5

Optical Cavities: Free-Space Laser Resonators

Robert C. Eckardt

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5.1 Introduction

Resonators provide the optical structure in which laser oscillations are established. Passive optical resonators can also be used to increase locally the power of coherent optical radiation or to filter optical radiation. An understanding of laser resonators is necessary in analysing the spatial beam characteristics and temporal coherence properties of the light output of laser systems. Optimizing the design of a laser system requires resonator analysis. In addition to the operation and design of lasers, an understanding of optical resonators is important in coupling the laser output to the application in which it is to be used. This chapter reviews the fundamental aspects of laser resonators and discusses some of the techniques of laser operation.

Optical resonators have resonant modes and these modes are important to the analysis of laser operation. An optical resonator mode is a field distribution that is resonant with the structure and that is reproduced in phase and in relative intensity after a round-trip transit of the resonator. The intensity of a mode may decrease due to resonator losses, or it may be increased by amplification in an active laser material or by light introduced from outside the resonator. Mode competition and selection usually occur when an oscillation builds up in a resonator. Typically, laser oscillations start from random quantum fluctuations and build to high intensity in a single mode or many simultaneous modes. After many resonator transits the mode or modes with minimum loss tend to dominate and to be reproduced after each cavity transit. An iterative numerical analysis that simulated this process was used in the seminal paper by Fox and Li [1] to analyse the field distributions of laser resonators. Numerical methods of resonator analysis augmented by fast Fourier transform techniques and computers of increasing capability are currently in wide use. The analysis
of resonators that have large dynamic changes, non-linearities or significant diffraction typically requires numerical methods rather than analytical techniques. However, certain analytical methods for resonator analysis also have wide application and are extensively used.

Many of the analytical techniques for describing stable open resonators originated in the work of Boyd and Gordon [2]. The Gaussian beams of stable open resonators can be characterized with analytical expressions and provide a widely used approach to the discussion of laser resonators. A Gaussian beam is the fundamental mode of a set of Hermite–Gaussian modes in rectangular coordinates, and the same Gaussian mode is also the fundamental mode of a set of Laguerre–Gaussian modes in cylindrical coordinates. Either of these sets of modes forms complete sets of orthogonal functions, which can be used in a series expansion to describe a transverse field distribution to an arbitrary degree of accuracy. These modes will usually be a very good approximation, although still an approximation, to the modes of any real laser resonator. Real resonators will always have some limiting aperture, which introduces a small loss, and possibly other perturbations that will slightly distort the modes and make them non-orthogonal.

A resonator is called ‘stable’ if a ray of a geometrical optical analysis always remains within the resonator rather than wandering off an increasing distance from the resonator centre with an increasing number of cavity transits. Unstable resonators, which do not satisfy this condition, nonetheless have modes that are reproduced through diffraction from one cavity transit to the next. The term ‘unstable resonator’ is somewhat misleading because these resonators can be quite stable in operation, and they are useful for efficient energy extraction in high-gain systems that can support the higher diffraction losses typical of unstable resonators. Siegman [3] prefers the descriptive name ‘geometrically unstable’ as being more accurate for these resonators. Stable resonators typically have small diffraction losses and support Gaussian transverse fundamental modes. They may also support many higher-order transverse modes giving a great deal of structure to the beam. Unstable resonators usually offer large loss discrimination between the lowest-loss transverse mode and other transverse modes. For this reason, they often provide good spatial beam quality in large-diameter laser output beams. Stable and unstable resonators are illustrated schematically in Figure 5.1.

Resonators simultaneously have axial modes in addition to their transverse modes. The condition that the phase distribution be reproduced after a round-trip cavity transit allows axial modes that cycle through a phase change of integer multiples of $$2\pi$$ on a round-trip transit. Higher-order transverse modes have more complex spatial amplitude distributions, which are also reproduced from one cavity transit to the next. The higher-order transverse modes have additional phase shifts that place them, in frequency, between the fundamental transverse modes. The axial or longitudinal resonator modes with a fundamental transverse distribution will be nearly equally spaced in frequency or wavenumber. The modes will only be precisely spaced if care is taken to compensate for dispersion and frequency-dependent phase shifts of the optical elements of the resonator. Such compensation becomes important for mode locking, in which case many axial modes are locked in phase to synthesize a single short pulse that propagates back and forth in the resonator. Other cavity control techniques include frequency selection or narrowing to restrict axial modes and spatial filtering to restrict transverse modes. Laser resonator losses can be controlled to hold off oscillation and then suddenly reduced in a technique called Q-switching, to produce an energetic pulse of only a few resonator transits duration.

The objective of this chapter is to present a basic description of resonator lasers at a level that will allow the reader to apply the material to practical systems. Derivations are not given in this review but are covered in detail in the references. There have been many excellent papers and reviews on this subject. The books by Siegman [3], by Hall and Jackson [4] and by Hodgson and Weber [5] cover the topic thoroughly. Other more general books dealing with lasers also have good discussions of the topic [6–8]. Two review papers by Siegman [9,10] detail the development of the analysis of laser resonators and laser beams and include extensive bibliographies. Many of the results, terminology and notation developed in the earlier papers have become standard and are followed in this chapter. The basic types of resonators that will be discussed include geometrically stable two-mirror cavities, simple unstable resonators and resonators with plane-parallel mirrors.

The next section of this chapter reviews the optics of Gaussian light beams. These beams are solutions of the wave equation for propagation in a homogeneous, isotropic medium and are the fundamental modes of geometrically stable open resonators. The diffraction of Gaussian beams can be treated directly with analytical methods, simplifying calculations of laser-beam propagation and mode-matching between resonators. The minimum diffraction of a Gaussian beam is used to define the diffraction-limited beam quality factor $$M^2$$ of one. The fundamental mode of a stable two-mirror cavity can be determined by matching the Gaussian-beam wavefront radii to the mirror radii. More complicated cavities are typically analysed with ABCD or ray-transfer-matrix techniques.
Such analysis and application of the ray transfer matrices to the propagation of Gaussian beams are discussed here. A description of higher-order transverse modes of stable resonators follows and is used for a further description of beam quality.

Unstable resonators are discussed here both from the geometrical optics and diffraction points of view. An illustrative example of numerical resonator analysis is used to discuss transverse-mode selection in several types of unstable resonators. The calculation is seeded with a transverse distribution obtained by randomizing the phases of the initial Fourier-spatial-frequency components to simulate development from quantum fluctuations. The comparison includes plane-parallel-mirror resonators and collimated-output unstable resonators with hard-edged and soft mirrors. The use of variable-reflectivity laser mirrors is important to mitigate diffraction effects in unstable resonators. Gain-guided laser resonators, common in laser-pumped lasers, offer another technique for controlling the diffraction effects in lasers with plane-parallel resonators and in unstable resonators.

The discussion of axial modes starts with the frequency spacing of fundamental and higher-order transverse modes. The topic of cavity finesse deals with the frequency width of individual resonator modes. The bandwidth of the active gain medium of the resonator limits the number of axial modes or the frequency bandwidth of the oscillation. Components such as prisms, gratings and etalons are added to resonators to further narrow the bandwidth. Frequency stability is achieved first through mechanical and thermal stabilization of resonators. Higher levels of frequency stabilization require active stabilization to a frequency standard and careful control of the pumping stability. Modulation within the resonator is used to achieve mode locking and Q-switching. Techniques for injection seeding and injection locking can be used to transfer the properties of highly coherent low-power resonators to high-power systems. It is necessary to match the properties of a beam to the cavity into which it is injected. Examples of mode-matching are presented with one lens and two lenses used to relay the laser beam. The optics of Gaussian beams is fundamental to all of these topics and that is where we begin.

5.2 Gaussian Beams

5.2.1 Conventions and Notation

The discussion here will be restricted to paraxial optics. In this limit, the sine of an angle is approximated by that angle expressed in radians. Spherical surfaces can be represented by a parabolic approximation: $z = (x^2 + y^2)/2R$ is used to describe a spherical surface of radius of curvature $R$. A positive wavefront radius of curvature indicates a beam that is convex in the direction of propagation or diverging. A negative wavefront radius of curvature indicates a beam that is concave in the direction of propagation or converging (Figure 5.2). Similarly, the radius of curvature of a concave spherical mirror surface is positive and that of a convex surface is negative. Coordinate rotations or reflections are assumed for refraction or reflection at arbitrary angles such that the beam remains centred on the $z$-axis and propagating in the $+z$-direction. There is no coordinate translation or scaling in the direction of propagation, and $z$ remains the cumulative physical distance of propagation. A plane wave travelling in the $+z$-direction is expressed as $\exp(-i(kz - \omega t))$. If the complex conjugate of this expression is used to describe this plane wave, appropriate changes of sign may be required in some expressions.

5.2.2 Description of Gaussian Beams

The derivation of the properties of a Gaussian beam is discussed in detail in texts such as those by Siegman [3] and Hodgson and Weber [5]. The early review article by Kogelnik and Li [11] standardized much of the notation for the treatment of Gaussian beams, and the article remains relevant. The results presented in these sources are reproduced here. Gaussian beams are solutions of the paraxial wave equation

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} - 2ik \frac{\partial E}{\partial z} = 0 \quad (5.1)$$

and, equivalently, Fresnel’s approximation to Huygen’s integral

![Figure 5.2](https://example.com/figure52.png) The sign convention for wavefront radii of curvature is positive when the beam is divergent and negative when the beam is convergent.
Here $E$ is the electric field, which is specified at coordinates $(x_0, y_0, z_0)$ and calculated at $(x, y, z)$, $i = \sqrt{-1}$ and $k = 2\pi/\lambda$, where $\lambda$ is the wavelength. For these equations to be valid, it is necessary that there is no sharp discontinuity, such as an aperture edge or a phase step, near the plane at which the field is evaluated.

A Gaussian beam has transverse amplitude distributions proportional to $\exp[-(r/\omega)^2]$, and the transverse intensity distribution is proportional to $\exp[-2(r/\omega)^2]$. The radial coordinate is the distance from the centre of the beam: $r = \sqrt{x^2 + y^2}$. The parameter $\omega$ is called the spot size and changes with distance as the beam propagates. Circularly symmetric beams are initially discussed here, but later, the discussion is expanded to include astigmatic beams with amplitude distributions expressed as $\exp[-(x^2/\omega_x^2) - (y^2/\omega_y^2)]$. Wavefronts of constant phase of circularly symmetric Gaussian beams form spherical surfaces, and the wavefronts of the astigmatic beams are circular in cross section along the principal axes of the astigmatic components, which are assumed here to be aligned for all components in the beam.

