On the Transition to Phase Synchronized Chaos

3.1 Introduction

Chaotic phase synchronization [3,11,22] denotes an interesting form of synchronization in which a chaotic attractor adjusts the frequencies of its internal dynamics to the rhythm of an external forcing signal, or to the dynamics of another chaotic oscillator, while the oscillator's amplitude continues to vary in an essentially uncorrelated manner. This form of chaotic synchronization is clearly distinguished from the complete or full synchronization one can observe in the interaction between two identical chaotic oscillators [30,34].

Complete synchronization has attracted significant attention both because of its potential application in areas such as chaos control [42,46] and secure communication [4,44] and because of the broad range of interesting nonlinear dynamic phenomena that can be observed as the synchronized state loses its stability [27]. Besides the so-called locally and globally riddled basins of attraction [2,21], these phenomena also include on–off intermittency [36], attractor bubbling, and blow-out bifurcations [29]. All these phenomena are derived from a situation in which, although the trajectories on the average are attracted to the synchronized state, any neighborhood of this state may contain a dense set of trajectories that are repelled from it. Under these conditions, the stability criterion takes a rather unusual form: all periodic orbits embedded in the synchronized chaotic state must be transversely stable [13]. The application of this condition for a system of two coupled identical Rössler oscillators has been demonstrated, for instance, by Yanchuck et al. [48].

In recent years, the interest in application of chaos synchronization for secure communication has led many investigators to consider different forms of the so-called projective synchronization by which the
requirement of complete identity of the interacting oscillators can be reformulated to allow for different forms of scaling [25]. At the same time, a significant effort to study the synchronization properties of different fractional order chaotic oscillators has been initiated [31,47].

Phase synchronized chaos, as described in this chapter, is characteristic for interacting nonidentical chaotic oscillators. This form of synchronization occurs in a wide variety of different physical, chemical, and biological systems, including coupled electronic oscillators [3], arrays of interacting chaotic lasers [9], plasma discharge tubes paced by a low-amplitude wave generator [40], coupled electrochemical reactors [50], sub-threshold behaviors of interacting nerve cells [26], and oscillatory dynamics in the blood flow regulation of neighboring functional units of the kidney [14].

In the animate world, the complex coordination of oscillatory modes associated with chaotic phase synchronization is likely to translate into a structure that can adapt to a wide range of functional states and rapidly switch from unsynchronized to almost coherent dynamics. It is well-established, for instance, that the activation of the smooth muscle cells in the arteriolar wall involves a transition from a state of seemingly random fluctuations of intracellular calcium concentrations into a state of nearly coherent cellular oscillations [12]. Detailed bifurcation analyses of chaotic phase synchronization in multidimensional biological systems have been performed, for instances, for chains of microbiological reactors [27], for interacting functional units of the kidney [23], and for scale-free networks of bursting neurons [7].

In several experiments with a single periodically driven chaotic oscillator, one can observe [6,27,33] how the frequency spectrum remains relatively unaffected by the variation of the forcing frequency until the system enters the region of chaotic phase synchronization where the mean spectral frequency locks to the forcing frequency. The width of the synchronization interval typically increases with the forcing amplitude and, as the system leaves this interval, the mean frequency of the chaotic oscillator again starts to slide away from the forcing frequency. This type of behavior is associated with the fact that, while the amplitude of the endogenous oscillations is controlled by a balance between the inherent instability of the system on one side and nonlinear restraining mechanisms on the other, there is no preferred value for the phase of an unforced oscillator. The application of a forcing signal breaks this time translational symmetry and, even for relatively small forcing amplitudes, the chaotic oscillator will adjust its frequencies and phases [33].

From its first observation, the phenomenon of chaotic phase synchronization has attracted significant theoretical interest [35,41,42], and the concepts and the methods developed through works in this area have been used to interpret experimental time series from many different sources [15]. Apart from the above-mentioned changes in the spectral distribution, the transition between phase synchronized and nonsynchronized chaos is reflected in the specific variation of the Lyapunov exponents, in characteristic changes of the spectral distribution of the oscillations, and through changes in the size and form of the Poincaré section associated with the attracting state [45].

For spiral-type chaos, it is known that the edge of the synchronization zone consists of a dense set of saddle–node bifurcations that delineate the range of existence for the different unstable periodic cycles, which constitute the chaotic state. Drawing on the analogy with the tangent bifurcation along the edge of the synchronization zone for a forced limit cycle oscillator, Pikovsky et al. [32] suggested that a large number of attractor–repellor collisions take place at the transition to chaotic phase synchronization. For systems of coupled period-doubling oscillators, Postnov et al. [37] have described a nested structure of phase synchronized regions for different attractor families associated with the transition to phase synchronization, and Vadivasova et al. [45] have provided a preliminary overview of the regions of existence for the main behavioral modes of the harmonically forced Rössler system. Moreover, by applying a lift to the phase variable (such that phase points separated by $2\pi$ no longer are considered the same), Rosa et al. [39] have described the transition to chaotic phase synchronization as a boundary crises mediated by an unstable–unstable pair bifurcation on a branched manifold. So far, however, the details of how the saddle–node bifurcations arise, how they are organized, and how the resonance tori that exist in the synchronization regime relate to the period-doubled ergodic tori that are known to exist outside the resonance zone [5,43], have remained largely unexplored.
An important contribution toward the development of a more detailed picture is the theory of cyclic (also called C-type) criticality developed by Kuznetsov et al. [16–18]. In particular, this work has demonstrated that the period-doubling transition to chaos along the edge of a resonance tongue displays an unusual scaling behavior, involving subsequent pairs of period-doubling bifurcations. Inspired by these results, we have recently applied continuation techniques [10,19] to perform a more complete analysis of the bifurcation structure involved [24,51]. In this way, we have demonstrated how the dynamics of the periodically forced Rössler system develop through cascades of period-doubling bifurcations of both the node and saddle cycles in a direction transverse to the original period-1 resonance torus, thereby producing a system of the so-called multilayered resonance tori [52–54].

After each period-doubling bifurcation, new saddle–node bifurcation curves are born on both sides of the synchronized zone in order to delineate the range of existence for the emerging resonance cycles. For a particular side of the synchronized zone, these saddle–node bifurcation curves emanate alternately from the stable and the unstable branches of the period-doubling curve and thereby causing the characteristic cyclic structure of the transition where the scaling relates to pairs of subsequent period-doubling transitions. Moreover, the new saddle–node bifurcation curves arise close to, but not in the so-called fold-flip bifurcation points [20] where the period-doubling curves are tangent to the saddle–node bifurcation curves produced in the preceding period-doubling bifurcation. Additional (local and/or global) bifurcations are, therefore, required to close the holes between the saddle–node bifurcation curves [24].

