of mathematics teachers. *Journal of Teacher Education*, *42*(4), 263–272

Bishop, A. J., Clements, K., Keitel, C., Kilpatrick, J., & Laborde, C. (Eds.) (1996). *International handbook of mathematics education*. Dordrecht, The Netherlands: Kluwer.

Bishop, A. J., Clements, K., Keitel, C., Kilpatrick, J., & Leung, F. K. S. (Eds.) (2003). *Second international handbook of mathematics education*. Dordrecht, The Netherlands: Kluwer.

Booth, L. R. (1988). Children's difficulties in beginning algebra. *The ideas of Algebra, K–12, NCTM Yearbook* (pp. 20-32). Reston, VA: NCTM.

Borasi, R. (1996). *Reconceiving mathematics instruction: A focus on errors*. Norwood, NJ: Ablex.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*, 247-285.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom*. New York: Teachers College Press.

Chinn, C. A., & Brewer, W. F. (1993). The role of anomalous data in knowledge acquisition: a theoretical framework and implications for science instruction. *Review of Educational Research*, *63*(1), 1–49.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York: Macmillan.

Cobb, P., & Bauersfeld, H. (1995). Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 1–16). Hillsdale, NJ: Erlbaum.

Cobb, P., & McClain, K. (1999). Supporting teachers' learning in social and institutional context. In F. Lin (Ed.), *Proceedings of the 1999 international conference on mathematics teacher education* (pp. 7–76). Taipei: National Taiwan Normal University.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *The Journal of the Learning Sciences, 10*(1&2), 113–163.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 117–148). New York: Springer-Verlag.
Collis, K. F. (1975). *The development of formal reasoning.* Newcastle, Australia: University of Newcastle.
D'Ambrosio, B. S., & Campos, T. M. M. (1992). Pre-service teachers' representations of children's understanding of mathematical concepts: Conflicts and conflict resolution. *Educational Studies in Mathematics*, *23*, 213–230.
Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. *Journal of Children's Mathematical Behavior*, *1* (3), 7–35.
Davis, R. B. (1989). Three ways of improving cognitive studies in algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 115–119). Reston, VA: NCTM.
Desforges, C. (1995). How does experience affect theoretical knowledge for teaching? *Learning and Instruction*, *5*(4), 385–400.
Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–123). Dordrecht, The Netherlands: Kluwer.
English, L. (Ed.) (2002). *Handbook of international research in mathematics education.* Mahwah, NJ: Erlbaum.
Even, R. (1999a). The development of teacher-leaders and in-service teacher educators. *Journal for Mathematics Teacher Education*, *2*, 3–24.
Even, R. (1999b). Integrating academic and practical knowledge in a teacher leaders' development program. *Educational Studies in Mathematics*, *38*, 235–252.
Even, R. (2005a). Integrating knowledge and practice at MANOR in the development of providers of professional development for teachers. *Journal of Mathematics Teacher Education, 8*(4), 343–357.
Even, R. (2005b). Using assessment to inform instructional decisions: How hard can it be? *Mathematics Educational Research Journal, 17*(3), 51–67.
Even, R., & Markovits, Z. (1993). Teachers' pedagogical content knowledge of functions: characterization and applications. *Journal of Structural Learning*, *12*(1), 35–51.
Even, R., & Schwarz, B. B. (2003). Implications of competing interpretations of practice to research and theory in mathematics education. *Educational Studies in Mathematics, 54*(2), 283–313.
Even, R., & Wallach, T. (2004). Between student observation and student assessment: A critical reflection. *Canadian Journal of Science, Mathematics, and Technology Education, 4*(4), 483–495
Fennema, E., Carpenter, T., Franke, M., Levi, L., Jacobs, V., & Empson, S. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*(4), 403–434.
Fischbein, E. (1983). The role of implicit models in solving elementary arithmeitc problems. In R. Hershkowitz (Ed.), *Proceedings of the 7th Conference of the International Group for the Psychology of Mathematics Education* (pp. 2–18). Shoresh, Israel: Weizmann Institute of Science.
Fischbein, E. (1993). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler, R. Scholz, R. Straser, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 231–245). Dordrecht, The Netherlands: Kluwer.
Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively-based misconceptions. *Journal for Research in Mathematics Education*, *28*, 96–105.
Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescence. *Journal for Research in Mathematics Education Monograph, 3.*
Goos, M. (2004). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education, 35*(4), 258–291.
Greeno, J. G., Collins, A. M., & Resnick, L. B. (1996). Cognition and learning. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 15–46). New York: Macmillan.
