3

THE GENERALIZED FAUSTMANN FORMULA

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Abstract

This chapter examines the four core areas of the generalized Faustmann formula – the management of even-aged natural stands, even-aged plantations and uneven-aged stands, as well as the development of Pressler's indicator rate formula. Under the generalized formula, stumpage prices, stand volumes, annual incomes, regeneration costs and interest rates could vary from timber crop to timber crop. As a result, the optimal management of even-aged and uneven-aged stands also could vary from timber crop to timber crop. The optimal conditions for the decision variables are derived and their economic meanings explained. Although similar to those obtained under the classic Faustmann formula, the optimal conditions under the generalized Faustmann formula offer much broader and richer interpretations. The increment in stumpage value is shown to consist of price increment, quality increment and quantity increment. The results of comparative statics analysis showed that under the generalized Faustmann formula it is possible to untangle the impacts of changes in current and future production parameters and produce much sharper results. Pressler's indicator rate formula is also shown to maximize the land expectation value under the generalized Faustmann formula. The chapter closes with observations on ongoing efforts and future research opportunities.

Keywords

Generalized Faustmann formula, dynamic programming, even-aged management, uneven-aged management, Pressler's indicator rate formula, price increment, quality increment, quantity increment, comparative statics analysis

Introduction

For nearly 150 years the literature on the determination of optimal rotation age (see Newman, 2002, for a comprehensive compilation of the literature until that time) has relied on the classic Faustmann formula first advanced by Martin Faustmann (1849). In the economic literature, the optimal rotation problem is known as the tree-cutting problem or the wine-storage problem.
Over the years, it has attracted the attention of two Nobel laureates (Ohlin, 1921; Samuelson, 1976). Recognizing that stumpage prices, stand volume, regeneration cost and interest rate do not stay the same rotation after rotation, Chang (1998) developed the generalized Faustmann formula by allowing these factors to vary from harvest period to harvest period. In this chapter, four core areas of the generalized Faustmann formula – the management of (1) even-aged natural stands, (2) even-aged plantations and (3) uneven-aged stands, plus (4) the development of Pressler’s indicator rate formula – will be addressed. As will be shown subsequently, these relaxations provide the generalized Faustmann formula with much greater flexibility and produce much richer analytical results.

Under the first topic, the question of optimal harvest age for even-aged natural stands will be examined. Given that about 93% of the world’s forests are some type of natural stand (FAO, 2012), this topic is highly pertinent. The condition of reaching optimal harvest age will be examined along with a graphic analysis of the impact of changes in various production parameters. In addition, the total increment in stumpage value will be separated into price increment, quality increment and quantity increment. The relationship among the various formulas of optimal harvest age determination will also be discussed.

The second topic addresses the determination of optimal planting density and harvest age. With most of the industrial roundwood coming from plantations, its proper management is becoming ever more important and deserves careful examinations. The impact of changes in both current and future production parameters on the management decision variables will then be examined through comparative statics analysis.

Under the third topic, the generalized Faustmann formula for uneven-aged management will be developed. It will be shown that the formula resembles that of even-aged plantation management. With both management systems sharing the same theoretical foundation, further analyses are no longer needed. All of the analytical results for the management of even-aged plantations can be readily applicable to that of uneven-aged stands.

Under the fourth topic, Pressler’s indicator rate formula will be shown to also represent the optimal condition for the generalized Faustmann formula. The chapter closes with observations on some current developments and future research opportunities.

The generalized Faustmann formula for even-aged natural stand management

Of the 4 billion hectares of forest in the world, 36% is primary forests and 57% is other naturally regenerated forests (FAO, 2012). Most of these forests are managed extensively as even-aged stands. After a clearcut, the stand is typically regenerated naturally, with or without incurring some expenses. The key management question thus revolves around how long one should wait before harvesting the new stand. As the simplest form of even-aged management, it will be discussed first.

Let

\[ V(t_i) = \sum_{j=1}^{n} P_{ij}(t_i) W_{ij}(t_i) Q_{ij}(t_i) \]

be the stumpage value of the \( i \)th timber crop at age \( t_i \), with \( \frac{\partial V(t_i)}{\partial t_i} > 0 \) and \( \frac{\partial^2 V(t_i)}{\partial t_i^2} < 0 \). \( P_{ij}(t_i) \) is the stumpage price of \( i \)th timber crop at age \( t_i \) for product class \( j \). For example, in the US South, southern pine timber stands typically consist of pulpwood, chip-and-saw timber and sawtimber.
$W_j(t)$ is the percentage of the product class $j$ at age $t$, of the $i$th stand volume, $Q_j(t)$ is the total stand volume at age $t_i$ and the volume of a particular product class, $Q_j(t) = W_j(t)Q_i(t)$, $A(s_i)$ is the net annual income for age $s_i$, $0 \leq s_i \leq t$, of the $i$th timber crop, $C_i$ is the regeneration cost for the $i$th timber crop, $r_i$ is the interest rate associated with the $i$th timber crop and $LEV_i$ is the land expectation value at the beginning of the $i$th timber crop.

To maximize the value of the land, we want to maximize the present value of profits from growing an infinite number of timber crops.

$$\text{Max } LEV_i = \sum_{j=1}^{k} \left[ V_j(t_i) + \sum_{i=1}^{n} A_i(s_i) \exp(r_i(t_i - s_j)) - C_i \exp(t_i) \right] \exp \left( \sum_{j=1}^{\infty} -r_i t_j \right)$$

(1)

Note that as a special case, if all $V_j(t_i)$, $A_i(s_i)$, $C_i$ and $r_i$ remain the same for all timber crops, then equation (1) can be expressed as

$$LEV_i = \left[ V_j(t_i) + \sum_{i=1}^{n} A_i(s_i) \exp(r_i(t_i - s_j)) - C_i \exp(t_i) \right] \left[ \exp(-r_i t_i) + \exp(-2r_i t_i) + \cdots \right]$$

$$= \left[ V_j(t_i) + \sum_{i=1}^{n} A_i(s_i) \exp(r_i(t_i - s_j)) - C_i \exp(t_i) \right] \left( \exp(r_i t_i) - 1 \right)$$

(2)

and collapses to equation (2) as the classic Faustmann formula. Note also that equation (1) includes the Hartman (1976) formula as a special case. For easy comprehension, equation (1) can also be written as

$$\text{Max } LEV_i = \left[ V_j(t_i) + \sum_{i=1}^{n} A_i(s_i) \exp(r_i(t_i - s_j)) - C_i \exp(t_i) \right] \exp(-r_i t_i)$$

$$+ LEV_{i+1} \exp(-r_i t_i)$$

(3)

In the previous equations the term ‘timber crop’ should be broadly interpreted. If future crops remain in forestry, they are naturally timber crops. If in the future, the land is switched to growing fruit trees, it would still be viewed as a timber crop. In this case, the income from annual fruit production becomes much more important, whereas that from the final harvest to replace the old fruit trees becomes far less important. Even in the case of conversion to annual crop production or real estate development, there are simply no timber crops in the future. Only the annual net incomes are involved. It should also be noted that over time, the timber crop species could change, for example, from southern pine to hardwood or from spruce to Douglas-fir. It could also change from timber production to fruit production or crop production and vice versa. The generalized Faustmann formula, therefore, could accommodate land-use changes by permitting different types of crops, may they be timber, fruit or grain, for different harvest periods. In the first case, the value of the timberland is determined endogenously, whereas in the latter cases, with land-use change under the generalized Faustmann formula, the value of the land in the future, as $LEV_i$ in equation (3), is determined exogenously as shown by Klemperer and Farkas (2001).

Equation (3) represents the famous recurrence relation of dynamic programming. In this equation, $LEV_1$ and $LEV_2$ represent the objective functions, and the expression
The generalized Faustmann formula

\[
V_i(t_i) + \sum_{s=1}^{s_i} A_i(s_i) \exp(r_i(t_i - s_i)) - C_i \exp(r_i t_i) \right] \exp(-r_i t_i)
\]

represents the payoff associated with the decision variable \( t_i \). Theoretically, equation (3) can be solved with the forward recursive solution method. However, such a solution would involve infinite numbers of stumpage prices, stand volumes, annual incomes or expenses, regeneration costs and interest rates, thus making it impractical. Fortunately, \( LEV_2 \) represents just a single value. It embodies all the optimal harvest age decisions for future timber crops that give rise to this specific value. Forest owners and/or managers need not know the details of these decisions, just that they give rise to the specific value. Therefore, solving for the optimal harvest age empirically would involve the insertion of a specific value of \( LEV_2 \) into equation (3) to solve for \( t_i \). Such a value could be gleaned from various timberland transactions if there is an active timberland market. Or it could be chosen judiciously to determine the resulting harvest age for the first timber crop under various future values for the timberland.