The mathematical description of the electric field of a Gaussian light beam or TEM$_{00}$ mode is a real function expressed as the sum of a complex expression and its complex conjugate:

$$E(x, y, z, t) = \frac{i}{2} E_0 \frac{\omega_0}{\omega(z)} \exp \left\{ -\frac{r^2}{\omega^2(z)} - i \left( \frac{r^2 k}{2R(z)} - \Phi(z) \right) \right\} \times \exp \{ -it(kz - \omega_0 t) \} + c.c.$$  \tag{5.3}

This is a solution to the wave equation representing a nearly collimated beam propagating in the $+z$-direction. The complex conjugate is not carried in the discussion, and it is sufficient to treat the field as a complex parameter. Only in a few special cases, not encountered here, is it necessary to retain the real number representation of the electric field. It is assumed that the transverse dimension of the beam is much larger than the wavelength, that is, $\omega_0 \gg \lambda$, a condition which is typical of most laser beams and laser resonators. The factor $\exp[-it(kz - \omega_0 t)]$, with $k = 2\pi/\lambda$, $n$ is the index of refraction of the medium in which the beam is propagating, $\lambda_0$ is the free-space wavelength, $\omega$ is the angular frequency and $t$ is the time, represents the plane wave component of the distribution. The remaining portion of the expression describes differences in field distribution and phase from that of the plane wave. The parameter $\omega(z)$ is the spot size and $R(z)$ is the wavefront radius of curvature. The change due to propagation with diffraction for these two parameters is

$$\omega^2(z) = \omega_0^2 \left[ 1 + \left( \frac{\lambda(z - z_0)}{\pi \omega_0^2} \right)^2 \right] = \omega_0^2 \left[ 1 + \left( \frac{z-z_0}{z_R} \right)^2 \right], \tag{5.4}$$

and

$$R(z) = (z - z_0) \left[ 1 + \left( \frac{\pi \omega_0^2}{\lambda(z - z_0)} \right)^2 \right] = (z - z_0) \left[ 1 + \left( \frac{z_R}{z-z_0} \right)^2 \right]. \tag{5.5}$$

The beam waist is located at $z_0$ where $\omega(z_0) = \omega_0$ has a minimum value. The parameter $z_R$ is the Rayleigh range or Rayleigh length:

$$z_R = \pi \omega_0^2 / \lambda. \tag{5.6}$$

The Rayleigh length is the distance necessary to travel from the beam waist for the spot size to increase by a factor of $\sqrt{2}$. The confocal parameter $b$ of a Gaussian beam, a commonly used parameter, is twice the Rayleigh range:

$$b = k \omega_0^2 = 2 \pi n \omega_0^2 / \lambda_0. \tag{5.7}$$

The parameter $\Phi(z)$ is the Gouy phase shift and gives the departure of the on-axis phase from that of a plane wave:

$$\Phi(z) = \arctan \left( \frac{\lambda(z - z_0)}{\pi \omega_0^2} \right). \tag{5.8}$$

When propagated into the far field, a Gaussian beam will expand with a divergence half-angle of $\theta = \lambda/\pi \omega_0$ (Figure 5.3). The constant $E_0$ in equation (5.3) is the peak electric field at the beam waist and is expressed in SI units in terms of the total power in the beam $P$ or peak intensity $I_0$ as

$$E_0 = \sqrt{2I_0 / (n\varepsilon_0 c)} = \sqrt{4P / (\pi \omega_0^4 n\varepsilon_0 c)} \tag{5.9}$$

where $E_0$ is in units of V m$^{-1}$, $P$ is in W, $I_0$ is in W m$^{-2}$, $\varepsilon_0 \approx 8.854 \times 10^{-12}$ CN m$^{-1}$ is the permittivity of free space and $c \approx 2.997 \times 10^8$ m s$^{-1}$ is the speed of light.

Expressing a Gaussian beam in terms of $\omega_0$, the beam waist spot size, and $z_0$, the position of the beam waist, offers the advantage of explicitly stating the spot size and wavefront radius of curvature as a function of position. An alternate description is given in terms of $q(z)$, the ‘complex beam parameter’ or ‘complex radius of curvature’,

$$1/q(z) = 1/R(z) - i\lambda/\pi \omega^2(z). \tag{5.10}$$

Using $q(z)$, the expression (5.3) for the Gaussian beam, becomes

$$E(x, y, z, t) = \frac{1}{2} E_0 \frac{\omega_0}{q(z)} \exp \left\{ -i \frac{k z^2}{2q(z)} \right\} \exp \{ -it(kz - \omega_0 t) \} + c.c. \tag{5.11}$$

where $q_0 = -i \lambda / (\pi \omega_0^2)$. It follows that $q_0/q(z) = \exp(i\Phi(z)) \times \omega_0/\omega(z)$, and the two forms equations (5.3) and (5.11) are equivalent. Using the complex beam parameter offers advantages in terms of mathematical simplicity. For example, the change of $q$ with propagation is simply
Optical Cavities

The use of the complex beam parameter and the $2 \times 2$ ABCD or ray transfer matrices greatly simplifies the analysis of complex resonators. Before discussing the ray transfer matrices, some additional relationships involving spot sizes and wavefront radii are given.

The parameters $\omega_0$ and $z_0$, along with the direction of propagation, wavelength, index of refraction and beam power, specify a Gaussian beam. Usually, the characterization of a beam depends on the determination of $\omega_0$ and $z_0$ with the other parameters known. Table 5.1 lists relationships that can be used to characterize a beam. Equations (5.14) can be used to determine the fundamental mode of a two-mirror laser cavity by matching the wavefront radius of curvature to the radii of curvature of the cavity mirrors (Figure 5.4). The stability condition for a two-mirror oscillator is implicit in equation (5.14) but difficult to extract in a simple form. Use of the complex beam parameter equation (5.10) and the ray transfer matrices provides this simplicity.

5.2.3 Ray Transfer Matrices

Ray transfer matrices or ABCD matrices provide concise and useful representations of both the geometrical propagation of paraxial rays and the propagation with diffraction of Gaussian beams. The propagation of rays through simple optical components is described by

$$
\begin{bmatrix}
  x_2 \\
  y_2
\end{bmatrix} = 
\begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
  x_2' \\
  y_2'
\end{bmatrix} = 
\begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
\begin{bmatrix}
  x_1' \\
  y_1'
\end{bmatrix}
$$

where $A = \omega_4 d (d - R_t) + (\lambda d R_t - d^2)/(\pi n)$, $B = \omega_4 (2d - R_t) + 2\lambda d (R_t - d)/(R_t - d) (R_t - d)/(\pi n)$, and

$$
c = \omega_4 d (d - R_t) + (\lambda d R_t - d^2)/(\pi n)
$$

if $\omega_1 = \omega(z_1)$ and $\omega_2 = \omega(z_2)$ are known at positions $z_1$ and $z_2$, $d = z_2 - z_1$.

$$
\omega_0^2 = \omega_0^2 [1 + \pi \omega_0^2 (1 + \lambda R(z))]/(\lambda R(z))
$$

if $R_1 = R(z_1)$ and $R_2 = R(z_2)$ are known, $d = z_2 - z_1$.

$$
\omega_0^2 = \frac{\lambda^2 (R_2 - d) (R_1 + d)(R_t - R_2 + d)}{\pi (R_t - R_2 + 2d)^2}
$$

if $R_1 = R(z_1)$ and $\omega_1 = \omega(z_1)$ are known, $d = z_2 - z_1$.

$$
z_0 = z_1 + \frac{b \pm \sqrt{b^2 - 4ac}}{2a},\ \omega_0^2 = \lambda \oint (z_2 - z_1) (R_t - z_1 + z_0) / \pi
$$

where $a = (\lambda (R_t - 2d)/\pi)^2 + \omega_0^2$, $b = \omega_0^2 (2d - R_t) + 2\lambda^2 d (R_t - d)/(R_t - d)/(\pi n)^2$, and

$$
c = \omega_0^2 d (d - R_t) + (\lambda d R_t - d^2)/(\pi n)^2
$$

if $\omega_1 = \omega(z_1)$ and $\omega_2 = \omega(z_2)$ are known at positions $z_1$ and $z_2$, $d = z_2 - z_1$.

$$
\omega_0^2 = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a},\ \omega_0^2 = z_1 [1 - (\omega_1^2 - \omega_0^2)/(\pi \omega_0^2 (\lambda^2 d^2 -(2d)^2)]/(2d)
$$

where $a = 1 + (\omega_0^2 - \omega_1^2)/(2d \lambda)^2$, $b = - (\omega_1^2 + \omega_0^2)/2$, and $c = (\lambda d/2 \pi)^2$.
The ray is assumed to be propagating nearly parallel to the \( z \)-axis. The distance of the ray from the \( z \)-axis is given by \((x, y)\), and the projections of the slope of the ray are \( x' = \frac{dx}{dz} \) and \( y' = \frac{dy}{dz} \) (Figure 5.5).

The ray position and slope before the optical element are identified by the subscript 1 and after the element by the subscript 2. Ray transfer matrices for six simple elements are given in table 5.2: propagation over a distance \( d \); a thin lens of focal length \( f \); propagation through a spherical interface from a medium of index \( n_1 \) to one of \( n_2 \); an element with flat parallel surfaces and an index distribution of \( n = -\frac{1}{202} z + \frac{1}{22} x^2 y^2 \) inside and \( n = 1 \) outside; a spherical mirror and a flat interface from index \( n_1 \) to index \( n_2 \). Ray transfer matrices can be combined by matrix multiplication to describe multiple-component systems:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
= 
\begin{bmatrix}
A_n & B_n \\
C_n & D_n
\end{bmatrix}
\begin{bmatrix}
A_{n+1} & B_{n+1} \\
C_{n+1} & D_{n+1}
\end{bmatrix}
\]

(5.18)

The multiplication must be performed as shown in equation (5.18) with subscript 1, representing the first element encountered and \( n \) the last. When the indices of refraction before the first element and after the last element of a system of components are equal, the determinant of the resultant matrix is unity:

\[
AB - CD = 1.
\]  

(5.19)

There is a generalization of the ray matrix formulation, namely, multiplying the slope by the local index of refraction, that results in the unity determinant even if the index changes [3]. For optical resonators and propagation of one round-trip of the resonator, the index of refraction at the start and finish must be the same. For an arbitrary number of cavity transits of the resonator represented by an \( ABCD \) matrix, the distance of the ray from the \( z \)-axis is

\[
r_n = r_{max} \sin(s\theta + \delta)
\]

(5.20)

where \( \theta = \cos^{-1}\left\{ (A + D)/2 \right\} \), \( \delta \) and \( r_{max} \) are determined by initial conditions and \( s \) is the number of cavity transits. If the quantity \((A + D)/2\) is in the range

\[
\frac{1}{2} < \frac{A + D}{2} < 1.
\]
The ray will be bound within the resonator never exceeding some maximum distance from the $z$-axis. If this condition is not met, the relationship describing the distance from the axis becomes a hyperbolic trigonometric function and the distance eventually expands without limit. Resonators that satisfy condition (5.21) are geometrically stable.