By following the development of the phase portrait for the various stable and unstable resonance cycles in a periodically forced Rössler system through the period-doubling cascade the present chapter illustrates the formation, reconstruction, and final breakdown of increasingly complicated multilayered resonance tori. This chapter also explains how the saddle–node bifurcation curves along the edge of the resonance zone are organized and illustrates the transition from multi-layered resonance torus to period-doubled ergodic torus.

### 3.2 Main Bifurcation Structure of the 1:1 Synchronization Zone

Let us consider the periodically forced Rössler system

\[
\dot{x} = -y - z + A \sin(\omega t); \quad \dot{y} = x + ay; \quad \dot{z} = b + z(x - c) \tag{3.1}
\]

that has also served as a vehicle for the investigation of chaotic phase synchronization in many previous studies [35,41,42]. Here, \(x, y,\) and \(z\) are the dynamic variables of the unforced oscillator, and \(A \sin(\omega t)\) represents the external forcing. The parameters \(a\) and \(b\) and the forcing amplitude \(A\) will be kept constant at the values \(a = b = 0.2\) and \(A = 0.1\), whereas the nonlinearity parameter \(c\) and the forcing frequency \(\omega\) are used as bifurcation parameters. For the above parameter values, the unforced Rössler system undergoes a Hopf bifurcation at \(c = 0.4\) and, for increasing values of \(c\), the system exhibits a Feigenbaum cascade of period-doubling bifurcations. When an external periodic forcing is applied in the regime of simple periodic oscillations, the Rössler system displays regions of two-mode quasiperiodic (ergodic) dynamics interrupted by a dense set of resonance zones where the internally generated periodic oscillations synchronize with the external forcing. The 1:1 resonance generally produces the most prominent synchronization regime, and the purpose of the current analysis is to examine how the structure of this region develops as the parameter \(c\) increases through the range, where the unforced Rössler system undergoes its period-doubling transition to chaos.

To illustrate some of the characteristic features of the transition to phase synchronized chaos, Figure 3.1a shows a scan across the main region of 1:1 phase synchronization [6,27,33]. Here, we have plotted the ratio between the mean spectral frequency \(\omega_1\) of the forced Rössler system in Equation 3.1 and the frequency \(\omega\) of the applied forcing as a function of the forcing frequency. The synchronization zone clearly stands out as an interval of forcing frequencies in which \(\omega_1\) coincides with \(\omega\). For forcing frequencies below this interval,
FIGURE 3.1 Main features of the transition to phase synchronized chaos. (a) Synchronization characteristic: Ratio of the mean spectral frequency $\omega_1$ to the forcing frequency $\omega$ as a function of the forcing frequency. Distribution of stroboscopically sampled phase space positions for the forced Rössler system (b) outside and (c) inside the range of chaotic phase synchronization. Projections of the forced Rössler oscillator are shown as background for the stroboscopic points in (b) and (c).

the mean spectral frequency exceeds the forcing frequency and, when the forcing frequency crosses the upper edge of the synchronization interval, $\omega_1$ no longer follows the variation of $\omega$.

Figure 3.1b and c provide an alternative illustration of the transition from unsynchronized to phase synchronized chaos in response to a shift in the forcing frequency. Here we have superimposed a projection of the phase space trajectory for the periodically forced Rössler system with a distribution of points obtained by stroboscopic marking of the position of the system after each completed period of the forcing signal. Figure 3.1b shows the typical point distribution observed outside the region of phase synchronization, and Figure 3.1c shows an example of the characteristic point distribution in the range of phase synchronized chaos. Each distribution involves the order of 1000 points. Note, how the point distribution for the unsynchronized state spreads nearly uniformly across the whole attractor, while the spread of the points is significantly smaller in the phase synchronized state. It is also interesting to note that, even for the unsynchronized attractor, the density of stroboscopic points is very low in a particular region. This is likely to be the region where the oscillator rapidly approaches its equilibrium point along the stable manifold (i.e., the $z$-axis).

To better understand what happens in the transition to phase synchronized chaos let us start by examining the transitions that occur for lower values of the nonlinearity parameter $c$ as the unforced Rössler system takes its first steps of the period-doubling cascade. The two-dimensional bifurcation diagram in Figure 3.2 provides an overview of the first four period-doubling bifurcations in the 1:1 resonance tongue [24].
As mentioned above, the bifurcation parameters are the forcing frequency $\omega$ and the nonlinearity parameter $c$ for the Rössler oscillator. $PD^S$ and $PD^U$ denote period-doubling bifurcation curves for stable (node) and unstable (saddle) resonance cycles, respectively, and $SN$ refers to the saddle–node bifurcation curves observed along the sides of the resonance zone. $TD$ locates torus-doubling bifurcations for the ergodic tori that exist outside the resonance zone, and the arrows $A \rightarrow A$ and $B \rightarrow B$ relate to one-dimensional bifurcations can be discussed in the following.

Below the first period-doubling bifurcation curve $PD^S_1$, the resonance zone is delineated to the left and right by the saddle–node bifurcation curves $SN^L_1$ and $SN^R_1$, respectively. In this region, the forced Rössler system displays a stable, synchronized period-1 cycle $N_1$ and a corresponding saddle cycle $S_1$, born together in the saddle–node bifurcation $SN^L_1$ (or $SN^R_1$) and both situated on a closed invariant curve that represents the resonance torus. Along the lower branch $PD^S_1$ of the first period-doubling curve, the stable period-1 cycle undergoes its first period-doubling bifurcation while the corresponding saddle cycle period doubles at $PD^U_1$. At the edge of the resonance zone, the two period-1 cycles merge, and the period-doubling bifurcations occur simultaneously. Above the curve $PD^U_1$ in Figure 3.2, the forced Rössler system displays a pair of saddle and doubly unstable saddle period-1 cycles together with a pair of saddle and stable node period-2 cycles.