**On reaching the optimal harvest age**

In addition to finding the optimal harvest age under equation (3), it is important to understand the economic meaning of reaching the optimal harvest age because it affords the opportunity to determine stepwise year by year the harvest decision by comparing the marginal benefit with the marginal cost of waiting. At the optimal \( t_i \)

\[
\frac{\partial LEV_i}{\partial t_i} = \left[ \frac{\partial V_i(t_i)}{\partial t_i} + \sum_{s=1}^{s_i} A_i(s_i) \exp(r_i(t_i - s_i)) + A_i(t_i) - C_i \exp(r_i t_i) \right] \exp(-r_i t_i)
\]

\[
+ |V_i(t_i) + \sum_{s=1}^{s_i} A_i(s_i) \exp(r_i(t_i - s_i)) - C_i \exp(r_i t_i)| (-r_i) \exp(-r_i t_i)
\]

\[
+ LEV_2(-r_i) \exp(-r_i t_i) = 0
\]

\[
\frac{\partial V_i(t_i)}{\partial t_i} + A_i(t_i) = r_i V_i(t_i) + r_i LEV_2
\]

Equation (5) states that at the optimal harvest age, the extra amount of stumpage value earned by waiting one more year plus the extra annual income on the left-hand side of the equation must equal the cost of holding the trees plus the cost of holding the land on the right-hand side of the equation. When the left-hand side of equation (5) is greater than the right-hand side, one should wait another year. Conversely, the stand should be harvested. In the interest of brevity, no empirical examples for this topic will be presented. Readers interested in such examples are referred to Chang (1998).

**The separation of the stumpage value increment**

What is the benefit of waiting? Pressler (1860) pointed out that the stumpage value increment \( \frac{\partial V_i(t_i)}{\partial t_i} \) consists of three types of increments when the harvest age is delayed one time period. They are the quantity increment (\( Quantitätzzuwachs \)), the quality increment (\( Qualitätzzuwachs \)) and, lastly, the price increment (\( Täuberungzatzuwachs \)). Over the years, these increments have been mentioned in various textbooks; however, it was Chang and Deegen (2011) who separated these satisfactorily both analytically and empirically. Given that
\[
\frac{\partial V_i(t_i)}{\partial t_i} = \sum_{j=1}^{n} \left[ \frac{\partial W_{ij}(t)}{\partial t_i} W_{ij}(t) Q_i(t_i) + P_{ij}(t) \frac{\partial W_{ij}(t)}{\partial t_i} Q_i(t_i) + P_{ij}(t) W_{ij}(t) \frac{\partial Q_i(t_i)}{\partial t_i} \right]
\]

(6)

\[
\sum_{j=1}^{n} P_{ij}(t) W_{ij}(t) Q'_i(t_i), \text{ the increase in stand value as a result of total stand volume increment, represents the quantity increment. The gain realized from changes in the composition of different product classes of the stand volume } \sum_{j=1}^{n} P_{ij}(t) W_{ij}'(t_i) Q_i(t_i) \text{ represents the quality increment. Finally, the gain realized from changes in prices of different product classes, } \sum_{j=1}^{n} P_{ij}'(t_i) W_{ij}(t_i) Q_i(t_i), \text{ represents the price increment. It should be noted that in some instances, the quality increment may not matter. For example, in the emerging biomass for energy market, sometimes no quality is recognized. In such a case, the quality increment simply falls out, and only price and quantity increments remain.}

In practice, the growth in stumpage value over time can be determined by

\[
V_i(t_i+1) - V_i(t_i) = \sum_{j=1}^{n} [P_{ij}(t_i+1) - P_{ij}(t_i)] W_{ij}(t_i+1) Q_i(t_i+1) + \sum_{j=1}^{n} P_{ij}(t_i) W_{ij}(t_i) Q_i(t_i+1) - W_{ij}(t_i) Q_i(t_i+1) + \sum_{j=1}^{n} P_{ij}(t_i) W_{ij}(t_i) | Q_i(t_i+1) - Q_i(t_i)|
\]

(7)

Dividing \(V_i'(t_i)\) by \(V_i(t_i)\) results in

\[
\frac{V_i'(t_i)}{V_i(t_i)} = \frac{\sum_{j=1}^{n} P_{ij}(t_i) W_{ij}(t_i) Q_i(t_i)}{V_i(t_i)} + \frac{\sum_{j=1}^{n} P_{ij}(t_i) W_{ij}'(t_i) Q_i(t_i)}{V_i(t_i)} + \frac{\sum_{j=1}^{n} P_{ij}'(t_i) W_{ij}(t_i) Q'_i(t_i)}{V_i(t_i)}
\]

(8)

with the three terms on the right-hand side of equation (8) being the rates of price increment, quality increment and quantity increment, respectively. Among them, the last two increments in equation (8) are usually positive and under the control of a forester. Price increment or the rate of price increment, however, as Pressler warned, could be either positive or negative depending on the overall economy, specific technological developments or market conditions. For an example of separating these three increments empirically, the readers are referred to Chang and Deegen (2011).

**Comparative statics analyses of the impact of changes in stumpage price levels, regeneration cost, annual income and regeneration cost**

How will the current versus future changes in stumpage prices, annual income, regeneration cost and interest rate affect the optimal harvest age of the current timber crop? These analyses are important because they will show a priori how the optimal harvest age will be affected before any empirical analyses. Here the impact of these changes will be analyzed graphically. Mathematical analyses of the impact of these changes are available in Chang (1998). To analyze graphically the impact of changes in production factors both currently and in the future, first rewrite equation (5) as

\[
\frac{V_i'(t_i) + A_i(t_i)}{V_i(t_i) + LEV_2} = t_i
\]

(9)
and name the left-hand side as the rate of marginal revenue growth (RMRG). As the timber stand ages, \( V'(t) \) gradually declines. The numerator of the RMRG approaches \( A_1(t) \), and the denominator increases and approaches the sum of the limit of \( V'(t) \) plus \( LEV_2 \). As shown in Figure 3.1, the RMRG curve gradually trends downward. On the other hand, the interest rate line is shown as a flat line. The point where these two curves cross is the optimal harvest age. With this graph, one can quickly see that a higher regeneration cost for the current timber crop, as a sunk cost, has no effect on the optimal harvest age of the current timber crop. On the other hand, a higher stumpage price level for the current timber crop would impact both the numerator and denominator of RMRG. When \( V'(t)/V(t) \) is greater than \( r \), higher stumpage prices would raise the current harvest age and vice versa. A higher annual income, on the other hand, would always move the RMRG curve up and raise the current harvest age. Finally, a higher current interest rate would simply move the interest rate line up and result in a lower harvest age for the current timber crop.

The impacts of all the changes in the production factors of future timber crops are reflected through \( LEV_2 \). For example, a higher stumpage price level for any of the future timber crops would result in a higher \( LEV_2 \) and consequently a smaller RMRG. A downward move of the RMRG curve will then lead to a lower harvest age for the current timber crop. The same is true for higher annual incomes for any of the future timber crops. On the other hand, a higher interest rate or a higher regeneration cost for any of the future timber crops would translate into a smaller \( LEV_2 \) and result in a bigger RMRG. As such, they will both lead to a higher harvest age for the current timber crop.

Table 3.1 summarizes the results of all of the comparative statics analyses and also compares these results with those under the classic Faustmann formula. Indeed, the generalized Faustmann formula yields much richer results. Under the classic Faustmann formula, a higher stumpage price level would always shorten the rotation. Yet under the generalized Faustmann formula, a higher current stumpage price level would either raise or lower the current harvest age, whereas a higher future stumpage price level would lower the current harvest age. Whereas

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**Figure 3.1** Rate of marginal revenue growth (RMRG) and interest rate \( r \).
a higher regeneration cost would raise the rotation age under the classic Faustmann formula, only a higher future regeneration cost would do so under the generalized Faustmann formula. Current regeneration cost under the generalized Faustmann formula, as a sunk cost, has no impact on the optimal harvest age. The impact of higher annual income levels under the classic Faustmann formula depends on whether $A(s)$ is an increasing or decreasing function of stand age. On the other hand, under the generalized Faustmann formula, the higher level of current annual income would raise the current harvest age, whereas higher levels of annual incomes in future timber crops would have the opposite effect. Lastly, a higher interest rate under the classic Faustmann formula lowers the optimal rotation age. Under the generalized Faustmann formula, a higher current interest rate lowers the optimal harvest age, whereas a higher future interest rate raises the optimal harvest age.

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**Table 3.1** The results of comparative statics analyses under the classic Faustmann formula and the generalized Faustmann formula.