Grouws, D. A. (Ed.) (1992). *Handbook of research on mathematics teaching and learning.* New York: Macmillan.
Gutiérrez, A., & Boero, P. (Ed.) (2006). *Handbook of research on the psychology of mathematics education: Past, present and future.* Rotterdam, the Netherlands: Sense.
Hershkowitz R. (1987). The acquisition of concepts and misconceptions in basic geometry—Or when "a little learning is a dangerous thing." In J. D. Novak (Ed.), *Proceedings of the second international seminar on misconceptions and educational strategies in science and mathematics* (Vol. 3, pp. 238–251). Ithaca, NY: Cornell University.
Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition* (pp. 70–95). Cambridge, MA: Cambridge University Press.
Hershkowitz, R., & Schwarz, B. B. (1999). The emergent perspective in rich environment: Some roles of tools and activities in the construction of sociomathematical norms. *Educational Studies in Mathematics, 39*(1–3), 149–166.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws, (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York: Macmillan.
Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). NJ: Erlbaum.
Hoffer, A. (1983). Van Hiele based research. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 205–227). New York: Academic Press.
Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgment of representativeness. *Cognitive Psychology*, *3*, 430-454.
Kahneman, D., & Tversky, A. (1973). Availability: A heuristic for judging frequency and probability. *Cognitive Psychology*, *5*, 207–232.
Lamon, S. J. (2006). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (2nd ed.). Hillsdale, NJ: Erlbaum.
Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*, 29–63.
Lampert, M. (2001). *Teaching problems and the problems of teaching.* New Haven, CT: Yale University Press.
Lampert, M., & Ball, D. L. (1998). *Teaching, multimedia, and mathematics: Investigations of real practice.* New-York: Teachers College.
Leinhardt, G., Young, K. M., & Merriman, J. (1995). Integrating professional knowledge: The theory of practice and the practice of theory. *Learning and Instruction*, *5*(4), 401–408.
Lerman, S. (1990). The role of research in the practice of mathematics education. *For the Learning of Mathematics*, *10*(2), 25–28.
Lerman, S. (2006). Socio-cultural research in PME. In A. Gutiérrez & P. Boero (Ed.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 347–366). Rotterdam, the Netherlands: Sense.
Llinares, S., & Krainer, K. (2006). Mathematics (students) teachers and teacher educators as learners. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 347–366). Rotterdam, the Netherlands: Sense.
Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, *21*, 16–32.
Markovits, Z., & Even, R. (1999). The decimal point situation: A close look at the use of mathematics-classroom-situations in teacher education. *Teaching and Teacher Education*, *15*(6), 653–665.
Mason, J., & Spence, M. (1999). Beyond mere knlowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics*, *38*, 135–161.
Mulligan, J., & Mitchelmore, M. (1997). Young children's intuitive models of multiplication and devision. *Journal for Research in Mathematics Education*, *28*, 309–330.
National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.
National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author.
Nesher, P. (1986). Are mathematical understanding and algorithmic performance related? *For the learning of mathematics*, *6*(3), 2–9.
Nesher, P., & Kilpatrick, J. (Eds.) (1990). *Mathematics and cognition.* Cambridge, MA: Cambridge University Press.
Owens, K., & Outhred, L. (2006). The complexity of learning geometry and measurement. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 83–116). Rotterdam, the Netherlands: Sense.
Pimm, D. (1987). *Speaking mathematically.* New York: Routledge and Kegan Paul.
Resnick, L. B., & Ford, W. W. (1981). *The psychology of mathematics for instruction.* Hillsdale, NJ: Erlbaum.
Rhine, S. (1998). The role of research and teachers' knowledge base in professional development. *Educational Researcher*, *27*(5), 27–31.
Robinson, N. (1993). *Connectedness in teaching: Equivalent algebraic expressions - by expert and novice teachers.* Unpublished master thesis. Tel Aviv, Israel: Tel Aviv University (in Hebrew).
Scherer P., & Steinbring, H. (2006). Noticing children's learning processes – teachers jointly reflect on their own classroom interaction for improving mathematics teaching. *Journal for Mathematics Teacher Education*, *9*, 157–185.
Schoenfeld, A. (1988). When good teaching leads to bad results: The disasters of "well taught" mathematics classes. *Educational Psychologist*, *23*, 145–166.
Schoenfeld, A. H. (1994). Reflections on doing and teaching mathematics. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 53–70). Hillsdale, NJ: Erlbaum.