The saddle–node bifurcation curves $SN^L_1$ and $SN^R_1$ continue up along the edge of the resonance zone to delineate the region of resonant period-1 dynamics. However, the synchronization zone for the new period-2 cycles is not the same as that for the period-1 cycles, and a new set of saddle–node bifurcation curves $SN^L_2$ and $SN^R_2$ are required to delineate the region of period-2 dynamics. These new saddle–node bifurcation curves originate from points on the period-doubling curve $PD_1$ ($PD^L_1$ or $PD^U_1$) near the so-called fold-flip bifurcation points [20] where the period-doubling curve is tangent to the saddle–node bifurcation curves $SN^L_1$ and $SN^R_1$. However, as closer inspection shows [24], the new saddle–node bifurcation curves do not emanate precisely from the fold-flip bifurcation points. There is a gap between the saddle–node bifurcation curves and, as we shall see, a number of additional local and global bifurcations are required to complete the border of the resonance zone for the period-2 cycles.
The stable period-2 solution undergoes a new period doubling at $PD^S_2$, and the saddle period-2 solution period doubles at $PD^U_2$. The saddle–node bifurcation curves $SN^L_2$ and $SN^R_2$ are tangent to these period-doubling curves, and close to the points of tangency a new pair of saddle–node bifurcation curves $SN^L_4$ and $SN^R_4$ are born to delineate the range of existence for the period-4 resonance dynamics (not visible in the scale of the figure). As the value of $c$ continues to increase, the same process repeats over and over again until the system undergoes a transition to phase synchronized chaos. Inspection of Figure 3.2 shows that, at least in the middle of the resonance zone, this transition occurs before the saddle solution has undergone its second period-doubling bifurcation. Inspection of Figure 3.2 also suggests that the torus-doubling bifurcations $TDL^2$ and $TDR^2$ that take place in the quasiperiodic regime outside the resonance tongue are coupled with the period-doubling bifurcations of the resonance cycles in the tongue. As an illustration to the two-dimensional bifurcation diagram in Figure 3.2, the one-dimensional bifurcation diagram in Figure 3.3 shows the transitions that take place along the direction A in Figure 3.2. Full curves represent stable periodic cycles, dashed curves the saddle cycles, and dotted curves the doubly unstable node solutions. Notice how the stable 1:1 solution that exists in the upper right corner of the figure undergoes a period-doubling at $PD^L_1$ while the corresponding 1:1 saddle solution suffers its first period doubling at $PD^U_1$. From here we can follow the two solutions (now as a saddle cycle and a doubly unstable node) to the saddle–node bifurcation $SN^L_1$ to the left in the figure. This saddle–node bifurcation defines the zone edge for the period-1 cycles. The saddle and stable node 2:2 resonance cycles merge at $SN^L_2$ to give birth to a period-doubled ergodic torus. Finally, when crossing the torus-doubling bifurcation curve $TD^L_2$, the ergodic torus undergoes a new period-doubling transition.

Figure 3.4 shows a similar one-dimensional bifurcation diagram for the direction B in Figure 3.1. After the first period-doubling bifurcations for the 1:1 resonance node and saddle cycles at $PD^S_1$ and $PD^U_1$, the interconnected period-doubling processes continue to the left in the figure. Here we can locate the period-doubling bifurcation $PD^S_2$ for the stable period-2 cycle and the bifurcation $PD^U_2$ for the corresponding saddle cycle. Each pair of saddle and doubly unstable node cycles born in these bifurcations can subsequently be followed to the saddle–node bifurcation that demarcates their synchronization zone.

In Figure 3.4, the boundary of the resonance zone consists of saddle–node bifurcations for the period-1, period-2, and period-4 cycles, but only the period-4 node is stable, and both the 1:1 and the 2:2 resonance tori have been destroyed. Hence, we observe that the period-4 resonance torus ends in a saddle–node bifurcation in which an ergodic period-4 torus is born. The period-1 saddle and doubly unstable node cycles continue to exist into the region of the stable period-4 ergodic torus. As the system moves further away from the resonance zone, the ergodic torus starts to fold and it finally undergoes torus destruction at the point $ETD$ where its different layers begin to mix.
On the Transition to Phase Synchronized Chaos

FIGURE 3.4 One-dimensional scan along the direction B in Figure 3.2. As the forcing frequency $\omega$ is reduced, we can follow two subsequent period-doubling bifurcations of both the synchronized period-1 node and saddle cycles. The resonant period-4 torus ends in a saddle–node bifurcation that gives birth to a period-4 ergodic torus. The torus-doubling process only occurs in a restricted region on the sides of the resonance tongue. At the point ETD, the ergodic torus is destroyed, and to the left of ETD the system displays nonsynchronous chaos.

3.3 Bifurcation Structure Near a Fold-Flip Bifurcation

To understand the above transitions in more detail we need to focus on the bifurcation structure close to the fold-flip bifurcation points where the saddle–node bifurcation is tangent to the corresponding period-doubling bifurcation curve. As explained above, the new saddle-node bifurcation curves do not emerge from the fold-flip bifurcation points, but from points on the period-doubling curve close to these points. The distance to the fold-flip bifurcation points decreases with increase in forcing amplitude, but never vanishes. The generic situation is that the new saddle–node bifurcation curves emerge from the stable branch of the period-doubling curve in one side of the resonance zone and from the unstable branch in the other side. Moreover, for a given side, the emergence of the new saddle–node bifurcation curve alternately takes place from the stable and the unstable branch of the period-doubling curves.

Figure 3.5a shows a slightly redrawn version of Figure 3.2 in which the regions of existence for the stable 1:1, 2:2, 4:4, etc., resonance cycles are displayed in different shades and the region of quasiperiodic (or nonsynchronized chaotic) dynamics outside the resonance zone is dark gray. Figure 3.5b and c shows magnifications of the regions around the two fold-flip bifurcation points for the period-1 cycles. Figure 3.5b considers the situation at the left-hand side of the resonance tongue where the new saddle–node bifurcation curve $SN^L_2$ emerges from the point $Q_2$ on the unstable branch $PDU^L_1$ of the period-doubling curve, and Figure 3.5c presents the situation near the right-hand side of the tongue where $SN^R_2$ emerges from a point on the stable branch $PDS^R_1$ of the period-doubling curve.

To close the gap between the saddle–node bifurcation curves $SN^L_1$ and $SN^L_2$ (and avoid that the period-2 cycle produced at $PD^S_2$ can escape from the resonance zone) the system makes use of a subcritical torus-birth (or Andronov–Hopf) bifurcation $T_2$ in conjunction with a complex set of global bifurcations $G_2$ through which the unstable two-branched torus generated in $T_2$ is transformed into a stable period-doubled ergodic torus [24]. The alternative case where the saddle–node bifurcation curve emerges from the stable branch of $PD_1$ does not leave a gap through which the 2:2 resonance cycles can escape. However, the section of $PD^S_1$ that connects the point at which $SN^R_2$ is born with the point of tangency with $SN^R_1$ must be subcritical. This allows saddle–node bifurcations to take place along that part of $SN^R_2$ that falls below its point of intersection with $SN^R_1$. We also note that $SN^R_2$ is born to the left of the already existing tongue edge. Hence, $SN^R_2$ must intersect $SN^R_1$ and thereafter, at least in beginning, proceed to the right of this saddle–node bifurcation curve.