<table>
<thead>
<tr>
<th>Cause</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Under classic Faustmann formula</strong></td>
<td></td>
</tr>
<tr>
<td>A one-time increase in $C$, generation cost</td>
<td>increase</td>
</tr>
<tr>
<td>$\alpha$ in $\alpha P(t)$, stumpage price level</td>
<td>increase</td>
</tr>
<tr>
<td>$\beta$ in $\beta A(s)$, annual income</td>
<td>increase if $A(s) &lt; A(t)$ for all $s$; no change if $A(s) = A(t)$ for all $s$; decrease if $A(s) &gt; A(t)$ for all $s$</td>
</tr>
<tr>
<td>$r$, interest rate</td>
<td></td>
</tr>
</tbody>
</table>

**Under generalized Faustmann formula**

<table>
<thead>
<tr>
<th>Cause</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current timber crop</td>
<td></td>
</tr>
<tr>
<td>$C$, regeneration cost</td>
<td>no change</td>
</tr>
<tr>
<td>$\alpha$ of $\alpha P(t)$, stumpage price level</td>
<td>increase</td>
</tr>
<tr>
<td>if $\frac{\partial V(t)}{\partial t} &gt; \eta V(t)$, increase</td>
<td></td>
</tr>
<tr>
<td>if $\frac{\partial V(t)}{\partial t} &lt; \eta V(t)$, decrease</td>
<td></td>
</tr>
<tr>
<td>$\beta$ of $\beta A(s)$, annual income</td>
<td>increase</td>
</tr>
<tr>
<td>$r$, interest rate</td>
<td>decrease</td>
</tr>
</tbody>
</table>

| Future timber crop |        |
| $C$, regeneration cost | increase |
| $\alpha$ of $\alpha P(t)$, stumpage price level | decrease |
| $\beta$ of $\beta A(s)$, annual income | decrease |
| $r$, interest rate | increase |
Other formulas of optimal harvest age determination and their relationship with the generalized Faustmann formula

Over the years, other formulas have been proposed to determine the optimal harvest age. Chief among them are the present net worth (PNW) formula, which maximizes the present value of the profit from growing just one crop of timber:

\[ PNW = V_1(t_i) \exp(-r t_i) - C_i \]

the forest rent (FR) formula of maximizing:

\[ FR = \frac{V_1(t_i) - C_i}{t_i} \]

and the biological formula of maximizing the mean annual increment (MAI):

\[ MAI = \frac{Q_1(t_i)}{t_i} \]

Regarding the relationship between \( LEV_1 \) and PNW, note that when all the annual incomes \( A(s) \) of the current timber crop as well as \( LEV_2 \) – the present value of all incomes and expenses from future timber crops – are ignored, then \( LEV_1 \) becomes PNW. Given that the PNW formula ignores the cost of holding the land, it will lead to an optimal harvest age that is higher than that from the generalized Faustmann formula.

The relationship between the generalized Faustmann formula and the FR formula is examined through the land rent (R). Note that when all the annual incomes \( A(s) \) are ignored,

\[ R = r_i \left[ V_1(t_i) - C_i \exp(r t_i) + LEV_2 \right] \exp(-r t_i) \]

Applying L’Hospital’s rule when \( r_i \) approaches 0,

\[ \lim_{r_i \to 0} R = \lim_{r_i \to 0} \frac{r_i \left[ V_1(t_i) - C_i \exp(r t_i) + LEV_2 \right] \exp(-r t_i)}{r_i \exp(r t_i)} \]

\[ = \lim_{r_i \to 0} \left[ \frac{V_1(t_i) - C_i \exp(r t_i) + LEV_2}{r_i} \right] \frac{\lim_{r_i \to 0} t_i \exp(r t_i)}{\lim_{r_i \to 0} t_i \exp(r t_i)} \]

\[ = (V_1(t_i) - C_i) / t_i = r_i R \text{ when } LEV_2 = 0 \]

That is to say, \( R \) collapses to \( FR \) when all the annual incomes are ignored, \( LEV_2 = 0 \) and interest rate \( r_i \) also equals 0. Given that when \( LEV_1 \) is maximized the land rent is also maximized, only when the previous conditions are satisfied will the FR formula result in the correct optimal harvest age.

For the biological formula of MAI maximization, note that when \( P_1(t_i) = k \) and \( C_i = 0 \), then

\[ FR = \frac{P_1(t_i) Q_1(t_i) - C_i}{t_i} \frac{k Q_1(t_i)}{t_i} - k MAI \]

That is to say, when all the annual incomes are ignored; \( LEV_2 \), interest rate \( r_i \) and regeneration cost \( C_i \), all equal to 0; and the stumpage prices of trees of different ages are all the same, implying that there is no premium for older and therefore larger diameter trees, then \( R \) collapses to \( MAI \), and the MAI formula results in the correct optimal harvest age.
The generalized Faustmann formula for even-aged plantation management

Timber plantations now account for 7% of the forests in the world (FAO, 2012). Despite this relatively small percentage, in recent decades these plantations have been producing an ever-increasing amount of industrial roundwood supplies. Large acreages of pine plantations have been established in the US South, Brazil, Chile and New Zealand, as well as extensive Chinese fir plantations in China. Eucalyptus plantations have been established in Brazil, China, Australia and several Southeast Asian countries. Red pine and spruce plantations have been established widely in Europe. In the future, energy plantations could also emerge to play an important role in sequestering carbon dioxide emissions. More importantly, these plantations with their high productivity assure the possibility of conserving natural forests and ecosystems.

For even-aged plantations, both the harvest age and the initial planting density must be determined simultaneously. In this section, the notations defined earlier are expanded as follows:

\[ P_i(t, n) \] is the stumpage price for the \( j \)th product class of the \( i \)th plantation established with an initial planting density of \( n_i \) at age \( t_i \).

\[ W_j(t, n) \] is the percentage of the \( j \)th product class of the \( i \)th plantation established with an initial planting density of \( n_i \) at age \( t_i \).

\[ Q_i(t, n) \] is the stand volume of the \( i \)th plantation established with an initial planting density of \( n_i \) at age \( t_i \).

\[ V_i(t, n) = \sum_{j=1}^{J} P_j(t, n_i) W_j(t, n_i) Q_j(t, n_i) \] is the stumpage value of the \( i \)th plantation with an initial planting density of \( n_i \) at age \( t_i \), with \( \frac{\partial V_i(t, n_i)}{\partial t_i} > 0, \frac{\partial^2 V_i(t, n_i)}{\partial n_i^2} < 0.\)

\( C_s \) stands for the site preparation cost for the \( i \)th plantation.

\( C_p \) stands for the cost of planting per seedling, including the cost of both the labor and seedling.

All other variables are as defined previously.

Following equation (3), the generalized Faustmann land expectation value formula for plantation management can be expressed as

\[ LEV_i = \int_{t_0}^{t_1} \left[ V_i(t, n_i) + \int_{t_0}^{t_1} A_i(s_i) \exp(\eta_i(t_i - s_i)) ds_i - (C_s + C_p n_i) \exp(\eta_i t_i) \right] \exp(-\eta_i t_i) \]
\[ + LEV_2 \exp(-\eta_i t_i) \]

\[ \frac{\partial LEV_i}{\partial t_i} = \left[ \frac{\partial V_i(t, n_i)}{\partial t_i} + \eta_i \int_{t_0}^{t_1} A_i(s_i) \exp(\eta_i(t_i - s_i)) ds_i + A_i(t_i) - (C_s + C_p n_i) \exp(\eta_i t_i) \right] \]
\[ \exp(-\eta_i t_i) + \left[ V_i(t, n_i) + \int_{t_0}^{t_1} A_i(s_i) \exp(\eta_i(t_i - s_i)) ds_i - (C_s + C_p n_i) \exp(\eta_i t_i) \right] \]
\[ (-\eta_i) \exp(-\eta_i t_i) - \eta_i \exp(-\eta_i t_i) LEV_2 = 0 \]

\[ \frac{\partial LEV_i}{\partial n_i} = \left[ \frac{\partial V_i(t, n_i)}{\partial n_i} - C_p \exp(\eta_i t_i) \right] \exp(-\eta_i t_i) = 0 \]
The generalized Faustmann formula

For notational simplicity, \( \sum_{i=1}^{n} A_i(s_i) \exp(r_i(t_i - s_i)) \) is replaced by \( \int_0^t A_i(s_i) \exp(r_i(t_i - s_i)) ds_i \). To maximize \( LEV_1 \), from equation (11):

\[
\frac{\partial V_1(t_1, n_1)}{\partial t_1} + A_1(t_1) - r_1 V_1(t_1, n_1) - r_1 LEV_2 = 0
\] (13)

from equation (12):

\[
\frac{\partial V_2(t_1, n_1)}{\partial n_1} - c_p \exp(r_1 t_1) = 0
\] (14)

Equation (13) states that at optimal harvest age, the extra stumpage value plus the extra annual income earned by waiting one more year must equal the cost of holding the trees plus the cost of holding the land, similar to the case of even-aged natural stand management discussed earlier. Equation (14) suggests that at the optimal planting density, the extra stumpage value earned by planting an additional tree must equal the extra cost of planting the extra tree compounded to the end of the harvest period.