The ray transfer matrix also describes the propagation of Gaussian beams by

$$q_z = (Aq_1 + B) / (Cq_1 + D). \quad (5.22)$$

Here, $A$, $B$, $C$ and $D$ are the components of a ray transfer matrix for a single component or a combination of components. The complex beam parameter (5.10) before the element or combination of elements is $q_1$, and after, it is $q_z$. Equation 5.22 is called the ABCD law. It is easily confirmed using matrices from Table 5.1 for propagation over a distance $d$ or for a thin lens of focal length $f$; the respective results are $q_z = q_1 + d$ and $1/R_2 = 1/R_1 - 1/f$.

### 5.2.4 Gaussian Resonant Modes

Modes of stable two-mirror resonators can be found by fitting a Gaussian beam to the curvature and spacing of the resonator mirrors. More complex multiple-element resonators require the use of ABCD matrix techniques to determine the modes of a stable cavity. The ray transfer matrix for a resonator satisfies the condition that the initial index is the same as the final index, and the determinant is unity, that is, equation (5.19) applies. If a resonator is to have a Gaussian beam as the fundamental mode, the complex beam parameter must be reproduced after each cavity round-trip transit. Starting at $z_1$, where $q_1 = q(z_1)$, and propagating through all $n$ resonator components and returning to the same position requires

$$q_n = (Aq_1 + B)/(Cq_1 + D) = q_1 \quad (5.23)$$

The ABCD matrix elements are obtained by ordered matrix multiplication as performed in equation (5.18).

The complex beam parameter at the position $z_1$ is obtained from equation (5.22) using equation (5.19):

$$q_1 = (D - A) / (2C) + i\sqrt{(A + D)^2 - 4(D - A)^2} / 2C \quad (5.25)$$

and obtain

$$\omega_n^2 = \lambda / [(4 - (A + D)^2) / (2\pi C)] \quad (5.26)$$

There are other cavity configurations that can support Gaussian modes, such as gain-guided resonators and those with apodized apertures but these are not treated with the simple ray transfer matrices used here. Siegman [3] describes higher-order matrices for generalized astigmatism and complex matrices to treat the apodized apertures. One generalization that is described later is an orthogonal astigmatic system in which the axes of the astigmatic components remain parallel and perpendicular. In this case, the sagittal and tangential characteristics of the resonator modes are treated separately with appropriate ray transfer matrices. This is illustrated schematically for a mirror in Figure 5.6. A tabulation of ray transfer matrices for astigmatic elements is given in Table 5.3.

### Table 5.2

Ray transfer matrices for circularly symmetric components

<table>
<thead>
<tr>
<th>Propagate distance $d$</th>
<th>Thin lens of focal length $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} 1 &amp; d \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ -1/f &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>Spherical interface of radius of curvature $R$ going from index $n_1$ to $n_2$</td>
<td>Thick slab of graded index $n(n) = n_0 - n_{10}(x^2 + y^2)/2$ with $n = 1$ outside</td>
</tr>
<tr>
<td>$\begin{bmatrix} 1 &amp; 0 \ n_2/n_1 &amp; n_1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} \cos(d\sqrt{n_0/n_1}) &amp; \sin(d\sqrt{n_0/n_1}) \ -\sqrt{n_{10}n_0}\sin(d\sqrt{n_0/n_1}) &amp; \cos(d\sqrt{n_0/n_1}) \end{bmatrix}$</td>
</tr>
<tr>
<td>Spherical mirror with radius of curvature $R_0$</td>
<td>Flat interface going from index $n_1$ to $n_2$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 1 &amp; 0 \ -2/R_0 &amp; 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 1 &amp; 0 \ 0 &amp; n_2/n_1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

The ray will be bound within the resonator never exceeding some maximum distance from the $z$-axis. If this condition is not met, the relationship describing the distance from the axis becomes a hyperbolic trigonometric function and the distance eventually expands without limit. Resonators that satisfy condition (5.21) are geometrically stable.

The ray transfer matrix also describes the propagation of Gaussian beams by

$$q_z = (Aq_1 + B) / (Cq_1 + D). \quad (5.22)$$

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The complex beam parameter at the position $z_1$ is obtained from equation (5.22) using equation (5.19):

$$q_1 = (D - A) / (2C) + i\sqrt{(A + D)^2 - 4(D - A)^2} / 2C \quad (5.25)$$

and obtain

$$\omega_n^2 = \lambda / [(4 - (A + D)^2) / (2\pi C)] \quad (5.26)$$

There are other cavity configurations that can support Gaussian modes, such as gain-guided resonators and those with apodized apertures but these are not treated with the simple ray transfer matrices used here. Siegman [3] describes higher-order matrices for generalized astigmatism and complex matrices to treat the apodized apertures. One generalization that is described later is an orthogonal astigmatic system in which the axes of the astigmatic components remain parallel and perpendicular. In this case, the sagittal and tangential characteristics of the resonator modes are treated separately with appropriate ray transfer matrices. This is illustrated schematically for a mirror in Figure 5.6. A tabulation of ray transfer matrices for astigmatic elements is given in Table 5.3.
5.3 Stable Resonators

5.3.1 Two Mirror Resonators

The previous section contains the analytic techniques necessary to characterize stable resonators. The simplest stable resonators have two mirrors. In a first approximation, the active laser material and other intra-cavity components can be accommodated by an effective cavity length change resulting from flat parallel plates of uniform refractive index. A graded index guide that approximates the effects of thermal loading in a laser rod could be used for a more accurate approximation. The resonator considered here simply has two spherical mirrors of radius of curvature $R_A$ and $R_B$ separated by distance $d$. The mirror radii are positive for concave mirrors and negative for convex mirrors. The first technique is to match the mirror curvatures to the wavefront curvatures by considering a Gaussian beam that passes through the two mirrors as shown.

![Top view showing the tangential focus (a) and side view showing the sagittal focus (b) of a spherical mirror. The axis of the mirror and incident beam are in a horizontal plane.](image)

**FIGURE 5.6**

TABLE 5.3

<table>
<thead>
<tr>
<th>Ray transfer matrices for astigmatic components at an angle of incidence $\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sagittal</strong></td>
</tr>
<tr>
<td>Thin lens of focal length $f$ with index $n_2$ surrounded by medium index $n_1$:</td>
</tr>
<tr>
<td>$\begin{bmatrix} 1 &amp; 0 \ \frac{\sqrt{n_1^2 - n_2^2 \sin^2 \varphi} - n_1 \cos \varphi}{(n_2 - n_1)f} &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 1 &amp; 0 \ \frac{-n_1 \cos \varphi}{(n_2 - n_1)f \cos^2 \varphi} &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>Thin cylindrical lens of tangential focal length $f$ and index $n_2$ in a medium of index $n_1$:</td>
</tr>
<tr>
<td>$\begin{bmatrix} 1 &amp; 0 \ \frac{0}{n_2} &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} \frac{n_1 \cos \varphi}{\sqrt{n_1^2 - n_2^2 \cos^2 \varphi}} &amp; 0 \ 0 &amp; \frac{n_1 \cos \varphi}{\sqrt{n_1^2 - n_2^2 \cos^2 \varphi}} \end{bmatrix}$</td>
</tr>
<tr>
<td>Flat interface going from index $n_1$ to index $n_2$:</td>
</tr>
<tr>
<td>$\begin{bmatrix} 1 \frac{0}{\frac{2 \cos \varphi}{R}} &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 1 &amp; 0 \ \frac{-2 \cos \varphi}{R \cos \varphi} &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>Concave spherical mirror of radius of curvature $R$:</td>
</tr>
<tr>
<td>$\begin{bmatrix} \frac{1}{\sqrt{n_1^2 - n_2^2 \sin^2 \varphi} - n_1 \cos \varphi} \frac{n_1}{n_2} \frac{1}{n_2} \ \frac{n_1}{n_2} \end{bmatrix}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} \frac{\sqrt{n_1^2 - n_2^2 \cos^2 \varphi}}{n_2 \cos \varphi} &amp; 0 \ \frac{1}{R \cos \varphi - \sqrt{n_1^2 - n_2^2 \cos^2 \varphi}} &amp; \frac{n_1 \cos \varphi}{\sqrt{n_1^2 - n_2^2 \cos^2 \varphi}} \end{bmatrix}$</td>
</tr>
<tr>
<td>Spherical interface of radius of curvature $R$ going from index $n_1$ to index $n_2$:</td>
</tr>
<tr>
<td>$\begin{bmatrix} \frac{1}{\sqrt{n_1^2 - n_2^2 \sin^2 \varphi} - n_1 \cos \varphi} \frac{n_1}{n_2} \frac{1}{n_2} \ \frac{n_1}{n_2} \end{bmatrix}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} \frac{\sqrt{n_1^2 - n_2^2 \cos^2 \varphi}}{n_2 \cos \varphi} &amp; 0 \ \frac{1}{R \cos \varphi - \sqrt{n_1^2 - n_2^2 \cos^2 \varphi}} &amp; \frac{n_1 \cos \varphi}{\sqrt{n_1^2 - n_2^2 \cos^2 \varphi}} \end{bmatrix}$</td>
</tr>
<tr>
<td>Thick plate of thickness $d$ and index $n_2$ with parallel surfaces and index $n_1$ outside:</td>
</tr>
<tr>
<td>$\begin{bmatrix} \frac{n_1 d}{\sqrt{n_1^2 - n_2^2 \cos^2 \varphi}} &amp; 0 \ \frac{n_1 d}{\sqrt{n_1^2 - n_2^2 \cos^2 \varphi}} &amp; 0 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\begin{bmatrix} \frac{dn_1 n_2 \cos \varphi}{n_2^2 - n_1^2 \sin^2 \varphi} &amp; 0 \ \frac{dn_1 n_2 \cos \varphi}{n_2^2 - n_1^2 \sin^2 \varphi} &amp; 0 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
in Figure 5.4. Use equation (5.14) with \( R_1 = -R_A \) and \( R_2 = R_B \) to obtain

\[
\alpha_0 = \frac{\lambda^2 d (R_B - d) (R_A - d) (d - R_A - R_B)}{\pi^2 (2d - R_A - R_B)^2}
\]

and

\[
z_0 = \frac{z_2 - d (d - R_A)}{2d - R_A - R_B}
\]

(5.27)

Here, \( z_1 \) is the position of the first mirror, \( z_2 \) is the position of the second mirror, \( d = z_2 - z_1 \) and \( z_0 \) is the location of the beam waist, which has spot size \( \alpha_0 \). There will be a real value for the beam waist only when the right-hand side of the first equation of equation (5.27) is positive. This is the condition for a stable cavity.