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FIGURE 3.5 Bifurcation structure for the 1:1 resonance tongue in the periodically forced Rössler system. (a) Overview of the first four period-doubling bifurcations. Only the first couple of saddle–node bifurcations along the zone edge can be resolved in the applied scale. (b) Enlargement of the region around PD₁ and SNL₁. The gap between the saddle–node bifurcations SNL₁ and SNL₂ is closed by the torus-birth bifurcation T₂ and the global bifurcations that take place near G₂. (c) Enlargement of the corresponding region to the right in the resonance zone. Here the new saddle–node bifurcation curve emerges from the stable branch of the period-doubling curve. Part of this branch (marked by small black squares) then has to be subcritical.

This completes our discussion of the detailed bifurcation structure near the fold–flip bifurcation point. Similar structures arise near the fold–flip bifurcation points of the subsequent period-doubling bifurcations and, as we have already indicated, the two different scenarios alternate in opposite phase along the sides of the resonance zone. This is the basis for the characteristic cyclic character of the period-doubling transition along the edge of a resonance tongue [16–18]. At the same time we can conclude that the above systematic leads the new saddle–node bifurcation curves to proceed alternately inside and outside the resonance zone delineated by the immediately preceding saddle–node bifurcation curves.

3.4 Period Doubling of the Ergodic Torus

Let us consider the above transitions once more, but now with a direct focus on the processes that lead to the birth and transformation of the large period-2 ergodic torus that arise in the torus-birth bifurcation T₂. In connection with Figure 3.5b we have already discussed how the saddle–node bifurcation curve SNL₂ emanates from the point Q₂ on the unstable branch of the period-doubling curve PD₁. This point does
not coincide with the point at which $PD_1$ is tangent to the edge of the synchronization region for period-1 dynamics, thus leaving a gap in the boundary of the resonance zone. However, as described in Section 3.3, additional local and global bifurcations are in place to ensure that the 2:2 cycles do not escape the resonance area through the gap left by the saddle–node bifurcation curves. Inspection of Figure 3.6 (which is a slightly redrawn version of Figure 3.5b) immediately allows us to locate the saddle–node bifurcation curves $SNL_1$ and $SNL_2$, the two branches of the period-doubling curve $PD_1$, and the point $Q_2$ where $SNL_2$ is born. The torus bifurcation curve $T_3$ is seen to bridge the gap between the two saddle–node bifurcation curves. At this torus bifurcation, the stable period-2 cycle produced at the lower branch of the period-doubling curve $PD_1$, and now transformed into a stable focus, loses its stability in a subcritical torus-birth bifurcation. As previously noted, the curve $G_2$ represents a set of closely situated local and global bifurcations that serve to stabilize the ergodic period-2 torus and thereby establish the required boundary for the synchronization zone. Finally, the torus-doubling curve $TDL_1$ represents the processes by which the period-2 ergodic torus transforms into the ergodic period-1 torus that exists outside the resonance tongue for values of the nonlinearity parameter $c$ below $TDL_1$.

In the presence of these additional bifurcations, the brute force bifurcation diagram calculated along the elliptic curve denoted $Ca$ in Figure 3.6 takes the form illustrated in Figure 3.7. At both ends of this diagram we observe the ergodic period-1 torus that exists below $TDL_1$ and to the left of the resonance zone. As we follow the transitions from left to right through the intermediate range of Figure 3.7 we first meet the saddle–node bifurcation $SNL_1$ at the edge of the resonance tongue where the 1:1 node cycle is born. At $PD_1$, this is followed by the period-doubling bifurcation in which the 1:1 node is transformed into a 1:1 saddle cycle while producing a stable 2:2 cycle transverse to the torus surface. Hereafter follows the sequence $G_2$ of closely situated local and global bifurcations that give birth to both the large period-2 torus that dominates most of the right-hand side of the diagram and to an unstable two-branch torus around the 2:2 focus cycle. The unstable two-branch torus again disappears in the subcritical torus-birth bifurcation $T_2$ while the large period-2 torus continues to exist until it undergoes the aforementioned reverse torus-doubling bifurcation at the point $TDL_1$.

The sketches in Figure 3.8 give a clearer account of the structure of local and global bifurcations observed in the region around the birth of the saddle–node bifurcation curve $SNL_2$. The curves along which these bifurcation diagrams are thought to be drawn are indicated in Figure 3.6 by the letters $Ca$, $Cb$, and $Cc$.

![FIGURE 3.6 Magnification of part of the bifurcation diagram near the fold-flip bifurcation point. $PDU_1$ and $PDU_2$ are the stable and unstable branches of the period-doubling bifurcation curve and $Q_2$ denotes the point in which the saddle–node bifurcation curve $SNL_2$ starts. $G_2$ represents the sequence of bifurcations that give rise to the large period-2 ergodic torus seen in Figure 3.7, and $TDL_1$ is the torus-doubling bifurcation in which this torus is transformed into an ordinary period-1 ergodic torus. Note that some of the substructure disappears when the forcing amplitude becomes sufficiently large.](image)
FIGURE 3.7 Brute force bifurcation diagram calculated along the elliptic curve $C_a$ in Figure 3.6. $\theta$ is a measure of the position along $C_a$ with $0$ (and $2\pi$) representing the outmost right point and $\pi$ the outmost left point. For increasing values of $\theta$ the diagram first shows the saddle–node bifurcation $SN^L_1$ that occurs when the system enters the 1:1 resonance zone. Hereafter follows first the period-doubling $PDS_1$ that produces the stable 2:2 resonant node and the subcritical torus-birth bifurcation $T^2_2$ in which the 2:2 cycle loses its stability. $G_2$ represents a sequence of bifurcations that give birth to the large ergodic period-2 torus and $TD^L_1$ denotes the torus-doubling bifurcation in which this torus transforms into an ordinary period-1 ergodic torus.

Starting from the bottom, panel $C_a$ first shows the saddle–node bifurcation through which the 1:1 node and saddle cycles are born as the system enters the resonance domain. Hereafter follows the period-doubling bifurcation $PDS_1$ on the 1:1 node. At $T_2$, the 2:2 cycle generated in this bifurcation (now a stable focus) undergoes a subcritical torus-birth bifurcation and turns into an unstable focus. In accordance with the brute force diagram in Figure 3.7, the unstable two-branch torus produced in this bifurcation disappears in a global bifurcation close to the point $G_2$. As we continue the scan, the unstable 2:2 focus cycle (now a doubly unstable saddle) undergoes a reverse period-doubling bifurcation in $PDU^L_1$ while the 1:1 saddle destabilizes into a doubly unstable cycle. This latter cycle finally disappears in a saddle–node bifurcation.