Table 3.2 presents an example of the simultaneous determination of optimal harvest age and planting density with an interest rate of 5.5% for the first harvest period, a site preparation cost of US$160 per acre and a planting cost of US$0.10 per tree, including the cost of the seedling and labor for planting, with no annual income and a future land value of US$800 per acre. Stumpage prices are US$80 per cord for chip-and-saw logs and US$28 per cord for pulpwood, with 76 cubic feet of solid wood per 128 cubic feet \((4' \times 8' \times 8')\) of stacked volume. Given these parameters, the optimal planting density will be 700 trees per acre and optimal harvest age will be 26 years.

**Comparative statics analysis of the generalized Faustmann formula for even-aged plantations**

To carry out comparative statics analyses, the second-order conditions for the optimal combination of \( t_1 \) and \( n_1 \) must be established first.

\[
\frac{\partial^2 LEV_1}{\partial t_1^2} = \left[ \frac{\partial^2 V_1(t_1, n_1)}{\partial t_1^2} + \frac{\partial A_1(t_1)}{\partial t_1} - r_1 \frac{\partial V_1(t_1, n_1)}{\partial t_1} \right] \exp(-r_1 t_1) + \left[ \frac{\partial V_1(t_1, n_1)}{\partial t_1} + A_1(t_1) - r_1 V_1(t_1, n_1) - r_1 LEV_2 \right] (-r_1) \exp(-r_1 t_1) = \left[ \frac{\partial^2 V_1(t_1, n_1)}{\partial t_1^2} + \frac{\partial A_1(t_1)}{\partial t_1} - r_1 \frac{\partial V_1(t_1, n_1)}{\partial t_1} \right] \exp(-r_1 t_1) < 0
\] (15)

Equation (15) is less than 0 because the terms inside the bracket on the second line are the first-order condition for optimal \( t_1 \) and equal 0.

\[
\frac{\partial^2 LEV_1}{\partial n_1^2} = \frac{\partial^2 V_1(t_1, n_1)}{\partial n_1^2} \exp(-r_1 t_1) < 0
\] (16)
Table 3.2 The simultaneous determination of optimal planting density and harvest age.

<table>
<thead>
<tr>
<th>Planting density (trees per acre)</th>
<th>Harvest age (years)</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>400 Cords of C&amp;S</td>
<td>15.46</td>
<td>17</td>
<td>18.72</td>
<td>20.92</td>
<td>22.63</td>
<td>24.2</td>
<td>26.14</td>
<td>27.35</td>
<td></td>
</tr>
<tr>
<td>400 Cords of pulpwood</td>
<td>12.6</td>
<td>12.61</td>
<td>12.99</td>
<td>12.17</td>
<td>12.08</td>
<td>12.68</td>
<td>12.28</td>
<td>12.29</td>
<td></td>
</tr>
<tr>
<td>400 Stumpage value/A</td>
<td>1,589.71</td>
<td>1,713.31</td>
<td>1,861.49</td>
<td>2,013.92</td>
<td>2,148.67</td>
<td>2,291.39</td>
<td>2,435</td>
<td>2,532.37</td>
<td></td>
</tr>
<tr>
<td>400 LEV/Acre</td>
<td>552.90</td>
<td>549.46</td>
<td>551.18</td>
<td>551.70</td>
<td>545.54</td>
<td>539.80</td>
<td>532.74</td>
<td>514.40</td>
<td></td>
</tr>
<tr>
<td>500 Stumpage value/A</td>
<td>1,611.59</td>
<td>1,749.72</td>
<td>1,897.42</td>
<td>2,085.16</td>
<td>2,223.08</td>
<td>2,399.61</td>
<td>2,516.34</td>
<td>2,680.58</td>
<td></td>
</tr>
<tr>
<td>500 LEV/Acre</td>
<td>549.79</td>
<td>530.32</td>
<td>551.32</td>
<td>560.73</td>
<td>554.35</td>
<td>555.70</td>
<td>541.16</td>
<td>536.17</td>
<td></td>
</tr>
<tr>
<td>600 Cords of C&amp;S</td>
<td>14.46</td>
<td>16.13</td>
<td>18.61</td>
<td>20.72</td>
<td>22.75</td>
<td>24.46</td>
<td>26.27</td>
<td>28.28</td>
<td></td>
</tr>
<tr>
<td>600 Cords of pulpwood</td>
<td>16.97</td>
<td>17</td>
<td>17.11</td>
<td>16.45</td>
<td>16.67</td>
<td>16.74</td>
<td>16.87</td>
<td>16.53</td>
<td></td>
</tr>
<tr>
<td>600 Stumpage value/A</td>
<td>1,632.24</td>
<td>1,766.37</td>
<td>1,968.05</td>
<td>2,117.95</td>
<td>2,286.24</td>
<td>2,425.91</td>
<td>2,573.94</td>
<td>2,725.03</td>
<td></td>
</tr>
<tr>
<td>600 LEV/Acre</td>
<td>546.30</td>
<td>545.28</td>
<td>561.25</td>
<td>559.49</td>
<td>560.32</td>
<td>551.99</td>
<td>544.21</td>
<td>535.70</td>
<td></td>
</tr>
<tr>
<td>700 Cords of C&amp;S</td>
<td>13.87</td>
<td>15.69</td>
<td>18.5</td>
<td>20.48</td>
<td>21.94</td>
<td>24.68</td>
<td>26.59</td>
<td>28.21</td>
<td></td>
</tr>
<tr>
<td>700 Stumpage value/A</td>
<td>1,639.46</td>
<td>1,797.58</td>
<td>1,996.58</td>
<td>2,146.29</td>
<td>2,282.91</td>
<td>2,509</td>
<td>2,641.03</td>
<td>2,763.8</td>
<td></td>
</tr>
<tr>
<td>700 LEV/Acre</td>
<td>538.57</td>
<td>544.59</td>
<td>559.30</td>
<td>557.06</td>
<td>549.08</td>
<td>561.87</td>
<td>549.40</td>
<td>534.01</td>
<td></td>
</tr>
<tr>
<td>800 Cords of pulpwood</td>
<td>20.68</td>
<td>21.11</td>
<td>20.24</td>
<td>20.49</td>
<td>20.7</td>
<td>20.66</td>
<td>20.16</td>
<td>19.91</td>
<td></td>
</tr>
<tr>
<td>800 Stumpage value/A</td>
<td>1,655.05</td>
<td>1,803.78</td>
<td>2,021.56</td>
<td>2,174.61</td>
<td>2,343.02</td>
<td>2,509.88</td>
<td>2,667.35</td>
<td>2,840.2</td>
<td></td>
</tr>
<tr>
<td>800 LEV/Acre</td>
<td>532.05</td>
<td>536.44</td>
<td>556.36</td>
<td>554.62</td>
<td>554.68</td>
<td>552.08</td>
<td>545.36</td>
<td>540.39</td>
<td></td>
</tr>
</tbody>
</table>

Stumpage price for chip and saw (C&S) = US$80/cord, pulpwood = US$28/cord, \( C_i = \) US$160/A, \( C_p = \) US$0.10/tree. Interest rate = 5.5%, and \( LEV_2 = \) US$800/acre.
and
\[
D = \begin{vmatrix}
\frac{\partial^2 LEV_i}{\partial t^2} & \frac{\partial^2 LEV_i}{\partial t \partial n_i} \\
\frac{\partial^2 LEV_i}{\partial t \partial n_i} & \frac{\partial^2 LEV_i}{\partial n_i^2}
\end{vmatrix}
= (\frac{\partial^2 LEV_i}{\partial t^2})(\frac{\partial^2 LEV_i}{\partial n_i^2}) - (\frac{\partial^2 LEV_i}{\partial t \partial n_i})^2 > 0
\]

(17)
as part of the second-order conditions.

It should be noted that
\[
\frac{\partial^2 LEV_i}{\partial t \partial n_i} = [\frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} - Cp_t \exp(r_t) \exp(-r_t)] + [\frac{\partial V_i(t_1,n_i)}{\partial n_i} \times -Cp_t r_t
\]

(18)
because the terms of the second line are the first-order condition for the optimal \( n_i \).

Thus, although equation (17) must be true, a priori nothing is said about the sign of \( \frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} \). Given that \( \frac{\partial V_i(t_1,n_i)}{\partial t} \) represents the current annual increment in revenue, \( \frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} \) represents changes in current annual increment in stumpage value as a result of changes in planting density. As Kent and Dress (1980) have shown, plantations of different initial planting densities eventually converge to the same random pattern. As such, given enough time, these stands of different initial planting densities will also converge to the same stand volume, and thus, value. Figure 3.2 shows two of the stumpage value curves and their corresponding current annual increments in stumpage value curves. For the stand with a higher planting density, its current annual increment (CAI) in stumpage value ascends faster, peaks at an earlier age and descends faster thereafter. For the stand with a lower planting density, its CAI ascends slower, peaks at a later age and descends slower thereafter. As shown in Figure 3.2, these two CAI curves will cross each other at an age \( T \). Because the area below the CAI in stumpage value curve stands for the stumpage value, the vertically shaded area represents that period when the higher planting density stand outgrows the lower planting density stand in value. The horizontally shaded area, on the other hand, would represent the opposite case. At an age \( T \), these two shaded areas would be equal in size, and the two stands would end up with the same stumpage value thereafter. Once the optimal planting density is determined, the relevant CAI in stumpage value curve will be uniquely defined. The critical question, then, is the position of the optimal harvest age \( t_1 \) relative to \( T \). If \( t_i \) is less than \( T \), \( \frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} > 0 \). If \( t_i \) is larger than \( T \), \( \frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} < 0 \). When \( t_i \) and \( T \) coincide, \( \frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} = 0 \). Thus, there are three possibilities.