To perform a ray-transfer-matrix analysis of the same resonator, pick a starting position, for example, just before the right-hand mirror \( R_B \). Then, the appropriate matrices from Table 5.2 are used for reflection from the mirror \( R_B \), propagation over the resonator length \( d \), reflection from mirror \( R_A \) and propagation back to just before the right-hand mirror.

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & d \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-2/R_A & 1
\end{bmatrix}
\begin{bmatrix}
1 & d \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-2/R_B & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 - \frac{2d}{R_A} - \frac{4d}{R_B} + \frac{4d^2}{R_AR_B} & 2d \left( 1 - \frac{d}{R_A} \right) \\
\frac{4d}{R_AR_B} - 2 \left( \frac{1}{R_A} + \frac{1}{R_B} \right) & \left( 1 - \frac{2d}{R_A} \right)
\end{bmatrix}
\]

(5.28)

The manipulation of the completed matrix multiplications in the second line of equation (5.28) could be continued to obtain equation (5.27). When the condition for a stable resonator (5.21) is applied to equation (5.28),

\[
0 < (1 - d/R_A) (1 - d/R_B) < 1
\]

(5.29)

is obtained. The quantities \((1-d/R_A)\) and \((1-d/R_B)\) are commonly referred to as the resonator \(g\) parameters \(g_1\) and \(g_2\), and the stability condition (5.29) in terms of these quantities becomes

\[
0 < g_1 g_2 < 1.
\]

(5.30)

The familiar stability diagram (shown in Figure 5.7) first used by Boyd and Gardner [2] is based on equation (5.29).

It is possible to include a large amount of information in the stability diagram. Resonators for which the \(g_1 g_2\) product is located in the unshaded area are stable, and those in the shaded area are unstable. The straight line from point a \((g_1 = 1, g_2 = 1)\) to point b \((g_1 = -1, g_2 = -1)\) represents symmetric, stable cavities. The end points of this line, the plane-parallel cavity at point a and the concentric cavity at point b, are on the border of the stability region. It is not possible to implement these geometries as stable cavities because an infinite spot size is required at the cavity mirrors. The centre point c \((g_1 = 0, g_2 = 0)\) representing the confocal cavity is also on the border of instability. A change from a symmetric geometry could make the confocal resonator unstable. The horizontal and vertical broken lines starting at point a \((g_1 = 1, g_2 = 1)\) represent stable resonators with one flat mirror. If a resonator is chosen for stability, it is useful to have \(g_1 g_2 = 1/2\). Other resonator choices could be based on other considerations such as the desired spot size at specific locations in the resonator. If a resonator is to accommodate a lensing that develops due to thermal loading of the active laser material, a good choice of resonator geometry might be to have the cold-cavity \(g_1 g_2\) product near the appropriate end of the stable-symmetric-cavity line of Figure 5.7 and have the thermal lensing bring the cavity towards the other end of the line. It is a geometrical property of stable resonators that the on-axis line from the surface of one mirror to its centre of curvature will partially overlap the corresponding line for the other mirror. If this condition is not met and there is no overlap or one line is completely contained within the other, the cavity is unstable.

5.4 Higher-order Modes of Stable Resonators

Stable lasers often run in multiple stable modes leading to mode beats. For laser operation on multiple longitudinal modes but restricted to a single transverse mode, the mode beating will result in temporal intensity and phase modulation that repeats with the cavity round-trip-transit frequency. If there are multiple transverse modes as well as multiple longitudinal modes, the mode beating becomes more complex. The transverse intensity and phase distribution will fluctuate as well. The average or time-integrated intensity distribution, however, may appear uniform, obscuring the instantaneous spatial structure.
Typically, special apertures or obstructions such as wires are required to force a laser resonator to operate in a single higher-order mode. Modelling of higher-order modes is useful because actual laser output is often closely represented by a superposition of several resonator transverse modes, and therefore, it offers a technique for dealing with actual laser oscillations and beams. Higher-order modes for open stable resonators or free-space propagation usually are derived mathematically by substitution of a trial solution in a paraxial wave (equation 5.1) or Fresnel's paraxial approximation to Huygen's integral (equation 5.2). Depending on the form of trial solution, the result can be equations that describe the Hermite–Gaussian modes for rectangular symmetry or the Laguerre–Gaussian modes for cylindrical symmetry. This procedure determines the scaling factors and relationship between modes of different order. A fundamental-mode spot size \( \omega(z) \) with propagation dependence given by equation (5.4) and a wavefront radius of curvature \( R(z) \) described by equation (5.5) will carry over unchanged as parameters of both sets of modes. The higher-order modes retain their shape but expand in proportion to \( \omega(z) \) with propagation. The Gouy phase shift changes with the order of the modes. The different phase shifts will cause the superposition of a series of modes to synthesize amplitude and phase distributions that change with propagation. Only individual transverse modes are guaranteed to retain their relative distribution with propagation. The more commonly used Hermite–Gaussian modes are described next, followed by the Laguerre–Gaussian modes.

**5.4.1 Cartesian Coordinates**

The Hermite–Gaussian modes, often called the transverse electromagnetic modes of order \( m \) and \( n \) or TEM\(_{mn}\), have the form

\[
E_{mn}(x,y,z,t) = \frac{1}{2} E_{m0} \sqrt{\frac{\omega_m \omega_n}{\omega_m(z) \omega_n(z)}} \times \exp\left[-i(kz - \omega t)\right] H_m\left(\sqrt{\frac{2x}{\omega_m(z)}}\right) H_n\left(\sqrt{\frac{2y}{\omega_n(z)}}\right) \exp\left[-x^2 \left(\frac{1}{\omega_m^2(z)} + \frac{ik}{2R_m(z)}\right) - y^2 \left(\frac{1}{\omega_n^2(z)} + \frac{ik}{2R_n(z)}\right)\right] + \text{c.c.}
\]

The constant \( E_{m0} \) in the above equation is given by

\[
E_{m0} = \frac{n e_0 c}{\omega_m \omega_n \pi^{m+n+1} m! n!} \left(\frac{2\lambda P_m}{n e_0 c}\right)^{1/2}
\]

where \( P_m \) is the power of the TEM\(_{mn}\) mode in watts. The Gouy phase shifts of the Hermite–Gaussian modes are given by

\[
\Phi_{mn}(z) = (m + 1/2) \arctan\left[\frac{\lambda(z - z_{s0})}{(\pi \omega_n^2)}\right] + (n + 1/2) \arctan\left[\frac{\lambda(z - z_{s0})}{(\pi \omega_m^2)}\right].
\]

Ellipticity and astigmatism are included in equations (5.31)–(5.33) by using separate spot sizes \( \omega_m(z) \) and \( \omega_n(z) \) and wavefront radii of curvature \( R_m(z) \) and \( R_n(z) \) for the orthogonal transverse directions. Implicit in the separate parameters for the two transverse directions is the possibility of different beam-waist positions \( z_{s0} \) and \( z_{c0} \). The functions \( H_m \) and \( H_n \) are Hermite polynomials [12] and, for order up to four, are

\[
H_0 = 1
\]

\[
H_1(x) = 2x
\]

\[
H_2(x) = 8x^3 - 12x
\]

\[
H_3(x) = 16x^4 - 48x^2 + 12.
\]

Higher-order Hermite polynomials can be obtained with the recurrence relation

\[
H_{n+1}(s) = 2sH_n(s) - 2nH_{n-1}(s)
\]

The Hermite–Gaussian-polynomial orthogonality integrals are

\[
\int_{-\infty}^{\infty} \exp(-s^2) H_n(s) H_m(s) ds = 0 \quad (n \neq m)
\]

and

\[
\int_{-\infty}^{\infty} \exp(-s^2) H_n^2(s) ds = \sqrt{\pi} 2^n n!.
\]

The orthogonality integrals permit the evaluation of the constant \( E_{mn} \) in equation (5.31), describing the TEM\(_{mn}\) mode amplitude. Another integral that is required shortly is the second moment.

\[
\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \exp(-s^2) H_n^2(s) ds / \int_{-\infty}^{\infty} \exp(-s^2) H_n^2(s) ds = (2n + 1)/2.
\]

This is obtained using the recurrence relation and orthogonality integrals.

The Hermite–Gaussian modes form the familiar rectangular array of transverse distribution intensity peaks that are demonstrated in laser outputs with cavity perturbations that favour a single mode. Normalized mode amplitudes and relative intensities are plotted for one dimension in Figure 5.8. The plots are for a beam waist or point where the beam is collimated to simplify the figure.

**5.4.2 Cylindrical Coordinates**

The modes of a resonator with strict circular symmetry can be described in cylindrical coordinates using generalized Laguerre polynomials. The fundamental mode in cylindrical coordinates is identical to the TEM\(_{00}\) Hermite–Gaussian mode with no astigmatism. Higher-order modes are given by
Optical Cavities

\[ E_{p}(r, \phi, z, t) = \frac{1}{2} E_{p0} \frac{\omega_0}{\omega(z)} \exp\left[-i(kz - \omega t)\right] \left(\sqrt{2} \frac{r}{\omega(z)}\right)^{m} \]
\[ \times L_{p}^{m}(2r^2/\omega^2(z)) e^{-i\theta} \]
\[ \times \exp \left\{ -r^2 \left( \frac{1}{\omega^2(z)} + \frac{i k}{2R(z)} \right) + \Phi_{p}(z) \right\} + \text{c.c.} \]
\[ \tag{5.39} \]

where \((r, \phi)\) are cylindrical coordinates and \(L_{p}^{m}(\rho)\) is a generalized Laguerre polynomial with \(m\) an integer or zero and \(p\) zero or a positive integer. The parameters \(\omega_0, \omega, k, \omega\) and \(R(z)\) are the same as those used with the Hermite–Gaussian modes.