FIGURE 3.8 Bifurcation diagrams drawn along parts of the curves $C_a$, $C_b$, and $C_c$ in Figure 3.6 and extended in both ends to the saddle–node bifurcation curve $SN^L_1$. Note how the 2:2 cycles in all cases are captured before they can escape from the resonance zone. Full lines represent stable node or focus solutions, dashed lines saddle solutions, and dotted lines doubly unstable node or unstable focus solutions. A more detailed description of the transformations that take place at $G_2$ may be found in our recent work [28].
at the tongue edge $SN_2^R$. We shall refer to the sequence of bifurcations that occur at $G_2$ as a torus-fold transition. A more detailed description of the processes associated with this transition has been presented in previous work [28].

If we denote the period-doubling bifurcation $PD_{1}^U$ that ends the life of the 2:2 cycle in the upper end of panel $Ca$ as subcritical, the corresponding period-doubling bifurcation in panel $Cb$ has become supercritical as the saddle–node bifurcation curve $SN_2^U$ has transformed the doubly unstable 2:2 saddle into a 2:2 saddle cycle with a single unstable direction. In this way the torus-birth bifurcation $T_2$ serves to degrade the stability of the 2:2 cycle so that it can annihilate with the 2:2 saddle cycle at the upper branch of the period-doubling curve. Finally, in panel $Cc$ the torus bifurcation on the 2:2 cycle no longer occurs, the saddle–node bifurcation $SN_2^U$ has overtaken the role of delineating the edge of the resonance tongue for the 2:2 cycles, and we recover the scenario discussed in Section 3.3.

In order to provide a clear impression the torus-doubling bifurcation that occurs along $TD_{1}^L$, Figure 3.9 presents a series of Poincaré sections of the ergodic torus observed in the one-dimensional brute force bifurcation diagram. As defined in the caption to Figure 3.7, the parameter $\theta$ is the angle along the curve $Ca$ in Figure 3.6. For $\theta = 3.8$, the Poincaré section shows an ordinary (i.e., period-1) ergodic torus. Note, however, that there is an uneven distribution of points along the periphery, indicating a "hesitation" of the system near the top of the section. This is the well-known indication that the system approaches a resonance zone, in this case it is the 1:1 resonance.

As $\theta$ is reduced, one can observe how the invariant curve starts to split into two different windings, as the quasiperiodic oscillator alternately chooses one route over the other. This is another indication of the fact that the resonance mode from which the torus originates has undergone a period-doubling transition in the direction transverse to the periphery of the closed invariant curve. As $\theta$ is further reduced, the separation between the two windings is seen to continue to grow, and for larger values of the parameter $c$, one can observe how the two windings of the period-2 torus move apart.

This completes our discussion of the main bifurcation structure near the fold-flip bifurcation point for the case where the new saddle–node bifurcation curve emerges from the unstable branch of the period-doubling curve. The alternative situation where the saddle–node curve emerges from the stable branch of $PD_{1}^U$, i.e., below the point of tangency, is found at the right-hand edge of the resonance zone. In this case the birth of $SN_2^R$ does not leave a gap between resonant and ergodic dynamics. This can easily be checked by trying to find a path from the 2:2 resonant region to ergodicity without crossing $SN_2^R$. Hence, no additional bifurcation is required. However, the period-doubling curve that connects the point where $SN_2^R$ is born to the point of tangency with $SN_4^R$ must be subcritical in order to account for that part of $SN_2^R$ that exists below $PD_{1}^U$. Moreover, $SN_2^R$ must cross $SN_4^R$ and at least at the beginning proceed along the right side of $SN_1^R$.

![FIGURE 3.9](image)

**FIGURE 3.9** Illustration of the transition that occurs as the forced Rössler system crosses the torus doubling bifurcation curve $TD_{1}^L$ in the direction indicated by the arrow $D$ in Figure 3.6. The figure presents a series of Poincaré sections of the ergodic torus for different positions $\theta$ along the curve $Ca$. Compare with the bifurcation diagram in Figure 3.7.
Similar bifurcation structures are observed at the points of tangency at the next period-doubling bifurcation, except that here the subcritical case is found at the left edge and the supercritical with the additional set of torus and global bifurcations at the right-hand edge.

### 3.5 Formation and Reconstruction of Multi-Layered Resonance Tori

After the above discussion of the bifurcation structure for a periodically forced period-doubling oscillator, the next problem is to describe the internal organization of the many different resonance cycles that arise as increasing values of the nonlinearity parameter $c$ takes the oscillator up through the period-doubling cascade. This leads us to study the formation and reconstruction of the so-called multilayered resonance tori [51] and to examine how these structures communicate with the ergodic tori that exist outside the resonance zone. In this discussion we shall again refer to the 2D bifurcation diagram in Figure 3.2.

Below the first period-doubling curve $PDS_1$, the forced Rössler system displays a stable synchronized period-1 cycle $N_1$ and a corresponding saddle cycle $S_1$, both situated on the closed invariant curve that represents the resonance torus. As shown in Figure 3.10a, this corresponds to the normal resonance structure for a forced limit-cycle oscillator. Along the curve $PDS_1$, the stable period-1 cycle undergoes its first period-doubling bifurcation. This gives rise to the formation of a stable period-2 resonance cycle $N_2$ while

![Phase portraits](https://example.com/phase_portraits)

**FIGURE 3.10** Phase portraits of the resonance structure at different values of the nonlinearity parameter along the scan line C in Figure 3.2. (a) Original 1:1 resonance torus with its stable node $N_1$ and saddle cycle $S_1$. (b) Phase portrait after the first period-doubling of the node cycle. (c) Phase portrait after the transverse period doubling of the original saddle cycle $S_1$. The system now displays a stable double-layered resonance torus consisting of the node $N_2$, the saddle $S_2$, and the unstable manifold of the saddle. (d) Phase portrait after the second period-doubling of the original resonance node cycle. $\omega = 1.08$. 

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the original resonance node is transformed into a period-1 saddle cycle $S_1^*$. As illustrated in Figure 3.10b, this period doubling takes place in a direction transverse to the resonance torus. The stable period-2 cycle is situated away from the resonance torus, and the unstable manifold of $S_1^*$ is transverse to this torus.

With further increase of the nonlinearity parameter $c$, the system crosses the curve $PD_1^U$. Here, the saddle solution $S_1$ undergoes a period-doubling bifurcation thus giving birth to the doubly unstable period-1 saddle cycle $S_0$ and to a new period-2 saddle $S_2$. As shown in Figure 3.10c, this transition again takes place in a direction transverse to the resonance torus. Note, however, how the unstable manifold of the new period-2 saddle cycle $S_2$ connects to the period-2 node $N_2$. The original period-1 resonance torus has now lost its transverse stability and a new, the so-called double-layered resonance torus [51,54], has been born. This torus consists of the period-2 node $N_2$ together with the corresponding saddle cycle $S_2$ and the unstable manifold of this cycle. With further increase of the nonlinearity parameter, the period-2 cycle $N_2$ undergoes a new period-doubling bifurcation in the direction transverse to the original resonance torus. As shown in Figure 3.10d, this bifurcation gives birth to the stable period-4 resonance cycle $N_4$ and to the period-2 saddle cycle $S_2^*$. 