Case 1, \( \frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} > 0 \) and \( [\frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i}] \exp(-r_t t_i) - Cp_t r_t < 0 \)

Case 2, \( \frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} > 0 \) and \( [\frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i}] \exp(-r_t t_i) - Cp_t r_t < 0 \)

Case 3, \( \frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} < 0 \)

As the subsequent analyses demonstrate, the sign of \( \frac{\partial^2 V_i(t_1,n_i)}{\partial t \partial n_i} \) plays an important role in discerning the impact of changes in site preparation cost, cost of planting, stumpage price and interest rate.
The impact of changes in current site preparation cost, \( Cs_1 \)

As shown in Appendix A-1, \( \frac{dt_1}{dCs_1} = 0 \) and \( \frac{dn_1}{dCs_1} = 0 \), suggesting that a change in the current site preparation cost, as a sunk fixed cost, affects neither the harvest age nor the planting density of the current timber crop.

The impact of changes in current planting cost, \( Cp_1 \)

As shown in Appendix A-2,

\[
\frac{dt_1}{dCp_1} = \frac{-\partial^2 LEV_i}{\partial t_i \partial n_i} / D
\]

(19)

and

\[
\frac{dn_1}{dCp_1} = \frac{\partial^2 LEV_i}{\partial t_i^2} / D < 0
\]

(20)

From equation (20), \( \frac{dn_1}{dCp_1} < 0 \), suggesting that a higher current planting cost always leads to a lower planting density for the current stand.
The generalized Faustmann formula

The effect of a higher current planting cost on the optimal harvest age of the current stand, on the other hand, depends on the sign and the magnitude of \( \partial^2 V_i(t, n_i) / \partial t_i \partial n_i \).

Under case 1, when \( \partial^2 V_i(t, n_i) / \partial t_i \partial n_i > 0 \) and \( [\partial^2 V_i(t, n_i) / \partial t_i \partial n_i] \exp(-r_i t_i) - C_{P_i}t_i > 0 \), \( dt_i / dC_{P_i} < 0 \). Higher planting cost for the current timber crop lowers the optimal harvest age for the current timber crop.

Under case 2, when \( \partial^2 V_i(t, n_i) / \partial t_i \partial n_i > 0 \) but \( [\partial^2 V_i(t, n_i) / \partial t_i \partial n_i] \exp(-r_i t_i) - C_{P_i}t_i < 0 \), and under case 3, when \( \partial^2 V_i(t, n_i) / \partial t_i \partial n_i < 0 \), \( dt_i / dC_{P_i} > 0 \), higher planting cost for the current timber crop raises the optimal harvest age for the current timber crop. Whether the impact of higher planting cost on the optimal harvest age is case 1, 2 or 3 can only be determined empirically.

**The impact of a higher current stumpage price level across the board, \( \alpha_i \)**

As shown in Appendix A-3,

\[
\frac{dt_i}{d\alpha_i} = \left\{ -\frac{\partial V_i(t_i, n_i)}{\partial t_i} + r_i V_i(t_i, n_i) \right\} \exp(-r_i t_i) \frac{\partial^2 V_i(t, n_i)}{\partial n_i^2} \exp(-r_i t_i) \\
- \frac{\partial^2 V_i(t, n_i)}{\partial n_i} \exp(-r_i t_i) \left[ \frac{\partial^2 V_i(t, n_i)}{\partial t_i \partial n_i} \right] \exp(-r_i t_i) / D
\]

(21)

\[
\frac{dn_i}{d\alpha_i} = \left\{ (\partial^2 LEV_i / \partial t_i^2) \left[ -\frac{\partial V_i(t_i, n_i)}{\partial t_i} + r_i V_i(t_i, n_i) \right] \exp(-r_i t_i) \right\} \\
- \left[ \frac{\partial^2 V_i(t_i, n_i)}{\partial t_i \partial n_i} \exp(-r_i t_i) - C_{P_i}t_i \right] \exp(-r_i t_i) / D
\]

(22)

From equation (21) and (22) we reach the following conclusions.

If \( \frac{\partial V_i(t_i, n_i)}{\partial t_i \partial n_i} \exp(-r_i t_i) - C_{P_i}t_i > 0 \) and \( \frac{\partial V_i(t_i, n_i)}{\partial t_i} - r_i V_i(t_i, n_i) \geq 0 \), both \( dt_i / d\alpha_i \) and \( dn_i / d\alpha_i \) are uncertain.

Higher stumpage price level across the board raises the harvest age and increases the initial planting density. Yet, when \( \frac{\partial V_i(t_i, n_i)}{\partial t_i} - r_i V_i(t_i, n_i) < 0 \), both \( dt_i / d\alpha_i \) and \( dn_i / d\alpha_i \) are uncertain.

If \( \frac{\partial^2 V_i(t_i, n_i)}{\partial t_i \partial n_i} \exp(-r_i t_i) - C_{P_i}t_i < 0 \), and \( \frac{\partial V_i(t_i, n_i)}{\partial t_i} - r_i V_i(t_i, n_i) \geq 0 \), both \( dt_i / d\alpha_i \) and \( dn_i / d\alpha_i \) are uncertain. When \( \frac{\partial V_i(t_i, n_i)}{\partial t_i} - r_i V_i(t_i, n_i) < 0 \), both \( dt_i / d\alpha_i \) and \( dn_i / d\alpha_i \) are uncertain, meaning higher stumpage prices across the board lower the harvest age and lower the planting density.

**The impact of higher annual income**

As shown in Appendix A-4, with \( \beta_i \) representing the level of annual income,

\[
\frac{\partial^2 LEV_i}{\partial t_i \partial \beta_i} = A_i(t_i) \quad \text{and} \quad \frac{\partial^2 LEV_i}{\partial n_i \partial \beta_i} = 0
\]

39
\[
\frac{dt_1}{d\bar{\beta}_1} = -A(t_i) \left( \frac{\partial^2 LEV_1}{\partial n_1^2} \right) / D > 0 \text{ and }
\]
\[
\frac{dn_1}{d\bar{\beta}_1} = -\left( \frac{\partial^2 LEV_1}{\partial t_i \partial n_1} \right) (-A(t_i)) / D .
\]

That is to say, a higher level of annual income for the current timber crop always raises the harvest age for the current timber crop. Whether such annual income will increase or decrease the planting density depends on the sign of \( \left( \frac{\partial^2 LEV_1}{\partial t_i \partial n_1} \right) \).

The impact of higher interest rate for the current timber crop

As shown in Appendix A-5,
\[
\frac{dt_1}{dr_t} = \left( V_1(t_i, n_i) + LEV_2 \right) \frac{\partial^2 LEV_1}{\partial n_1^2} - \left( C_{p_1, t_i} \exp(t(t_i)) \frac{\partial LEV_1}{\partial t_i \partial n_1} \right) / D
\]
and
\[
\frac{dn_1}{dr_t} = \left( \frac{\partial^2 LEV_1}{\partial t_1^2} (C_{p_1, t_i} \exp(t(t_i)) \right) - \left( \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \left( V_1(t_i, n_i) + LEV_2 \right) \right) / D
\]

Whether a higher interest rate for the current timber crop would lower the optimal harvest age depends on the sign of \( \left( \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \right) \). When it is greater than 0, the optimal harvest age will be lowered. Otherwise, the impact is uncertain and would depend on the magnitude of \( \left( \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \right) \).

Similarly, the optimal planting density also depends on the sign of \( \left( \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \right) \). When it is greater than 0, a higher interest rate leads to a lower planting density. Otherwise, the impact is uncertain and would depend on the magnitude of \( \left( \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \right) \).

The impact of higher future land value

As shown in Appendix A-6,
\[
\frac{dt_1}{dLEV_2} = t \left( \frac{\partial^2 LEV_1}{\partial n_1^2} \right) / D < 0 \text{ and }
\]
\[
\frac{dn_1}{dLEV_2} = \left( \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \right) t / D .
\]

Higher future land value always lowers the optimal harvest age for the current timber crop. Its impact on the optimal planting density for the current timber crop depends on the sign of
The generalized Faustmann formula\[\left(\frac{\partial^2 V_i}{\partial t_i \partial n_i}\right)\]. When it is greater than 0, a higher future land value will decrease the planting density. Otherwise, it will increase the planting density.