Schoenfeld, A. H., Smith, J. P., III, & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 55–175). NJ: Erlbaum.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22* (1), 1–36.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4–13.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Simon, M., & Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics*, *22*, 309–331.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher*, *26*(3), 9–15.

Smith, J. P. III, diSessa, A. A., & Roschelle, J. (1993). Misconceptions reconceived: a constructivist analysis of knowledge in transition. *Journal of the Learning Sciences*, *3*, 115–163.

Stacey, K., & MacGregor, M. (1994). Algebraic sums and products: Students' concepts and symbolism. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* (Vol. 43, pp. 289–296). Lisbon, Portugal.

Stavy, R. & Tirosh, D. (1996). Intuitive rules in science and mathematics: The case of "more of A-more of B." *International Journal of Science Education*, *18*, 653–667.

Stavy, R., & Tirosh, D. (2000). *How students (mis)-understand science and mathematics*. New York: Teachers College Press.

Streefland, L. (1993). Fractions: A realistic approach. In T. R. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 289–325). Hillsdale, NJ: Erlbaum.

Tall, D. O., & Thomas, M. O. J. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics*, *22*, 125–147.

Tirosh, D. (2000). Enhancing propsective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31*, 5–25.

Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics*, *35*, 51–64.

Tirosh, D., & Graeber, A. (2003). Challenging and changing mathematics teaching classroom practices. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 643–688). The Netherlands: Kluwer.

Tversky, A., & Kahneman, D. (1982). Judgments of and by representativeness. In D. Kahneman, P. Slovic, & A. Tversky (Eds.), *Judgement under uncertainty: Heuristics and biases* (pp. 84–100). Cambridge, UK: Cambridge University Press.

Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjunction fallacy in probability judgment. *Psychological Review*, *90*, 293–315.

Urbanska, A. (1993). On the numerical competence of six-year-old children. *Educational Studies in Mathematics, 24*, 265–275.

van Hiele P. M., & van Hiele-Geldof, D. (1958). A method of initiation into geometry. In H. Freudenthal (Ed.), *Report on methods of initiation into geometry* (pp. 67–80). Groningen: Walters.

Vosniadou, S, & Verschaffel, L. (2004). Extending the conceptual change approach to mathematics learning and teaching. *Learning & Instruction, 14,* 445–451.

Wallach, T, & Even, R. (2005). Hearing students: The complexity of understanding what they are saying, showing, and doing. *Journal of Mathematics Teacher Education, 8*(5), 393–417

Wood, T. (2004). In mathematics classes what do students' think? *Journal of Mathematics Teacher Education, 7,* 173–174.

Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education, 37*(3), 222–255.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 9–24). Utrecht, the Netherlands.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*, 458–477.