Inspection of Figure 3.10d reveals an interesting structure of the unstable manifold from the period-1 saddle $S_1^*$ as it approaches the stable period-4 cycle $N_4$. In order to visualize this structure in more detail, Figure 3.11 shows a magnification of the region around $S_1^*$ and $N_4$. Note, how the unstable manifold from $S_1^*$ approaches the stable period-4 cycle $N_4$ in an oscillatory manner that appears to be controlled by the eigenvalues of the saddle cycle $S_2^*$. These eigenvalues are positive and numerically less than unity in the direction of the stable manifold and negative and numerically larger than unity in direction of the unstable manifold.

With further increase of the nonlinearity parameter $c$, the stable resonance node continues its transverse period-doubling process with the formation of period-8 and period-16 resonance nodes. This is illustrated in Figure 3.12a and $c$ for $c = 3.936$ and $c = 3.947$, respectively. Figure 3.12b demonstrates how the folded approach from the unstable period-1 resonance cycle $S_1^*$ continues to develop finer and finer details as the periodicity of the stable node cycle increases. With even further increase in $c$, the system crosses into a regime where the original resonance node $N_1$ has completed its transverse period-doubling cascade whereas the original saddle cycle $S_1$ has period doubled only once. As illustrated in Figure 3.13, the dynamics around the original node cycle is now chaotic while the resonance saddle cycle continues its transverse period-doubling cascade with the formation of saddle cycles of period-4 Figure 3.13a, period-8 Figure 3.13b and period-16 Figure 3.13c.

![FIGURE 3.11 Magnification of part of the phase portrait in Figure 3.10d around the saddle cycle $S_1^*$ and the stable period-4 node $N_4$. Note, how the unstable manifold from $S_1^*$ oscillates as it approaches $N_4$. A similar structure is observed in Figure 3.6 for higher levels of the period-doubling cascade.](image-url)
FIGURE 3.12  Phase portraits after the third (a) and the fourth (c) transverse period-doubling of the original period-1 resonance node. (b) Magnification of part of the phase portrait in Figure 3.6a between the saddle cycle \( S^*_1 \) and the stable period-8 node \( N_8 \). While developing finer and finer structures, the folded approach of the unstable manifold from \( S^*_1 \) to the stable node cycle continues to occur as the period-doubling process proceeds. \( \omega = 1.08 \).

The above description has presented some of the main structures that one can observe in a vertical scan of the 2D-bifurcation diagram in Figure 3.2. Let us now consider the phenomena that arise in a scan along the direction \( A \rightarrow A \) in Figure 3.2, i.e., along the line \( c = -51\omega + 58 \). Starting at a point between the period-doubling curves \( PDU_1 \) and \( PDS_2 \), this scan takes the system across the saddle–node bifurcation curves \( SNL_2 \) and \( SNL_1 \) and into the region of quasiperiodicity. A view on the bifurcation structure to the left of the resonance zone in Figure 3.2 shows that the scan leads the system into a region where the ergodic torus has undergone a first period-doubling bifurcation [5,43]. The idea of the scan is, therefore, to examine how the double-layered resonance torus in the synchronization zone transforms into the period-doubled ergodic torus that exists to the left of this zone.

Figures 3.14a is similar to Figure 3.10c and shows the organization of the resonance cycles after the first period-doubling transitions of the period-1 node and saddle cycles \( (N_1 \text{ and } S_1) \) transversely to the period-1 resonance torus. These transitions have given birth to the formation of a double-layered resonance torus, and the original period-1 resonance torus is now unstable. As shown in Figure 3.14b, breakdown of the period-1 torus involves a splitting of the unstable manifold of \( S_0 \) into two branches that are both attracted by the double-layered resonance cycle. This implies that a trajectory starting close to \( S_0 \) may approach the stable double-layered torus in an alternating manner. As the system hereafter crosses the saddle–node bifurcation curve \( SNL_2 \), the resonance cycles \( N_2 \) and \( S_2 \) merge and disappear leaving a period-doubled ergodic torus together with a pair of singly and doubly unstable saddle cycles \( S^*_1 \) and \( S_0 \) (Figure 3.14c). In this way, communication across the zone boundary from resonant period-2 dynamics to period-doubled quasiperiodicity takes the form of a transformation of a double-layered resonance torus into a period-doubled ergodic torus of similar shape and size. Finally, \( S^*_1 \) and \( S_0 \) also merge and disappear.
in the saddle–node bifurcation $SN_1^L$ (Figure 3.14d). However, this happens only after the stable dynamics of the system has turned ergodic.

### 3.6 Bifurcation Structure in the Period-3 Window

Let us complete our bifurcation analysis for the region of period-doubling bifurcations in the forced Rössler system by demonstrating that similar bifurcation phenomena take place in the periodic windows that exist in the chaotic regime for higher values of the nonlinearity parameter $c$. Figure 3.15 shows the main bifurcation structure of the period-3 window. Here, $SN_3^B$ denotes the saddle–node bifurcation in which the 3:3 resonant node and saddle cycles are born. This curve continues up along the two sides of the resonance tongue, now denoted $SN_3^T$ and $SN_3^R$, respectively. The closed curve $PD_3$ represents the first period-doubling bifurcation. The resonant 3:3 node cycle period doubles along the lower branch of this curve, and the 3:3 saddle cycle doubles its period along the upper branch. Similarly, the closed curve $PD_6$ represents the second period doubling with the node cycle period doubling its period at the lower branch and the saddle cycle at the upper branch. With its pronounced cusp structure, the bifurcation curve $SN_3^C$ represents a couple of saddle–node bifurcations in which the 3:3 saddle cycle generated in the period-doubling bifurcation $PD_3$ first loses and subsequently regains stability in a secondary direction. Finally, $T_3$ is a torus-birth bifurcation curve that serves a purpose similar to that of the torus bifurcation curve $T_2$ in Figure 3.5c. This curve closes a hole in the edge of the resonance tongue between $SN_3^L$ and the saddle–node bifurcation curve that delineates the sides of the resonance zone for the 6:6 cycles.
Stages in the transition from double-layered resonance torus to period-doubled ergodic torus in a scan along the direction $A \rightarrow A$ in Figure 3.2. (a) Double-layered resonance torus produced through period doubling of the original period-1 node and saddle cycles. The corresponding period-1 resonance torus is unstable. (b) Destruction of the period-1 resonance torus. (c) Disappearance of the period-2 resonance cycles $N_2$ and $S_2$ in the saddle–node bifurcation $SNL_2$. This marks the transition to ergodic dynamics. (d) Disappearance of the saddle cycles $S_0$ (doubly unstable) and $S_1^*$ in the saddle–node bifurcation curve $SNL_1$. This final bifurcation takes place in the quasiperiodic regime outside the resonance zone.