The first two generalized Laguerre polynomials are
\[ L_{0}^{0}(\rho) = 1, \quad L_{1}^{0}(\rho) = l+1 \]
\[ \tag{5.40} \]

Higher-order generalized Laguerre polynomials can be obtained from the recurrence relation
\[ (p+1)L_{p+1}^{m}(\rho) = (2p+1l+1-\rho)L_{p}^{m}(\rho) - (p+1l) L_{p-1}^{m}(\rho). \]
\[ \tag{5.41} \]

The Gouy phase shifts for the cylindrical modes are given by
\[ \Phi_{p} = (2p+1l+1) \tan^{-1} \left( \lambda z/\pi \omega^2_0 \right). \]
\[ \tag{5.42} \]

The orthogonality integral for the generalized Laguerre polynomials is
\[ \int_{0}^{\infty} \exp(-\rho)\rho^{(m+1)/2} L_{p}^{m}(\rho) L_{p'}^{m}(\rho) d\rho = \delta_{m0} \delta_{p,p'} (l+1)!/p! \]
\[ \tag{5.43} \]

where the Kronecker delta \(\delta_{\alpha\beta}\) is 1 for \(\alpha = \beta\) and 0 if \(\alpha \neq \beta\). The coefficient \(E_{p0}\) in equation (5.39) is given in terms of \(P_{p0}\), the power of the \(p, l\) mode in watts, by
\[ E_{p0} = \frac{2P_{p0}}{n e_0 c \pi \omega_0} \frac{1}{(l+1)!} \left( \frac{2}{l+1} \right). \]
\[ \tag{5.44} \]

In practice, it is difficult to eliminate astigmatism completely and attain the degree of circular symmetry necessary to produce cylindrical modes.

5.4.3 Beam Quality

Actual resonators will have imperfections that distort a laser oscillation from the ideal Hermite–Gaussian modes, even though the distortion may be small. Laser oscillations may also consist of multiple transverse modes. Siegman’s ‘M squared’ or \(M^2\) parameter has been established as a measure of beam quality [13,14]. \(M^2\) is the ratio of the product of the square root of the second moment of the time-averaged transverse spatial distribution and the square root of the angular distribution of a beam and the corresponding value for an ideal Gaussian beam. The transverse spatial distribution second moment or variance changes with propagation, and the minimum values are used in the measurement of \(M^2\). The \(M^2\) value can be specified separately for the \(x\) and \(y\) coordinates as \(M^2_x\) and \(M^2_y\), or a combined value for the total beam can be specified. The variance for the \(x\) coordinate of the fundamental or TEM\(_{00}\) Gaussian mode is \(\sigma^2_{x,x=0}(z) = \omega_x^2(z)/4\), which has a minimum value at the beam waist of \(\sigma^2_{x,x=0}(z_0) = \omega_{x0}^2(\pi)^2/4\). The angular distribution can be obtained by Fourier transform or, equivalently, by propagating to the far field, where the angular half width in the \(x\)-direction at the 1/e maximum amplitude is \(\theta_{x,x=0} = \lambda/(\pi w_0, 0)\). It follows that the angular variance of the
A similar expression describes the angle with respect to the $xz$ plane. The angular variance in the $x$-direction is

$$\sigma_{\theta_x}^2 = \sum_{q_1} \sum_{q_2} (\theta_{x,q_1} - \theta_{x,q_2})^2 \tilde{E}_{q_1,q_2} \tilde{E}_{q_1,q_2}^* / \sum_{q_1} \sum_{q_2} \tilde{E}_{q_1,q_2} \tilde{E}_{q_1,q_2}^*.$$  

(5.53)

The $y$ spatial variance is

$$\sigma_y^2 = \sum_{n_1} \sum_{n_2} \left( \frac{n_1 \Delta x}{N_1} - \bar{x} \right)^2 E_{n_1,n_2} E_{n_1,n_2}^* / \sum_{n_1} \sum_{n_2} E_{n_1,n_2} E_{n_1,n_2}^*.$$  

(5.54)

It is necessary to propagate to a $z$ position where the spatial variance is minimum to obtain $M^2$, or, equivalently, to remove the spherical curvature [17] from the wave front before calculation of the spatial and angular variances. When this is done, the $M^2$ values are given by

$$M_x^2 = 4 \pi \sigma_{\theta_x} \sigma_{\lambda, \min} / \bar{\lambda} \text{ and } M_y^2 = 4 \pi \sigma_{\theta_y} \sigma_{\lambda, \min} / \bar{\lambda}.$$  

(5.55)

There are as many ways to obtain the $M^2$ of an actual beam experimentally as there are for numerically modelled beams. One experimental technique involves estimating a Gaussian spot size $w_n$ of the beam at many positions $z_n$ along the beam and fitting these to functions of the form:

$$W_n^2(z) = M_n^2 [\omega_{n,0}^2 + (z - z_{n,0})^2 (\bar{\lambda}/\sigma_{\lambda,0})^2]$$

$$= W_{n,0}^2 [1 + M_n^2 (z - z_{n,0})^2 (\bar{\lambda}/\sigma_{\lambda,0})^2]$$  

(5.56)

to obtain the $M^2$ value, the embedded Gaussian beam-waist spot size $w_n$ and the beam-waist position $z_n$. The capital letter $W$ is used to indicate the spot of an actual beam, which is larger than the spot size of the embedded Gaussian beam. In performing the least-squares fit to a set of experimental data, it is helpful to weight the individual measurements of $W^2$ by $1/W^2$ to place more significance on the measurements near the beam waist and reduce the possibility of a fit that predicts meaningless negative values of $W^2$. The spot sizes $W_n$ may be estimated by taking the difference of knife-edge positions that transmit 16% and 84% of the total beam. A refinement could involve obtaining best-fit Gaussian distributions from several knife-edge positions at each of many propagation distances, $z_m$. Scanning pinholes, scanning slits and array detectors also can be used to measure beam distributions and second moments. It is necessary to consider the properties of the measurement and application. For example, a small amount of energy or error in measuring at a large distance from the central lobe of a beam will increase $M^2$. However, the small amount of energy at a large distance from the central lobe may or may not be significant in the application. The $M^2$ parameter provides both a measure of beam quality and a mechanism to use Gaussian beam propagation methods to deal with the propagation of actual laser beams.
5.5 Mode-Matching

In many applications it is necessary to be precise in the spatial distribution of a laser beam delivered on a target. Knowledge of the beam’s intensity distribution at the target is obtained from the beam power and the beam propagation parameters. The size and beam quality parameters are critically important when coupling laser beams into optical fibres. It is desirable to match the transverse distribution of a pump beam to the mode size of a resonator for laser-pumped lasers. In some applications, mode-matching is extended to matching the confocal parameters of a pump beam and an external resonant cavity. With a single lens or spherical mirror, it is possible to control either the size of the beam waist or its location. Two lenses adjustable in position are required to control both the beam-waist position and size simultaneously.

Configurations for single-lens mode-matching with an ideal Gaussian beam are described next. The application of the beam quality parameter and the concept of an embedded beam waist of spot size \( w_0 \), formed by the lens (Figure 5.9). The ABCD law (5.22) is applied to the complex beam parameter (5.10). The real and imaginary parts of the resulting equation are separated to yield [11]

\[
\begin{align*}
(d_1 - f) b_2 &= (d_2 - f) b_1 & (5.57) \\
(d_1 - f)(d_2 - f) &= f^2 - b_1 b_2/4. & (5.58)
\end{align*}
\]

Here, \( b_1 = 2\pi \omega_{0,1}^2 / \lambda \) and \( b_2 = 2\pi \omega_{0,2}^2 / \lambda \) are the confocal parameters of the beam of wavelength \( \lambda \) before and after the lens. The quantity \( b_1 b_2/4 \) is sometimes labelled \( \omega_0^2 \). It is necessary that \( f^2 \geq b_1 b_2/4 = \omega_0^2 \) for there to exist distances \( d_1 \) and \( d_2 \) that will yield a confocal parameter \( b \) from an initial beam with confocal parameter \( b_{0,1} \); that is, the absolute value of the lens’ focal length must be longer than a minimum value. In this case, when \( \omega_{0,1} \) and \( \omega_{0,2} \) and \( f \) are specified, the distance from the first waist to the lens and the distance from the lens to the second waist are given by

\[
d_1 = f \pm (\omega_{0,1}/\omega_{0,2}) \sqrt{f^2 - b_1 b_2/4} \tag{5.59}
\]

and

\[
d_2 = f \pm (\omega_{0,2}/\omega_{0,1}) \sqrt{f^2 - b_1 b_2/4}. \tag{5.60}
\]

Here, either + or − signs should be used in both equations (5.59) and (5.60).