The results of the previous comparative statics analyses are summarized in Table 3.3. A comparison of these results with those under the classic Faustmann formula (Chang, 1983) would indicate that the former produces much richer results regarding changes in the current parameters and clear-cut results regarding the future parameters unavailable under the classic Faustmann formula.

**Table 3.3** Summary of comparative statics analyses of the impact of changes on the optimal \(t\) and \(n\) for even-aged plantation management under the generalized Faustmann formula.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(\frac{\partial^2 V_i(t_i, n_i)}{\partial t_i \partial n_i} &lt; 0)</th>
<th>(\frac{\partial^2 V_i(t_i, n_i)}{\partial t_i \partial n_i} &gt; 0)</th>
<th>(\frac{\partial^2 V_i(t_i, n_i)}{\partial t_i \partial n_i} &gt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(dt_i / dC_s)</td>
<td>(= 0)</td>
<td>(= 0)</td>
<td>(= 0)</td>
</tr>
<tr>
<td>(dn_i / dC_s)</td>
<td>(= 0)</td>
<td>(= 0)</td>
<td>(= 0)</td>
</tr>
<tr>
<td>(dt_i / dC_p)</td>
<td>(&gt; 0)</td>
<td>(&gt; 0)</td>
<td>&lt;0</td>
</tr>
<tr>
<td>(dn_i / dC_p)</td>
<td>&lt;0</td>
<td>&lt;0</td>
<td>&lt;0</td>
</tr>
<tr>
<td>(\frac{\partial V_i(t_i, n_i)}{\partial t_i} - r_i V_i(t_i, n_i) \geq 0) If</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(dt_i / d\alpha_i)</td>
<td>uncertain</td>
<td>uncertain</td>
<td>&gt;0</td>
</tr>
<tr>
<td>(dn_i / d\alpha_i)</td>
<td>uncertain</td>
<td>uncertain</td>
<td>&gt;0</td>
</tr>
<tr>
<td>(\frac{\partial V_i(t_i, n_i)}{\partial t_i} - r_i V_i(t_i, n_i) &lt; 0) If</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(dt_i / d\alpha_i)</td>
<td>&lt;0</td>
<td>&lt;0</td>
<td>uncertain</td>
</tr>
<tr>
<td>(dn_i / d\alpha_i)</td>
<td>&lt;0</td>
<td>&lt;0</td>
<td>uncertain</td>
</tr>
<tr>
<td>(dt_i / d\beta_i)</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>&gt;0</td>
</tr>
<tr>
<td>(dn_i / d\beta_i)</td>
<td>&lt;0</td>
<td>&lt;0</td>
<td>&gt;0</td>
</tr>
<tr>
<td>(dt_i / dr_i)</td>
<td>uncertain</td>
<td>uncertain</td>
<td>&lt;0</td>
</tr>
<tr>
<td>(dn_i / dr_i)</td>
<td>uncertain</td>
<td>uncertain</td>
<td>&lt;0</td>
</tr>
<tr>
<td>(dt_i / d\alpha V_i)</td>
<td>&lt;0</td>
<td>&lt;0</td>
<td>&lt;0</td>
</tr>
<tr>
<td>(dn_i / d\alpha V_i)</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>&lt;0</td>
</tr>
</tbody>
</table>
The generalized Faustmann formula under uneven-aged management

For various reasons, uneven-aged forest management has become the preferred method in many regions. For example, ‘near natural’ forest management favoring mixed stands of site-adapted tree species has become a dominant type of forest management in Europe (see, e.g. Pommerening, 2001; Detten, Wurz and Schraml, 2009). Given that uneven-aged management represents possibly the oldest form of forest management, extending the generalized Faustmann formula to uneven-aged management thus represents more than just an intellectual curiosity. Under uneven-aged management, instead of harvest age and planting density as in even-aged management, cutting cycle and residual growing stock level are the decision variables. As such, let

\[ FV_0 = \text{the value of the uneven-aged forest before any management activities.} \]

\[ S = \text{the existing stand volume.} \]

\[ Q(t_i, g_i), i = 1 \text{ to } \infty, = \text{the volume of the } i\text{th uneven-aged stand with an initial residual growing stock of } g_i \text{ and a cutting cycle of } t_i \text{ years.} \]

\[ V_i(S) = \text{the stumpage value of the existing uneven-aged stand.} \]

\[ v_i(g), i = 1 \text{ to } \infty, = \text{the convex cost function for the residual growing stock value at the beginning of the } i\text{th cutting cycle. As such, } (\partial v_i(g)/\partial g) > 0 \text{ and } (\partial^2 v_i(g)/\partial g^2) \geq 0. \]

These conditions imply that the value of the residual growing stock is always increasing as the level of residual growing stock increases. Further, the value is increasing either at an increasing rate or a constant rate as the level of residual growing stock increases.

\[ V_i(Q(t_i, g_i)), i = 1 \text{ to } \infty, = \text{the quasi-concave stumpage value associated with the } i\text{th cutting cycle (timber harvest) before timber harvest. As such, over the relevant range } \partial V_i(Q(t_i, g_i))/\partial t_i > 0 \text{ and } \partial^2 V_i(Q(t_i, g_i))/\partial t_i^2 < 0, \text{ suggesting that the stumpage value of the uneven-aged stand increases with the elapsed time at a decreasing rate, and } \partial V_i(Q(t_i, g_i))/\partial g_i > 0 \text{ and } \partial^2 V_i(Q(t_i, g_i))/\partial g_i^2 < 0, \text{ indicating that the stumpage value of the uneven-aged stand also increases with the residual growing stock level at a decreasing rate.} \]

\[ K_i, i = 1 \text{ to } \infty, = \text{the fixed cost, for example the cost of obtaining a timber harvest permit, associated with the } i\text{th timber harvest.} \]

\[ LEV_i, i = 1 \text{ to } \infty, = \text{the land expectation value at the beginning of the } i\text{th cutting cycle under the generalized Faustmann formula.} \]

Others are as defined earlier under even-aged management.

Under uneven-aged management, if \( t_0 \) years must elapse before the existing stand can be brought under management, then instead of \( V_0(S) \) being the stumpage value for the existing stand, we need to wait \( t_0 \) years for the stand value to reach \( V_0(Q(t_0, S)). \) As such,

\[ FV_0 = \left[ V_0(t_0, S) - v_1(g_1) - K_1 + \int_{s_0}^{s_1} A_h(s_0) \exp(t_0(s_0 - s_0))ds_0 \right. \]

\[ + \sum_{n=1}^{\infty} \left[ V_1(Q(t_n, g_n)) - v_{-1}(g_{n+1}) \right. \]

\[ \left. K_{-1} + \int_{s_n}^{s_{n+1}} A_s(s_n) \exp(t_n(s_n - s_n))ds_n \right] \exp(-s_n(t_0)) \]

\[ = \left[ V_0(t_0, S) + \int_{s_0}^{s_1} A_h(s_0) \exp(t_0(s_0 - s_0))ds_0 + LEV_1 \right] \exp(-t_0) \]  

(23)
Similar to earlier presentations,

\[ LEV_1 = \left\{ V_1(Q_t(t_1, g_1)) + \int_0^1 A_1(s) \exp(t(s - s_1)) ds \mid \nu_1(g_1) + K_1 \right\} \exp(-r t_1) \\
+ LEV_2 \exp(-r t_1) \]  

(24)

Because the value of the forest consists of the value of the land and that of the trees, equation (23) is appropriately called the forest value.

If the existing stand can be brought under management immediately, \( t_0 = 0 \), then, as a special case, equation (23) can be expressed as

\[ FV_0 = \{ V_0(S) + LEV_1 \} \]  

(25)

Thus, uneven-aged management consists of two subproblems, that of determining the optimal cutting cycle for the existing stand and that of determining the cutting cycle and residual growing stock for future stands. Because the volume of the existing stand \( S \) is a given figure, the determination of its optimal cutting cycle becomes a simple problem once the value of \( LEV_1 \) is known, similar to the problem of even-aged natural stand management. The more interesting problem, therefore, is to solve for \( LEV_1 \) in equation (23). Note that equation (24) is similar to equation (10) for even-aged plantation management presented previously. For example, the fixed cost \( K_1 \) is the equivalent of the site preparation cost \( C_s \), and the value of the residual growing stock \( \nu_1(g_1) \) is the equivalent of the planting cost \( C_p \). Given their similarity, expositions on solving for both \( t_1 \) and \( g_1 \), the meaning of their optimal conditions and the comparative statics analyses are no longer necessary. Readers interested in such topics should consult the relevant sections under even-aged plantation management and check out the article by Chang and Gadow (2010) for an empirical example.