In close accordance with the bifurcation structure observed for the 1:1 resonance regime, examination of the bifurcation structure in the 3:3 region confirms that

1. The period-doubling cascades for the node and saddle solutions are interconnected. At the edge of the synchronization tongue, the two bifurcations are simultaneous, but away from the tongue edge the node solution (with the applied forcing type) bifurcates before the saddle solution.
2. Each pair of period-doubling bifurcations generates a new pair of saddle–node bifurcations that define the edges of the resonance zone at the next level in the cascade.
3. Additional torus and global bifurcations serve to close the gap between the new and the previous borders of the resonance zone.

### 3.7 Accumulating Set of Saddle–Node Bifurcations

Let us finally focus on the problems that relate to the accumulation of the saddle–node bifurcation curves along the edges of the resonance zone and to the continuation of these curves into the range of phase synchronized chaos. Except, perhaps, for the first two to three bifurcations that are model dependent,
FIGURE 3.15 Main bifurcation structure in the period-3 window. The lower saddle–node bifurcation curve marks the onset of the period-3 resonance solutions. For increasing values of the nonlinearity parameter $c$, the same saddle–node curve extends up along both tongue edges. $PD_3$ and $PD_6$ represent the first and second period doubling with the node solution bifurcating at the lower branch and the saddle solution at the upper branch.

scaling analyses performed by Kuznetsov et al. [16] have demonstrated that the alternation of the bifurcation processes between the two sides of the synchronization zone can be observed for many generations of period-doubling bifurcations, at least up to period-256. This represents the basis for their formulation of a special scaling theory for C-type criticality [16–18].

Based on the bifurcation analyses presented in Section 3.2, we have argued that, at least immediately after their formation, saddle–node bifurcation curves emerging from the stable branch of a period-doubling curve cross out the zone boundary as defined by the preceding saddle–node bifurcation curve along the same edge. On the other hand, saddle–node bifurcation curves that emerge from the unstable branch proceed inside the resonance zone defined by the former saddle–node bifurcation curve. Hence, we can represent the characteristic cyclic behavior associated with the generation of saddle–node bifurcation curves in the period-doubling cascade on the form outlined in Figure 3.16.

The sketch in Figure 3.17 addresses the same problem from a more global point of view. Use of continuation techniques [10,19] has allowed us to follow the saddle–node bifurcations that delineate the left-hand

FIGURE 3.16 Sketch to illustrate the alternation of two different scenarios for the birth of new saddle–node bifurcation curves along the edge of the resonance zone. Segments denoted $T_2$, $T_4$, and $T_8$ represent torus-birth bifurcations and segments denoted $PD_1$, $PD_2$, and $PD_4$ represent subcritical period-doubling bifurcations. Note, how new saddle–node bifurcation curves along a given side alternately move to the right and the left of the preceding saddle–node bifurcation curve.
FIGURE 3.17 Sketch illustrating the cascade of saddle–node bifurcations along the left-hand side of the 1:1 resonance zone for the periodically forced Rössler oscillator. While distorting the scales, the sketch preserves the observed systematic of the structure. The curves denoted 1, 2, 4, 8, etc., represent the saddle–node bifurcation curves $SN_1, SN_2, SN_4, SN_8$, etc. The gray zone represents the region of phase synchronized chaos, and the hatched gray zone is the region of nonsynchronous chaos.

side of the resonance zone through seven to eight generations of period-doubling bifurcations and thus to outline the structure of the accumulating set of saddle–node bifurcations. The curves denoted 1, 2, 4, 8, etc., represent the saddle–node bifurcation curves $SN_1, SN_2, SN_4, SN_8$, etc., and $PD_1, PD_2$ locate the first period-doubling curves. $SN_\infty$ denotes the numerically determined accumulation curve for the saddle–node bifurcation curves, $PD_\infty$ similarly represents the accumulation curve for the period-doubling cascade, and $ETD$ represents the curve of ergodic torus destruction. The resonance zone is presented with a white background, the region in which ergodic tori (of different periodicity) exist is hatched, the region of phase synchronized chaos (neglecting the occurrence of periodic windows) is gray, and the region of nonsynchronous chaos is hatched with a gray background. As before, the bifurcation parameters are the forcing frequency $\omega$ and the nonlinearity parameter $c$ of the Rössler oscillator.

Closer inspection of Figure 3.17 reveals that although the first few saddle–node bifurcation curves follow their own courses, a systematic structure soon materializes in which the saddle–node bifurcation curves alternate around a final accumulation curve $SN_\infty$. It is also interesting to note that the hatched region of ergodic torus dynamics, unaffected by several of the saddle–node bifurcation curves, consistently penetrates the structure until it finds the saddle–node bifurcation curve of the highest periodicity for the given value of $c$. This is also the saddle–node bifurcation that provides the boundary of existence for the stable node cycle. As mentioned above, this implies that the period-doubled ergodic torus that exists for a given value of $c$ communicates with a multi-layered resonance torus of the same periodicity, form, and size. To complete this description it is important to note that the well-organized structure of alternating saddle–node bifurcation curves tends to dissolve as the system progresses deeper into the chaotic regime. Hence, we cannot provide a description of the organization of the saddle–node bifurcation curves for the fully developed phase synchronized chaos.

3.8 Conclusions

Chaotic phase synchronization is an extremely interesting area of research with obvious applications to many different problems in physics, chemistry, biology, and other fields of science and technology.
However, although work on chaotic phase synchronization has already contributed many useful results [6, 27, 33], our understanding of this area is still far from complete. In particular, many questions related to the synchronization of multi-mode oscillations remain unsolved. Already the definition of the phase for a given chaotic oscillator can present a complicated problem, and in many situations it may not even be possible to find a satisfactory solution [8, 33]. This is the case, for instance, for the Lorenz attractor where the dynamics shifts, in an apparently random manner, between rotation around one or the other of the two symmetric equilibrium points [49].