Another set of useful equations comes from combining the last two equations. What is the focal length of the required lens when the beam waist is \( \omega_{0,1} \) and \( \omega_{0,2} \) and their separation \( d = d_1 + d_2 \) are specified? This will aid in the choice of a lens from available focal lengths to provide approximately the desired waist separation. The resulting equation is quadratic in \( f \):

\[
\begin{align*}
4 - (\omega_{0,1}^2 + \omega_{0,2}^2) / \omega_{0,1}^2 &- (\omega_{0,1}^2 + \omega_{0,2}^2) f^2 \\
- 4df + (\omega_{0,1}^2 + \omega_{0,2}^2) f^2 &+ d^2 = 0 \tag{5.61}
\end{align*}
\]

which has either two or no real solutions. Even with real solutions, it is necessary to check that the lens positions are physically realizable, e.g. not beyond the target. The position of a lens of focal length \( f \) determined in equation (5.61) is

\[
d_1 = (\omega_{0,1}^2 (d - f) + \omega_{0,2}^2 f) / (\omega_{0,1}^2 + \omega_{0,2}^2). \tag{5.62}
\]

5.5.2 Two-lens Mode-matching

One way to approach calculations of two-lens mode-matching is to step through a series of lens positions until the desired mode-matching is found. Each iteration involves a different placement of the first lens, for which the position of the waist and the confocal parameter of the beam formed by the first lens of focal length \( f_1 \) are given by

\[
d_1 = f_1 (b_1^2 + 4(d_4 - f_1 d_4))/\{b_1^2 + 4(d_4 - f_1)^2 \}. \tag{5.63}
\]

FIGURE 5.9 Parameters for mode-matching with one lens.
b_2 = b_1 (d_0 - f_1) / (d_3 - f_1). \quad (5.64)

This leaves a known distance \( d \) from the position of the waist formed by the first lens to the desired position of the final waist. The second lens is then numerically placed at a position that produces a third beam waist at the desired position. The numerical analysis is stepped through a range of positions for the first lens. The parameters are illustrated in Figure 5.10. A cubic equation is obtained for the position of the second lens when the distance between beam waists \( d = d_1 + d_3 \), \( w_{02} \), the size of the intermediate beam waist and \( f_2 \), the focal length of the second lens, are specified:

\[
d^3 - (f_2 + d)d^2 + (2f_0 d + b^2_2/4)d_2 - (d - f_2)b^2_2/4 - f^2_2 d = 0
\]

(5.65)

There are either one or three real roots to equation (5.65) and the confocal parameter of each of the resulting beams is given by

\[
b_3 = b_2 (f_2 - d_2) / (f_2 - d_1)
\]

(5.66)

Single-lens mode-matching is simpler than two-lens mode-matching. In cases where the distance between the initial and final beam waists can be adjusted, a single lens will work well. However, if that distance is fixed, two-lens mode-matching may be required. It is possible to add two more adjustable parameters by tilting the lenses, but any astigmatism or ellipticity in the beam might best be removed before mode-matching.

FIGURE 5.10 Mode-matching with two lenses may be necessary when there is a fixed distance \( D \) between the initial and final beam waists.

\[
N_F = a^2 / (L\lambda)
\]

(5.67)

where \( a \) is the radius of the limiting aperture of the resonator, \( L \) is the propagation distance from one encounter of the limiting aperture to the next and \( \lambda \) is the wavelength of the resonated light. The limiting aperture could be a laser rod, a cavity mirror or an actual aperture placed in the resonator. Resonators with the same Fresnel number will have equivalent diffraction properties. A circular aperture is used here, and the resonator has circular symmetry. The circular symmetry is lost, however, when the Fourier components of the initial amplitude distribution are given a random phase.

The diffraction for various apertures and beams provides some insight into the significance of the Fresnel number. In many cases, there is a diffraction spread of a collimated beam that is approximately \( \lambda / a \) rad. For example, the diffraction half-angle of a Gaussian beam is \( \lambda / (\pi w_0) \). The centre to first minimum angle of the Airy diffraction pattern of a uniformly illuminated circular aperture is \( 1.22\lambda / (2a) \), and the full width at half maximum of the diffraction pattern from a slit of width \( 2a \) illuminated by a plane wave is \( 1.39\lambda / (\pi a) \). With a Fresnel number of \( N_F = 1 \), a nearly collimated beam will spread by diffraction in a resonator length \( L \), to slightly overfill the aperture for significant loss on the next encounter with the aperture.

Numerical calculations show that a resonator with a Fresnel number of 1 will have about 18% loss for each cavity transit from aperture to aperture, whereas the loss will be approximately 0.88% for a Fresnel number of 10. The Fresnel number is a useful and general concept. For example, the \( N_F \) of a stable resonator is approximately the number of Hermite–Gaussian modes that the resonator will support.

Iterative computer techniques are commonly used to determine the cavity modes of plane-parallel resonators. In the original paper by Fox and Li [1], iterative solutions to Huygen’s integral in cylindrical symmetry starting with a uniform plane wave were used. It is more common now to use Hankel transform techniques for cylindrical symmetry and Fourier transform techniques for rectangular symmetry [15,16]. A discrete Fourier transform pair such as equations (5.50) and (5.51) is used. The spatial amplitude and phase distribution transmitted through the aperture are transformed into a sum of plane waves propagating at regularly spaced directions with respect

5.6 Plane Parallel Resonators

The analysis of ideal stable resonators provides a useful background for understanding practical laser resonators including plane-parallel-mirror resonators and unstable resonators. We begin with plane-parallel resonators in this section and continue with unstable resonators in the next section. Plane-parallel resonators are characterized by their Fresnel number.
to the central direction of propagation. The propagation of the plane waves to the next encounter with the aperture is straightforward, each having a relative phase shift dependent on the direction of propagation. The plane waves are then summed to synthesize the spatial distribution, which is modified by transmission through the aperture. Routines for calculating the Fourier transformations are available \[18,19\].

Calculated diffraction losses as a function of Fresnel number for plane-parallel resonators of circular symmetry are shown in Figure 5.11. Starting with a uniform distribution and propagating through 300 cavity transits yielded the plotted values. The power in the beam was re-normalized after each cavity transit. The plotted line shows transmission on the 300th transit. The process is more slowly converging for Fresnel numbers greater than ten, typically requiring more than 300 transits to converge. Virtual source techniques \[20\] are useful for the analysis of resonators with large Fresnel numbers.

A modification of the Fox and Li iterative cavity transit technique was used for the calculations illustrated in Figures 5.12–5.14. A two-dimensional FFT technique was used to calculate beam propagation. In this case, the initial spatial frequency or angular components all had equal power but the phases were random. A Fresnel number of 10 was used for the calculation. An initial intensity distribution transmitted through an aperture is shown in Figure 5.12a. This is intended to simulate a laser oscillation that is growing out of zero-point quantum fluctuations. The initial distribution is limited by the number of spatial sampling points used in the calculation, and the distribution changes for a new calculation with different random phases. After each transit, the intensity was again re-normalized. After 10 cavity transits (Figure 5.12b), the high spatial frequencies are greatly attenuated but the beam is still strongly structured with an \(M^2\) of 9.5. Spatial filtering continues with fewer high-frequency components and an \(M^2\) of 5.3 after 20 transits (Figure 5.12c). Many high-gain Q-switched lasers reach their peak output power in 10 or 20 cavity transits, and this type of distribution could be present in an instantaneous sampling of the output. Averaging over the total Q-switched pulse could give the appearance of a uniform intensity beam with an \(M^2\) between 5 and 10. The loss per transit behaviour and the evolution of \(M^2\) during the iterative calculation are illustrated in Figure 5.13.

After several hundred cavity transits, the intensity and phase distribution become constant, as shown in Figure 5.14. At this point, the beam has an \(M^2\) of 1.5 and the intensity loss per transit is 0.88%. In practice, such a distribution would be difficult to obtain, even with hundreds of cavity transits, due to the sensitivity of the plane-parallel resonator to misalignment. Injection seeding of a field distribution close to that of the fundamental mode of the resonator would quickly produce a dominant oscillation of that mode in the resonator. The use of injection seeding, however, is more commonly used to achieve single-frequency oscillation in resonators with large gain. Selection of the fundamental transverse mode is easily achieved in unstable resonators.

### 5.7 Unstable Resonators

Unstable resonators offer advantages of good energy extraction efficiency from a large-volume active laser material and reasonably good spatial beam quality in the laser output. Typically, unstable resonators have large loss or output coupling that must be offset by high laser gain. The high-gain requirement usually restricts operation to the pulsed mode because of the difficulty in maintaining high can gain. Frequency control and narrow spectral bandwidth operation are more difficult with high laser gain. Unstable resonators have transverse modes that reproduce from one cavity transit to the next. These modes, however, are not orthogonal or even nearly orthogonal.
as in the case of stable resonators. The non-orthogonality of
the modes leads to some perhaps surprising effects, such as a
mode is most efficiently seeded with a conjugate beam; that is,
the backward propagating beam, converging where the output
beam was diverging, will most effectively seed the oscillation.
Transverse mode discrimination is usually large, and oscilla-
tions usually resolve to a single transverse mode in a small
number of cavity transits.

5.7.1 Hard-Edged Apertures

Unstable resonators can be classified as either hard- or soft-
edged. In a hard-edged resonator, output coupling is typically
by expansion of the resonated beam beyond the edge of some
limiting aperture. This could be a hole cut in the centre of a
mirror used for output coupling, a high-reflectivity mirror
smaller than the expanded beam or a high-reflectivity spot on a
substrate used to reflect and output couple the resonated beam.
Soft-edged resonators can be created with apodized apertures,
variable reflectivity mirrors (VRMs) and gain guiding such
as is common in laser-pumped lasers. Unstable resonators are
also classified as either negative or positive branch according
to the sign of the product \( g_1g_2 = (1 - d/R_A)(1 - d/R_B) \). Two-mirror
negative-branch unstable resonators have a beam focus inside
the resonator, whereas a two-mirror positive-branch resonator
can have either none or two. The absence of an intra-cavity
beam focus is an advantage for high-power laser oscillations.
Confocal unstable resonators are useful because the resonated
beam is collimated in the output part of the resonator round
trip [21]. This can be useful in producing a collimated output
or region of collimated propagation inside the resonator.

Unstable resonators are usually analysed on two levels.
Geometrical analysis of unstable resonators provides a fair
first approximation. Some soft-edge unstable resonators can
be analysed using Gaussian optics and an extension of the
ABCD matrix techniques to include complex matrix elements.
Numerical beam propagation calculations, however, are usu-
ally required for a detailed understanding of the diffraction
effects. Fourier transform techniques are commonly used to
perform these numerical calculations.

When equation (5.24),

\[
q = (Aq + B)/(Cq + D),
\]

is solved for \( 1/q \)
in the case of an unstable resonator, the result is real:

\[
q = \frac{B}{A - \frac{D}{C}}.
\]
This result gives a non-physical interpretation of spherical waves since \(1/\omega^2 = 0\). When the accurate solutions with limiting apertures and diffraction are considered, it is found that the two solutions for \(R\) represent a stable solution of a divergent beam and an unstable solution for a convergent beam. The convergent beam becomes smaller only for a few cavity transits until diffraction begins to dominate and it quickly changes to become a divergent beam.