**The generalized Faustmann formula and Pressler’s formula**

In 1860 Max Robert Pressler published his famous indicator rate (Weiserprozent) formula, here shown with his then-used notations (Pressler, 1860, p. 190):

\[(a + b + c)[k/(k + 1)], \text{ with } k = h/g\]

where \( a \) is the rate of quantity increment (Quantiszuwachs), \( b \) is the rate of quality increment (Qualitatszuwachs), \( c \) is the rate of price increment (Teuerungszuwachs) discussed previously in detail, \( h \) is the variable timber capital and \( g \) is the fixed land capital.

As Johansson and Löfgren (1985) pointed out, Pressler’s indicator rate formula represents the earliest solution to maximizing the classic Faustmann land expectation value in its simplest form to determine the optimal rotation age.

\[ \max \text{ } LEV = \frac{|V(t) - C \exp(rt)|}{\exp(rt) - 1} \]  

(26)

with all the variables as defined previously. At the optimal rotation age,

\[ V'(t) = rV(t) + r \text{ } LEV \]  

(27)
where $V'(t) = dV(t)/dt$.

Equation (27) can also be written as Pressler’s indicator rate (Weisprozent) formula

$$\left[ \frac{V'(t)}{V(t)} \right] \frac{k}{k+1} = r$$  

(28)

where $k = V(t)/LEV$.

The more relevant question, therefore, for forest economics and management is the following: Can Pressler’s indicator rate formula be used fruitfully under the generalized Faustmann formula?

Note that without $A_i(t_i)$, equation (5) as the first-order condition for the optimal harvest age $t_i$ can also be expressed as

$$\frac{V'_i(t_i)}{V_i(t_i)} = r_i \left[ 1 + LEV \cdot \frac{V_i(t_i)}{V_i(t_i)} \right]$$  

(29)

With $V_i(t_i)$ as the variable timber capital $h_i$, $LEV \cdot V_i$ the fixed land capital $g_i$ and $V_i(t_i)/LEV \cdot V_i = k$, equation (29) can be transformed into equation (28) as the famous Pressler’s indicator rate formula. Thus, Pressler’s indicator rate formula is also relevant under the generalized Faustmann formula. Moreover, in a recent article, Chang and Deegen (2011) showed how the price increment, quality increment and quantity increment can be combined with Pressler’s indicator rate formula to determine the optimal harvest age in a dynamic world of constantly changing prices.

**Conclusion**

This chapter examines the four core areas of generalized Faustmann formula: the management of an even-aged natural stand, the management of a plantation, the management of an uneven-aged stand and the development of Pressler’s indicator rate formula under the generalized Faustmann formula. Freed of the stringent assumptions about stumpage prices, stand volumes, regeneration costs and interest rates, harvest age and planting density or cutting cycle and residual growing stock level are allowed to vary under the generalized Faustmann formula. The ability to separate current and future production parameters under the generalized Faustmann formula makes it possible to untangle the impact of changes in these parameters.

Current efforts are being made to incorporate payment of carbon sequestration benefits and other ecological services as part of the $A_i(s)$. A manuscript on the generalized version of the van Kooten formula (van Kooten, Binkley and Delcourt, 1995) will soon be published (Susaeta, Chang, Carter and Lal, 2013). Its extension under uneven-aged management has recently been published by Parajuli and Chang (2012). Furthermore, a manuscript on extensions of the generalized Faustmann formula to incorporate catastrophic risk similar to the work of Reed (1984) and Reed and Errico (1986) is also under review (Susaeta, Carter, Chang and Adams 2013).

Incorporating various forms of forest taxation into the generalized Faustmann formula to examine their impact on the optimal management represents a promising line of research. Such an effort will also open the opportunity in forest valuation to determine the value of the forest, the timber stand and the land value.

It should be noted that the generalized Faustmann formula presented here addresses forest management under uncertainty. Despite recent progress in addressing optimal management under uncertainty (see, e.g. Alvarez and Koskela, 2006; Chaladná, 2007), this area remains fertile ground for additional research, particularly when it comes to the management of even-aged
The generalized Faustmann formula

plantations and uneven-aged stands involving both harvest age and planting density or cutting cycle and residual growing stock as decision variables. Lastly, the relationship between the generalized Faustmann formula and the literature on the reservation price strategy (Braze and Mendelsohn, 1988; Gong and Löfgren, 2007), as well as the real options theory, needs to be fully explored.

References


Hartman, R. (1976). ‘The harvesting decision when a standing forest has value’, *Economic Inquiry*, vol. 14, no. 1, pp. 52–58.


Appendix A-1: The impact of a higher site preparation cost for the current timber crop

Applying the implicit function theorem,

\[ d \left( \frac{LE V_i}{\partial t_i} \right) = \left( \frac{\partial^2 LE V_i}{\partial t_i \partial n_i} \right) dt_i + \left( \frac{\partial^2 LE V_i}{\partial t_i \partial C_i} \right) dC_i = 0 \]

\[ d \left( \frac{LE V_i}{\partial n_i} \right) = \left( \frac{\partial^2 LE V_i}{\partial n_i \partial C_i} \right) dC_i + \left( \frac{\partial^2 LE V_i}{\partial n_i \partial t_i} \right) dt_i + \left( \frac{\partial^2 LE V_i}{\partial n_i^2} \right) d n_i = 0 \]

As such,

\[ \left[ \frac{\partial^2 LE V_i}{\partial t_i \partial C_i}, \frac{\partial^2 LE V_i}{\partial t_i \partial n_i}, \frac{\partial^2 LE V_i}{\partial n_i \partial C_i}, \frac{\partial^2 LE V_i}{\partial n_i^2} \right] \left[ \frac{dt_i}{dC_i}, \frac{dn_i}{dC_i} \right] = \left[ \frac{\partial^2 LE V_i}{\partial t_i \partial n_i} \right] \]

Applying Kramer's rule,

\[ \frac{dt_i}{dC_i} = \frac{\left| \begin{array}{c} -\frac{\partial^2 LE V_i}{\partial t_i \partial C_i} \\ -\frac{\partial^2 LE V_i}{\partial t_i \partial n_i} \\ \frac{\partial^2 LE V_i}{\partial n_i \partial C_i} \\ \frac{\partial^2 LE V_i}{\partial n_i^2} \end{array} \right|}{D = 0} \]

\[ \frac{dn_i}{dC_i} = \frac{\left| \begin{array}{c} \frac{\partial^2 LE V_i}{\partial t_i \partial C_i} \\ \frac{\partial^2 LE V_i}{\partial t_i \partial n_i} \\ -\frac{\partial^2 LE V_i}{\partial n_i \partial C_i} \\ -\frac{\partial^2 LE V_i}{\partial n_i^2} \end{array} \right|}{D = 0} \]

because both \( -\frac{\partial^2 LE V_i}{\partial t_i \partial n_i} \) and \( -\frac{\partial^2 LE V_i}{\partial n_i \partial C_i} \) equal 0.

Appendix A-2: The impact of a higher planting cost per seedling for the current timber crop

\[ \frac{dt_i}{dC_p} = \frac{\left| \begin{array}{c} -\frac{\partial^2 LE V_i}{\partial t_i \partial C_p} \\ -\frac{\partial^2 LE V_i}{\partial t_i \partial n_i} \\ \frac{\partial^2 LE V_i}{\partial n_i \partial C_p} \\ \frac{\partial^2 LE V_i}{\partial n_i^2} \end{array} \right|}{D} \]

\[ \frac{dn_i}{dC_p} = \frac{\left| \begin{array}{c} \frac{\partial^2 LE V_i}{\partial t_i \partial C_p} \\ \frac{\partial^2 LE V_i}{\partial t_i \partial n_i} \\ -\frac{\partial^2 LE V_i}{\partial n_i \partial C_p} \\ -\frac{\partial^2 LE V_i}{\partial n_i^2} \end{array} \right|}{D} \]
Because $-\partial^2 LEV_i / \partial t_i \partial Cp_i = 0$ and $-\partial^2 LEV_i / \partial n_i \partial Cp_i = 1$,

\begin{align*}
\frac{dt_i}{dCp_i} &= \left[ \frac{\partial^2 LEV_i / \partial t_i \partial Cp_i}{\partial Cn_i / \partial n_i} - \frac{\partial^2 LEV_i / \partial t_i \partial n_i}{\partial Cn_i / \partial n_i} \right] / D = \frac{\partial^2 LEV_i / \partial t_i \partial n_i}{\partial Cn_i / \partial n_i} / D \\
\frac{dn_i}{dCp_i} &= \left[ \frac{\partial^2 LEV_i / \partial t_i \partial Cp_i}{\partial Cn_i / \partial n_i} - \frac{\partial^2 LEV_i / \partial t_i \partial n_i}{\partial Cn_i / \partial n_i} \right] / D = \frac{\partial^2 LEV_i / \partial t_i \partial n_i}{\partial Cn_i / \partial n_i} / D < 0.
\end{align*}