Other chaotic systems display more or less random shifts between small loops and loops with much longer transition times, and it may not be possible, in a consistent manner, to decide which loops constitute a cycle [33]. This situation arises even for the Rössler system if the parameters are chosen in the regime of the so-called funnel attractor. However, problems of this type do not arise for the parameter values we have used in the present study. As illustrated in Figure 3.1b, the projection of the Rössler attractor onto the xy-plane for these parameter values produces a regular rotation of the phase point around origo. Hence, the angle between any fixed direction in the xy-plane and the direction from origo to the instantaneous phase-point is often sufficient [33]. More theoretical studies may also involve determination of the neutrally stable direction, i.e., the direction of vanishing Lyapunov exponent [8].

In spite of these complications it is clear that many chaotic systems react to an external forcing in a manner that in many ways resembles the reaction observed for the simple Rössler system. Moreover, one can often use some of the diagnostic tools described in the beginning of this chapter without actually introducing the concept of a phase [33]. The mean frequency, for instance, typically displays a variation similar to the synchronization characteristics shown in Figure 3.1a. This means that one can delineate a restricted range of forcing frequencies where the mean frequency of the forced chaotic system approximately follows the variation of the forcing frequency. The lack of precise frequency synchronization may then be revealed by the rounding of the otherwise sharp edges of the synchronization characteristic at the synchronization thresholds. In experimental studies, a similar rounding phenomenon may be associated with the presence of noise.

When the stroboscopic projection technique illustrated in Figure 3.1b and c is applied, one might find that the distribution of dots across the forced state from time to time turns unstable and undergoes a major reorganization to again return the same basic distribution. In this connection the distribution of return times of the different loops in the unforced chaotic oscillator may also play a role. If this distribution is sufficiently inhomogeneous, the distribution of stroboscopic points may break up into several groups.

In view, primarily, of analyzing the phase relationships for experimental time series, Pikovsky et al. [35] have suggested the use of a Hilbert transformation approach to determine the degree of phase synchronization between two interacting chaotic oscillators. Provided the series is sufficiently long, the Hilbert transformation allows one to define the instantaneous phase (and amplitude) of an experimentally observed time series. With similar definitions for other locking ratios, 1:1 phase synchronization between two interacting, chaotic oscillators may then be said to occur if, with a sufficiently small margin \( \mu \), the difference between the two phases continues to fall near some constant value. Theoretically, phase synchronization may be said to occur if the small margin \( \mu \ll 2\pi \). In experimental studies of interacting chaotic oscillators one often finds that the phase difference remains constant within a relatively small margin \( \mu \) for a certain period of time to then jump one or more \( 2\pi \) steps as the oscillators for a moment lose synchronization. The question whether phase synchronization occurs or not then becomes a matter of the frequency of such \( 2\pi \) steps. We have successfully used the Hilbert transformation approach to demonstrate, for instance, the occurrence of phase synchronization between the proximal tubular pressure oscillations for a pair of neighboring nephrons of the kidney [14].

The purpose of this chapter has been to present the results of a relatively detailed bifurcation analysis of the transition to chaotic phase synchronization in a periodically forced Rössler oscillator. With this purpose, we have applied a standard continuation method [10] to follow the bifurcations that occur within...
and along the edges of the 1:1 synchronization zone as increasing values of the so-called nonlinearity parameter $c$ take the system up through the cascade of period-doubling bifurcations.

This approach has allowed us to examine the interplay between a significant number of interesting phenomena, including (i) the relation between the interconnected period-doubling bifurcations for the node and saddle cycles that occur in the resonance zone, (ii) the formation, transformation, and elimination of different forms of multi-layered resonance tori, (iii) the communication between the resonance tori and the quasiperiodic oscillations that exist outside the resonance zone, (iv) the torus doubling bifurcations that occur along the edges of the resonance zone, and (v) the alternating involvement of torus-birth bifurcations and subcritical period-doubling bifurcations in restricting the stable resonance modes from escaping from their designated regions of parameter space. It is precisely this alternation of different bifurcations that produce the special scaling structure (cyclic criticality) that characterizes the period-doubling bifurcation along the edge of a resonance tongue [17,18].

If one extends the investigation to consider the transition to chaotic phase synchronization for a pair of coupled, nearly identical period-doubling systems, the first thing to notice is that the presence of a coupling splits the synchronized modes into symmetric and anti-symmetric states [38]. If synchronization takes place into a symmetric state, the two oscillators move in phase, the explored part of phase space remains relatively narrow, and the subsequent bifurcations in general maintain the original symmetry. If, on the other hand, the synchronization occurs into an anti-symmetric state, the two oscillators move with opposite phases, excursions are made to a much larger part of phase space, and the subsequent bifurcations typically involve torus-birth processes and homoclinic bifurcations.

The bifurcation structure one can observe along the edge of a synchronization tongue that leads to symmetric states is essentially the same as the structure we have described for the periodically forced Rössler attractor.

However, along the other edge of the synchronization zone, where the phase locking typically leads to anti-symmetric states, one can observe a very different bifurcation structure [23]. Here, the edge of the resonance zone involves cascades of bifurcations of different type. A cascade of saddle–node bifurcations serves to delineate the range of existence of the resonance cycles. Like in the bifurcation diagram (Figure 3.2) for the periodically forced Rössler system, the saddle–node bifurcations curves emerge from points on the corresponding period-doubling curve and continue up along the edge of the resonance zone. Moreover, this process still occurs in an alternating manner to the left and to the right of the former saddle–node bifurcation curve. However, the saddle–node bifurcation curves no longer mark the transition from stable periodic dynamics to quasiperiodicity.

This function has been taken over by a cascade of torus bifurcation curves that stretch up along the saddle–node bifurcation curves, but inside the resonance zone. In this way, a region is formed where stable quasiperiodic oscillations coexist with doubly and triply unstable resonance modes. The torus bifurcation curves emerge from the points of intersection between the corresponding period-doubling curve and the former torus bifurcation curve. They are supported, close to the transition to chaos, by points on the corresponding saddle–node bifurcation curves.

From each of these points, a curve of homoclinic bifurcations stretches between the torus bifurcation curve and the corresponding saddle–node bifurcation curve all the way down to the point where the torus bifurcation curve was born.

The torus bifurcation curves produce a set of period doubled quasiperiodic states, and the overall result of this reconstruction of the bifurcation diagram for the anti-symmetric modes is that part of the resonance zone is invaded by different quasiperiodic modes. This includes the occurrence of phase-modulated quasiperiodicity. Let us finally note that a recent study of the transition to chaotic phase synchronization for two coupled, nearly identical two-oscillator systems (nephrons) [23] has demonstrated essentially the same bifurcation structures for synchronization into symmetric, respectively anti-symmetric states. This result applies, provided that the 1:5 or 1:4 synchronization for the internal modes of the individual oscillator remained unaffected by the coupling.
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References