Geometrical magnification of the unstable resonator is also obtained from the ABCD matrices. For positive-branch unstable resonators, \(m = (A+B)/2 > 1\); the values for magnification are

\[
M = m + \sqrt{m^2 - 1} \quad (5.69a)
\]

for the expanding beam and

\[
1/M = m - \sqrt{m^2 - 1} \quad (5.69b)
\]

for the convergent beam. For negative-branch unstable resonators, \(m = (A+B)/2 < -1\); the values for magnification are

\[
M = m - \sqrt{m^2 - 1} < -1 \quad (5.70a)
\]

and

\[
-1 < 1/M = m + \sqrt{m^2 - 1} < 0. \quad (5.70b)
\]

A parameter important for characterization of hard-edged unstable resonators is the equivalent Fresnel number:

\[
N_{eq} = \frac{a^2}{2\lambda}. \quad (5.71)
\]

Here, \(M\) is the magnification of the resonator as given earlier; \(B\) is the component of the ABCD matrix of the resonator; \(a\) is the radius of the limiting circular aperture and \(\lambda\) is the wavelength of the light circulating in the resonator.

We now restrict the discussion to an illustrative example of a positive-branch, confocal, unstable resonator as shown in Figure 5.1b. This resonator is formed by a concave mirror of radius of curvature \(R_A > 0\) and a convex mirror of radius of curvature \(R_B < 0\) separated by distance \(d\). The output mirror has a central highly reflecting spot of radius \(a\) and is transmitting for a radius greater than \(a\). Such an output mirror could be a small suspended mirror, as shown, or a reflecting spot deposited on a meniscus substrate to preserve the collimation of the output beam. The confocal property of the resonator specifies that \(R_A = 2d - R_B\). In the special case of a confocal unstable resonator, the magnification is given by \(M = -R_A/R_B\) and the equivalent Fresnel number is \(N_{eq} = \sigma^2/(\lambda R_B)\). The magnification is a positive value because \(R_B < 0\).

A calculation of loss per round-trip transit as a function of \(N_{eq}\) for a positive-branch confocal resonator with magnification \(M = 3\) is shown in Figure 5.15. At larger values of \(N_{eq}\), the loss converges on the geometrical value of \((1-1/M^2) = 0.89\). The broken line in Figure 5.15 represents this value. At smaller values, loss is minimum at half-integer values of \(N_{eq}\). a
feature common to hard-edged unstable resonators. This feature is due to the transverse-mode properties of the resonator. At half-integer values of \( N_{eq} \), the loss difference between modes is large, and the oscillation of a single transverse mode dominates after a few cavity transits. At integer values of \( N_{eq} \), there are two transverse modes of nearly equal loss and many cavity transits are required to obtain a reproducible loss and transverse field distribution.

The initial distribution and distribution after eight transits for a resonator with magnification \( M = 3 \) and equivalent Fresnel number \( N_{eq} = 5.5 \) (Figure 5.16) illustrate that the lowest loss mode can be resolved quickly both in calculations and in the actual build-up of oscillation in an unstable resonator laser. The calculation used to generate Figure 5.16 used random phasing of the initial Fourier components all of equal power. The total power was normalized on each cavity transit, preserving the phase and relative intensity of the individual components in the method of the Fox and Li calculation. The lowest loss distribution of this resonator is numerically propagated outside the cavity in Figure 5.17. In a distance equal to four times the cavity length, a strong spot of Arago develops in the centre of the beam. The development of the spot of Arago, with a peak intensity many times that of the other portions of the beam, is a disadvantage of hard-edged unstable resonators.

A short description of the numerical techniques used in the calculations is appropriate. A direct application of Fourier beam propagation methods to the unstable resonator would require prohibitively large arrays of sampling points to handle the diverging portions of beam propagation in the cavity. This problem is avoided with a simple transformation that reduces the problem of calculating the propagation of the collimated beam. The technique is illustrated with propagation in a Galilean telescope equivalent to reflections from the convex mirror followed by reflection from the concave mirror in our resonator. The beam, just before the convex mirror, is first magnified by setting

\[
E_{mag}(x, y) = \frac{1}{M}E_{in}(x/M, y/M) \tag{5.72}
\]

where \( E_{in} \) is the incident electric field, \( E_{mag} \) is the magnified field, \( M = -R_s/R_a \) is the magnification and the factor \( 1/M \) is needed for conservation of energy. Next, the field is propagated over an effective distance of the magnification times the mirror separation:

\[
d_{eff} = M \times d. \tag{5.73}
\]
Finally, it is necessary to retrieve the spacing of the original array sampling points by some method such as interpolation using first, second and cross-derivatives. Siegman [3] provides a justification of this transformation based on Fermat’s principle. The technique is more general than that described here and can be applied to an open optical system described by an ABCD matrix.

The value of the calculated beam quality parameter $M^2$ is dependent on the sharpness of the edge of the aperture and on the number of sampling points used in the calculation. For example, the distributions shown in Figures 5.16 and 5.17 were obtained using a rather sharp edge of intensity reflectivity on the output mirror given by $R_M = \exp\left\{-\left(r^2/a^2\right)^{64}\right\}$, and the value $M^2 = 7.2$ was obtained for the $N_{eq} = 5.5$ resonator. When the edge is softened to $R = \exp\left\{-\left(r^2/a^2\right)^{32}\right\}$, $M^2 = 5.6$ results. In the limit of an infinitely sharp edge, the paraxial approximation will fail. The number of sampling points also limits the resolution of the sharpness of the edge. The improvement of beam quality with softening of the reflector or aperture edge leads to VRMs.

### 5.7.2 Soft-edged Apertures

An example of a VRM with an intensity reflectivity of $R = 0.34 \exp(-r^2/a^2)$ is described here. The magnification of the resonator is reduced to 1.75 to yield a loss per transit of 0.89, the same as the resonator with the sharp-edged reflector discussed earlier. The reflectivity profiles of the hard-edged reflectors and the variable reflector used in these examples are shown in Figure 5.18. A positive-branch confocal resonator with an equivalent Fresnel number of $N_{eq} = 2.3$ is used with the VRM and a magnification of $M = 1.75$. There is nothing unique about these values. They were chosen for comparison because the cavity transit losses were the same as the hard-edged aperture example and the output distribution was reasonable. Again, the phases of the Fourier components are initially random and the initial amplitudes are equal; the total power is renormalized after each cavity transit. After only eight resonator transits, the calculated intensity distribution is near the final form with a beam quality of $M^2 = 1.4$ (Figure 5.19). Good beam quality can develop in relatively few cavity transits in such a resonator.

These examples are for ideal conditions. An actual laser resonator would have a gain saturation that would make the distribution more ‘top-hat’ like and less like a Gaussian distribution. Laser-pumped laser resonators can also produce good beam quality. Some laser-pumped lasers have a transverse gain distribution that is nearly Gaussian.

### 5.8 Distortion Effects

There are a number of practical problems in actual lasers that lead to beam distortion. Heat deposition in the laser gain medium resulting from pumping can cause thermal lensing
and thermally induced stress birefringence. Master oscillator power amplifier techniques may be useful. A high-quality beam is generated in a low-power oscillator where beam quality is more easily controlled. Power amplification is then obtained with a single or double pass through a high-power laser amplifier. This avoids multiple passes in a high-power resonator where greater beam distortion could accumulate. A technique that is finding wide use in these systems is phase-conjugate reflection. After the first pass through a laser amplifier, the beam acquires sufficient intensity that relatively efficient phase-conjugate reflection in a Brillouin cell is possible. The reflection of the conjugate mirror traverses the path through the laser amplifier in reverse, cancelling the distortion acquired on the first transit. These topics are beyond the scope of this discussion. Next, we turn to an overview of axial modes and temporal properties of laser resonators.

5.9 Axial Modes

5.9.1 Stable-resonator Axial-mode Spectral Separation

Up to this point, we have only discussed the phase difference between transverse resonator modes and plane waves propagating in the same direction. For a discussion of axial modes, it is necessary to add the additional constraint that the resonator mode must reproduce itself both in amplitude and phase except for uniform amplification or attenuation. This means the wavelength of a resonator mode must satisfy the condition that the optical length of a round-trip cavity transit is an integer multiple of that wavelength with adjustment for Gouy and other possible phase shifts.

Often it is the case that the optical length of a resonator is not known precisely on the scale of a fraction of a wavelength of light. This is commonly the case for resonators used as interferometers to determine the relative spectral distribution of an optical beam. It may be sufficient to consider only the spectral spacing of modes. For example, the free spectral range of fundamental modes of an open, stable resonator is given to a high degree of accuracy by

$$\Delta \lambda_{FSR} = \Delta \lambda_{FSR}/\lambda^2 = 1/\lambda_{n+1} - 1/\lambda_n = 1/(2d). \quad (5.74)$$

Here, $\lambda_{n+1}$ and $\lambda_n$ are the wavelengths of adjacent modes. The free spectral range is given as a wavenumber separation $\Delta \nu_{FSR}$ and as a wavelength separation $\Delta \lambda_{FSR}$ in equation (5.74). The mirror separation of the resonator is $d$, and $\lambda$ is the central or average wavelength. A symmetric confocal interferometer is commonly used in the spectral analysis of laser beams. In typical use, there are four reflections before the beam path inside the resonator is closed (Figure 5.20), and the free spectral range is $\Delta \nu_{FSR} = 1/(4d)$. Usually, the accuracy of these expressions is limited by the precision to which the mirror separation is known.

The optical length is the integral of the refractive index over the round-trip transit path followed by the centre of the transverse field distribution:

$$\text{optical length} = \int_{\text{round trip}} n(\lambda, z) \, dz. \quad (5.75)$$

The condition that the phase of a resonator mode change by an integer multiple on a cavity round trip is

$$\frac{\text{optical length}}{\lambda} - \frac{\Phi}{2\pi} = \text{integer} \quad (5.76)$$
where $\Phi$ represents the sum of the Gouy phase shifts on the complete cavity transit. The axial or longitudinal cavity modes associated with a transverse mode will be nearly equally spaced in wavenumber, but the spacing will not be exactly equal because of the dispersion of the refractive index. Transverse modes of different order will also usually be spectrally positioned between the fundamental transverse modes. Secondary changes such as additional wavelength-dependent phase shifts in optical components and wavelength-dependent path differences must be considered in critical applications such as the propagation of femtosecond-duration pulses.