**Appendix A-3: The impact of a higher stumpage price level for the current timber crop**

The impact of a higher current stumpage price level for the current timber crop can be examined with the introduction of a price level variable $\alpha_i$. As such, equations (13) and (14) will be rewritten as

\begin{align*}
\frac{\partial L E V_i}{\partial t_i} &= \left[ \frac{\partial \alpha_i V_i(t_i, n_i)}{\partial t_i} + A_i(t_i) - t_i \alpha_i V_i(t_i, n_i) - r_i L E V_i \right] \exp(-r_i t_i) - 0 \quad (A-3-1) \\
\frac{\partial L E V_i}{\partial n_i} &= \left[ \frac{\partial \alpha_i V_i(t_i, n_i)}{\partial n_i} - Cp_i \exp(r_i t_i) \right] \exp(-r_i t_i) = 0 \quad (A-3-2)
\end{align*}

From equation (A-3-1)

\begin{equation}
\frac{\partial^2 L E V_i}{\partial t_i \partial \alpha_i} = \left[ \frac{\partial V_i(t_i, n_i)}{\partial t_i} - t_i V_i(t_i, n_i) \right] \exp(-r_i t_i) \quad (A-3-3)
\end{equation}

From equation (A-3-2)

\begin{equation}
\frac{\partial^2 L E V_i}{\partial n_i \partial \alpha_i} = \frac{\partial V_i(t_i, n_i)}{\partial n_i} \exp(-r_i t_i) > 0 \quad (A-3-4)
\end{equation}

Applying the implicit function theorem and the Kramer's rule,

\begin{align*}
\frac{dt_i}{d\alpha_i} &= \left[ \frac{\partial^2 L E V_i / \partial t_i \partial \alpha_i}{\partial Cn_i / \partial n_i} - \frac{\partial^2 L E V_i / \partial t_i \partial n_i}{\partial Cn_i / \partial n_i} \right] / D \\
&= \left\{ \left. \left[ \frac{\partial V_i(t_i, n_i)}{\partial t_i} - t_i V_i(t_i, n_i) \right] \exp(-r_i t_i) \frac{\partial^2 V_i(t_i, n_i)}{\partial n_i^2} \exp(r_i t_i) \right\} / D \right. \\
&\quad \left. - \left[ \frac{\partial V_i(t_i, n_i)}{\partial n_i} \exp(-r_i t_i) \right] \left\{ \left. \frac{\partial^2 V_i(t_i, n_i)}{\partial t_i \partial n_i} \exp(r_i t_i) - C_p t_i \right\} / D \right. \quad (A-3-5)
\end{align*}
\[
\frac{d n_i}{d \alpha_i} = \left[ \frac{\partial^2 LEV_i / \partial t_i^2}{\partial t_i, \partial n_i} - \frac{\partial^2 LEV_i / \partial t_i, \partial \alpha_i}{\partial t_i} \right] / D \\
= \left\{ \left( \frac{\partial^2 LEV_i / \partial t_i^2}{\partial t_i, \partial n_i} \right) \left[ - \frac{\partial V_i(t_i, n_i)}{\partial t_i} - n V_i(t_i, n_i) \exp(-r t_i) \right] \right\} \\
- \left\{ - \frac{\partial V_i(t_i, n_i)}{\partial t_i, \partial n_i} \exp(-r t_i) \right\} \left[ - \frac{\partial V_i(t_i, n_i)}{\partial n_i} \exp(-r t_i) \right] / D \\
(A-3-6)
\]

**Appendix A-4: The impact of a higher current annual income**

The impact of a higher annual income for the current timber crop can be examined with the introduction of a price level variable \( \beta \). As such, equations (13) and (14) will be rewritten as

\[
\frac{\partial LEV_i}{\partial t_i} = \left[ \frac{\partial V_i(t_i, n_i)}{\partial t_i} + \beta A_i(t_i) - n V_i(t_i, n_i) - r LEV_i \right] = 0 \quad (A-4-1)
\]

\[
\frac{\partial LEV_i}{\partial n_i} = \left[ \frac{\partial V_i(t_i, n_i)}{\partial n_i} - C_p \exp(r t_i) \right] = 0 \quad (A-4-2)
\]

From equations (A-4-1) and (A-4-2),

\[
\frac{\partial^2 LEV_i}{\partial t_i \partial \beta_i} = A_i(t_i) \quad \text{and} \quad \frac{\partial^2 LEV_i}{\partial n_i \partial \beta_i} = 0
\]

Take the total derivatives of equations (A-4-1) and (A-4-2), and applying Kramer's rule,

\[
\frac{d t_i}{d \beta_i} = \left[ \frac{\partial^2 LEV_i / \partial t_i \partial \beta_i}{\partial t_i, \partial n_i} - \frac{\partial^2 LEV_i / \partial t_i \partial \beta_i}{\partial t_i} \right] / D = (-\partial^2 LEV_i / \partial n_i) / D > 0
\]

\[
\frac{d n_i}{d \beta_i} = \left[ \frac{\partial^2 LEV_i / \partial n_i \partial \beta_i}{\partial t_i, \partial n_i} - \frac{\partial^2 LEV_i / \partial n_i \partial \beta_i}{\partial n_i} \right] / D = (-\partial^2 LEV_i / \partial t_i \partial n_i) / D
\]

**Appendix A-5: The impact of a higher interest rate for the current timber crop**

Applying the implicit function theorem to equations (13) and (14),

\[
\frac{d \partial LEV_i}{\partial t_i} = \frac{\partial^2 LEV_i}{\partial t_i} dt_i + \frac{\partial^2 LEV_i}{\partial t_i \partial n_i} dn_i + \frac{\partial^2 LEV_i}{\partial t_i \partial r_i} dr_i = 0
\]
The generalized Faustmann formula

\[ d \frac{\partial LEV_1}{\partial n_1} = \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} dt_1 + \frac{\partial^2 LEV_1}{\partial n_1^2} dn_1 + \frac{\partial^2 LEV_1}{\partial n_1 \partial r_1} dr_1 = 0 \]

Given that

\[ \frac{\partial^2 LEV_1}{\partial t_1 \partial r_1} = -V(t_1, n_1) - LEV_2 \quad \text{and} \quad \frac{\partial^2 LEV_1}{\partial n_1 \partial r_1} = -C_{p.t_1} \exp(t_1), \text{applying}\]

Kramer's rule

\[ \frac{dt_1}{dr_1} = \frac{\left| \begin{array}{cc} -\frac{\partial^2 LEV_1}{\partial t_1 \partial r_1} & \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \\ -\frac{\partial^2 LEV_1}{\partial n_1 \partial r_1} & \frac{\partial^2 LEV_1}{\partial n_1^2} \end{array} \right|}{D} = \{(V(t_1, n_1) + LEV_2)(\frac{\partial^2 LEV_1}{\partial n_1^2} - C_{p.t_1} \exp(t_1)(\frac{\partial^2 LEV_1}{\partial t_1 \partial n_1}) - \partial^2 LEV_1 / \partial t_1 \partial n_1) \} / D \]

\[ \frac{dn_1}{dr_1} = \frac{\left| \begin{array}{cc} \frac{\partial^2 LEV_1}{\partial t_1^2} & -\frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \\ \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} & \frac{\partial^2 LEV_1}{\partial n_1^2} \end{array} \right|}{D} = \{(\frac{\partial^2 LEV_1}{\partial t_1^2})(C_{p.t_1} \exp(t_1)) - (\frac{\partial^2 LEV_1}{\partial t_1 \partial n_1})(V(t_1, n_1) + LEV_2) \} / D \]

**Appendix A-6: The impact of a higher future land value**

Applying the implicit function theorem to equations (13) and (14) and Kramer's rule,

\[ \frac{\partial^2 LEV_1}{\partial t_1 \partial LEV_2} = -t_1 \quad \text{and} \quad \frac{\partial^2 LEV_1}{\partial n_1 \partial LEV_2} = 0 \]

Applying Kramer's rule,

\[ \frac{dt_1}{\partial LEV_2} = \frac{\left| \begin{array}{cc} -\frac{\partial^2 LEV_1}{\partial t_1 \partial LEV_2} & \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \\ -\frac{\partial^2 LEV_1}{\partial n_1 \partial LEV_2} & \frac{\partial^2 LEV_1}{\partial n_1^2} \end{array} \right|}{D} = \frac{\frac{\partial LEV_1}{\partial n_1}}{D < 0} \]

\[ \frac{dn_1}{\partial LEV_2} = \frac{\left| \begin{array}{cc} \frac{\partial^2 LEV_1}{\partial t_1^2} & -\frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} \\ \frac{\partial^2 LEV_1}{\partial t_1 \partial n_1} & \frac{\partial^2 LEV_1}{\partial n_1^2} \end{array} \right|}{D} = \frac{-\frac{\partial^2 LEV_1}{\partial t_1 \partial n_1}}{D} \]

49