The spectral width of a resonator mode depends on several factors. Losses typically determine the spectral width in a resonator used passively as a multi-pass interferometer. It is useful to consider an ideal case of a two-mirror stable resonator with identical mirror reflectivity $R_m$ for lossless mirrors that have transmissions and reflections that sum to one: $T_m \cdot R_m = 1$. The transmission of a monochromatic Gaussian beam mode matched to the resonator is

$$\frac{P_t}{P_i} = \frac{1}{1 + 4 R_m \sin^2(\phi/2)/(1 - R_m)^2}. \hspace{1cm} (5.77)$$

Here, $P_i$ is the incident power in the beam, $P_t$ the transmitted power and $\phi$ is the phase shift encountered on one round-trip transit of the resonator. When the reflectivity is high, the shape of the interferometer transmission peaks can be obtained by using the approximation $\sin(\Delta \phi/2) = \Delta \phi/2$, where $\Delta \phi$ is the small difference between $\phi$ and a multiple of $2\pi$, and $\phi$ is nearly equal to a multiple of $2\pi$. In this approximation, the full width at half maximum of the resonance is given by

$$\phi_{\text{FWHM}} = 2(1 - R_m)/\sqrt{R_m}. \hspace{1cm} (5.78)$$

The finesse of the resonator is the ratio of the free spectral range divided by the resonance width:

$$\text{finesse} = 2\pi/\phi_{\text{FWHM}} = \pi \sqrt{R_m}/(1 - R_m). \hspace{1cm} (5.79)$$

For less than ideal cavity mirrors, the finesse can be stated as a function of total resonator loss

$$\text{finesse} = \pi/(1 - \text{loss}/2). \hspace{1cm} (5.80)$$

Such resonators, when illuminated by a single-frequency laser, can provide accurate measurements of low levels of loss in components inserted in the resonator.

High-gain pulsed lasers, however, typically have insufficient time to resolve single-mode operation. Effects such as spectral and spatial 'hole burning' can also limit frequency selection.

In mode-locked lasers, many modes are locked in phase to synthesize a single short pulse. The time–bandwidth limit specifies the minimum pulse width attainable for a given bandwidth or the minimum bandwidth required to support a given pulse duration. The minimum time–bandwidth product is obtained for pulses that have a Gaussian shape in time with no further amplitude or phase modulation. Frequently, the time–bandwidth product is given in terms of pulse width in seconds, $\Delta t_{\text{FWHM}}$, and spectral width, $\Delta \lambda_{\text{FWHM}}$, in hertz:

$$\left( \Delta t_{\text{FWHM}} \Delta \lambda_{\text{FWHM}} \right)_{\text{min}} = 0.44. \hspace{1cm} (5.81)$$

In terms of wavenumber, it is $\left( \Delta t_{\text{FWHM}} \Delta \lambda_{\text{FWHM}} / \lambda \right)_{\text{min}} = 14.72 \times 10^{-6} \text{ps nm}^{-1}$.

where $\lambda$ is the central wavelength. The time–bandwidth product can also be stated as an uncertainty principle $(\Delta t_{\text{FWHM}} \Delta E_\text{h})_{\text{min}} = \hbar/2$ given in terms of the variance.

The properties of the laser gain have an effect on frequency selection. When the gain is inhomogeneously broadened, gain in a narrow spectral region can be depleted without depleting the gain in neighbouring regions. An example is a gas discharge where gain broadening is due to Doppler shifting in a distribution of velocities. Multiple modes can oscillate, each drawing gain from a different velocity population. The familiar helium–neon laser exhibits this type of behaviour with typically two or three modes oscillating simultaneously. The depletion of gain in a narrow spectral region of an inhomogeneously broadened laser is called spectral hole burning. It is also possible to have spatial hole burning in a standing-wave laser resonator. The gain is not depleted at the nodes of the standing-wave laser oscillation and remains to provide gain for modes of a different frequency with different node locations. Spatial hole burning can occur with either inhomogeneous or homogeneous gain broadening. When the laser gain is inhomogeneously broadened, energy extraction in a narrow spectral region will uniformly decrease gain over the entire gain bandwidth. Gain broadening by lifetime-limiting collisions or thermal vibrations in solids are examples of homogeneous broadening.

Techniques for frequency selection include the addition of dispersive elements such as prisms and gratings in the resonator. Etalons within the resonator are frequently used for gain narrowing as are multiple-element birefringent filters. Individual elements of birefringent filters are wavelength-dependent high-order waveplates providing a retardation between orthogonal polarizations. These elements are placed between polarizers or Brewster-angle surfaces to provide favoured transmission for wavelengths that have integer orders of retardation. The order or retardation is adjusted by rotating the plate: using plates of different thickness can provide a single region of high transmission in a wider spectral range. Etalons can be as simple as an uncoated plane-parallel plate of a transmitting material or a precise assembly of two closely
spaced surfaces with multi-layer-dielectric reflective coatings to provide higher finesse. It is more difficult to control multiple etalons, and it is common to use only a single etalon. It is also common to use combinations of these techniques.

To obtain frequency-stable output from a laser, it is first necessary to make the laser as stable as possible before proceeding to active control techniques. Mechanical stability and the reduction of mechanical vibrations are a first essential step. Temperature stability must be considered next. At this point, it is necessary to consider the stability of the pump source. This is true even for a cw laser pumped by another cw laser. Active control of laser frequency requires an external reference and feedback control of the resonator. This is usually done with piezoelectric control of the cavity length. Frequency reference standards can be a stable external etalon for relative frequency stabilization or an atomic transition reference for absolute stabilization. Sub-Doppler spectroscopy techniques may be appropriate for the absolute frequency standard. Even greater absolute frequency accuracy is obtained by reference to cryogenically cooled atoms held in a trap or transiting through an atomic fountain.

Stable single-frequency operation is significantly easier to attain in low-power lasers. Injection locking is a technique for transferring the frequency characteristics of the low-power laser to a higher-power laser. To accomplish this, the resonances of the locking and locked laser must be actively controlled and locked together. One technique for accomplishing this is called Pound–Dreier locking. The seeding laser beam is modulated to produce fm sidebands along with the central frequency. The sidebands are outside the resonance of the higher-power laser and are reflected. The central-frequency portion of the beam is partially transmitted into the second laser and drives the oscillation of that laser. The phase of the combined reflected locking beam and transmitted oscillation of the second laser change slightly if the two resonances are not exactly matched in frequency. This produces an amplitude modulation in the combined beam with the fm sidebands. The phase and amplitude of this amplitude modulation provide the error signal used to control the cavity length of the second laser. Ring resonators are suited to this type of locking. A unidirectional oscillation is established in the second laser. The combined transmitted and reflected beams from the second laser are deflected away at an angle from the incident locking beam. This greatly helps in the isolation of the laser generating the locking radiation.

### 5.11 Temporal Resonator Characteristics

Mode-locking many cavity modes requires some modulation technique that couples the modes in phase. Acousto-optic modulators are used for active mode-locking of cw laser systems. An acoustic modulation driven at a frequency that matches the resonator mode spacing is established in a transparent material. This modulation produces sidebands on the modes, which couple to the adjacent modes, locking many modes in phase. Saturable absorbers are normally used with higher-peak-power pulsed mode-locked lasers. Saturable absorbers appropriate for mode locking must have short relaxation times much less than the round-trip cavity transit times. As the laser oscillation builds up with random fluctuations, the strongest fluctuation will begin to saturate the absorption. This fluctuation will build up more rapidly and come to dominate the laser oscillation. Gain will be depleted by the strong fluctuation reducing the amplitude of secondary fluctuations. The rapid saturation of the absorption on each cavity transit is a modulation that couples additional modes. The depth of modulation, gain, gain bandwidth and dispersion in the resonator determine the width on the mode-locked pulses. Kerr-lens-mode locking is an additional technique for producing very short mode-locked pulses. This technique uses the optical Kerr effect or change in refraction index that is produced by very high-intensity pulses.

In a Q-switched laser, oscillation is held off by introducing high losses in the resonator, while an inverted population is built up by some means of laser pumping. When the gain has reached a high level, the loss is abruptly removed. Laser oscillation develops in an intense short pulse that may be as short as a few cavity transits of the resonator. Electro-optic and acousto-optic Q-switches are frequently used to control cavity losses for the purpose of Q-switched operation. An acoustic wave reflects a beam by Bragg diffraction in the acousto-optic modulator. When the rf signal to a piezo-electric transducer is turned off, the resonator returns to the low-loss condition. Pockels cells, and waveplates and polarizers are used for electro-optic modulation. A double-pass through a quarter-wave-retardation waveplate rotates the polarization of the resonator beam, and the beam is deflected out of the resonator at the polarizer. When voltage is applied to the Pockels cell, a polarization retardation is produced that cancels the retardation of the waveplate, loss is minimized and a Q-switched laser oscillation rapidly develops.

### 5.12 Fibre Laser Resonators

Brief mention is made here of fibre laser resonators. In fibre lasers, the resonator beam is primarily a guided wave inside a fibre. Pump radiation, often generated by semiconductor diode lasers, is coupled into the cladding of the optical fibre. The pump radiation is absorbed over a relatively great length in a fibre core that is doped with the active laser ion. The fibre core is manufactured with a higher refractive index than that of the cladding, and the core is usually chosen to be of a size that will support only a single-fibre mode. Bragg reflectors can be established in the fibre by techniques such as processing with ultraviolet radiation. It is necessary to engineer the coupling of the pump radiation into the fibre carefully, predict the free-space propagation of the fibre laser radiation outside the fibre and to optimize the placement of discrete elements of the fibre laser resonator not incorporated in the fibre itself. Gaussian optics is useful in each of these areas. Fibre lasers have notable properties such as high efficiency and simplicity of operation. Optical properties are also remarkable, with high powers exceeding 100W having been demonstrated. Broad wavelength availability and high coherence are also being achieved.
5.13 Conclusion

The goal of this chapter has been to present an overview of laser resonators and the techniques of resonator analysis and design. Most of the detail of this discussion has focused on the fundamental aspects of Gaussian optics and stable resonators. Unstable resonators were discussed with illustrative examples. Other topics were briefly mentioned. It is hoped that the information presented is sufficient to address basic issues in resonator design and the management of laser beams. There has been a substantial amount of work performed on the topics of optical resonators since the first demonstrations of lasers in the early 1960s, and the investigation on optical resonators and beam propagation still continues. Several of the references have more extensive presentations and detailed bibliographies.

REFERENCES