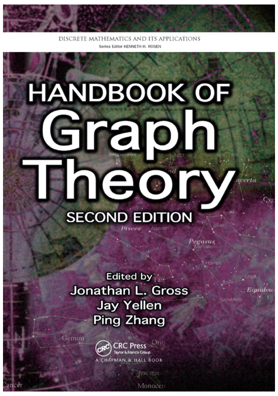

Handbook of Graph Theory

Jonathan L. Gross, Jay Yellen, Ping Zhang

History of Graph Theory

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Robin J. Wilson
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Section 1.3
History of Graph Theory

Robin J. Wilson, Pembroke College, Oxford University, UK

INTRODUCTION

Although the first mention of a graph was not until 1878, graph-theoretical ideas can be traced back to 1735 when Leonhard Euler (1707–83) presented his solution of the Königsberg bridges problem. This chapter summarizes some important strands in the development of graph theory since that time. Further information can be found in [BiLlWi98] or [Wi99].

1.3.1 Traversability

The origins of graph theory can be traced back to Euler's work on the Königsberg bridges problem (1735), which subsequently led to the concept of an *eulerian graph*. The study of cycles on polyhedra by the Revd. Thomas Penyngton Kirkman (1806–95) and Sir William Rowan Hamilton (1805–65) led to the concept of a *Hamiltonian graph*.

The Königsberg Bridges Problem

The *Königsberg bridges problem*, pictured in Figure 1.3.1, asks whether there is a continuous walk that crosses each of the seven bridges of Königsberg exactly once — and if so, whether a closed walk can be found. See §4.2 for more extensive discussion of issues concerning *eulerian graphs*.

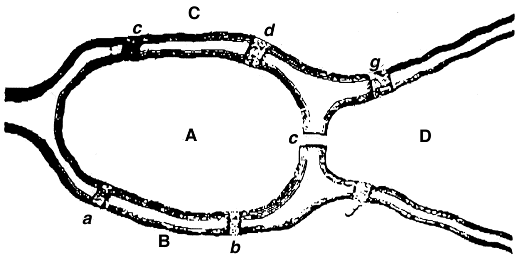

Figure 1.3.1: The seven bridges of Königsberg.

FACTS [BiLlWi98, Chapter 1]

F1: On 26 August 1735 Leonhard Euler gave a lecture on "The solution of a problem relating to the geometry of position" to the Academy of Sciences of St. Petersburg, Russia, proving that there is no such continuous walk across the seven bridges.

F2: In 1736, Euler communicated his solution to several other mathematicians, outlining his views on the nature of the problem and on its situation in the geometry of position [HoWi04].

F3: Euler [Eu:1736] sent his solution of the problem to the *Commentarii Academii Scientiarum Imperialis Petropolitanae* under the title "Solutio problematis ad geometriam ad geometriam situs pertinentis". Although dated 1736, it did not appear until 1741, and was later republished in the new edition of the *Commentarii* in 1752.

F4: Euler's paper is divided into 21 sections, of which 9 are on the Königsberg bridges problem, and the remainder are concerned with general arrangements of bridges and land areas.

F5: Euler did not draw a graph in order to solve the problem, but he reformulated the problem as one of trying to find a sequence of eight letters A, B, C, or D (the land areas) such that the pairs AB and AC are adjacent twice (corresponding to the two bridges between A and B and between A and C), and the pairs AD, BD, and CD are adjacent just once (corresponding to the remaining bridges). He showed by a counting argument that no such sequence exists, thereby proving that the Königsberg bridges problem has no solution.

F6: In discussing the general problem, Euler first observed that the number of bridges written next to the letters A, B, C, etc. together add up to twice the number of bridges. This is the first appearance of what some graph-theorists now call the "handshaking lemma", that the sum of the vertex-degrees in a graph is equal to twice the number of edges.

F7: Euler's main conclusions for the general situation were as follows:

- If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.
- If the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these two areas.
- If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished starting from any area.

These results correspond to the conditions under which a graph has an eulerian, or semi-eulerian, trail.

F8: Euler noted the converse result, that if the above conditions are satisfied, then a route is possible, and gave a heuristic reason why this should be so, but did not prove it. A valid demonstration did not appear until a related result was proved by C. Hierholzer [Hi:1873] in 1873.

Diagram-Tracing Puzzles

A related area of study was that of *diagram-tracing puzzles*, where one is required to draw a given diagram with the fewest possible number of connected strokes. Such puzzles can be traced back many hundreds of years – for example, there are some early African examples.

FACTS [BiLlWi98, Chapter 1]

F9: In 1809 L. Poinsot [Po:1809] wrote a memoir on polygons and polyhedra in which he posed the following problem:

> Given some points situated at random in space, it is required to arrange a single flexible thread uniting them two by two in all possible ways, so that the two ends of the thread join up and the total length is equal to the sum of all the mutual distances.

Poinsot noted that a solution is possible only when the number of points is odd, and gave a method for finding such an arrangement for each possible value. In modern terminology, the question is concerned with eulerian trails in complete graphs of odd order.

F10: Other diagram-tracing puzzles were posed and solved by T. Clausen [Cl:1844] and J. B. Listing [Li:1847]. The latter appeared in the book *Vorstudien zur Topologie*, the first place that the word "topology" appeared in print.

F11: In 1849, O. Terquem asked for the number of ways of laying out a complete ring of dominoes. This is essentially the problem of determining the number of eulerian tours in the complete graph K_7, and was solved by M. Reiss [Re:1871–3] and later by G. Tarry.

F12: The connection between the Königsberg bridges problem and diagram-tracing puzzles was not recognized until the end of the 19th century. It was pointed out by W. W. Rouse Ball [Ro:1892] in *Mathematical Recreations and Problems*. Rouse Ball seems to have been the first to use the graph in Figure 1.3.2 to solve the problem.

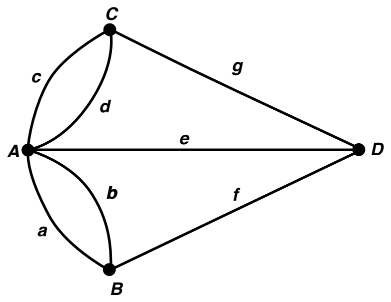

Figure 1.3.2: Rouse Ball's graph of the Königsberg bridges problem.

Hamiltonian Graphs

A type of graph problem that superficially resembles the eulerian problem is that of finding a cycle that passes just once through each vertex of a given graph. Because of Hamilton's influence, such graphs are now called *hamiltonian graphs* (see §4.5), instead of more justly being named after Kirkman, who, prior to Hamilton's consideration of the dodecahedron, as discussed below, considered the more general problem.

FACTS [BiLlWi98, Chapter 2]

F13: An early example of such a problem is the *knight's tour problem*, of finding a succession of knight's moves on a chessboard, visiting each of the 64 squares just once and returning to the starting point. This problem can be dated back many hundreds of years, and systematic solutions were given by Euler [Eu:1759], A.-T. Vandermonde [Va:1771], and others.

F14: In 1856 Kirkman [Ki:1856] wrote a paper investigating those polyhedra for which one can find a cycle passing through all the vertices just once. He proved that every polyhedron with even-sided faces and an odd number of vertices has no such cycle, and gave as an example the polyhedron obtained by "cutting in two the cell of a bee" (see Figure 1.3.3).

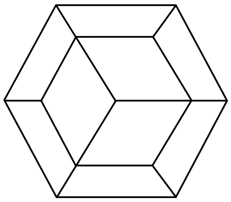

Figure 1.3.3: Kirkman's "cell of a bee" example.

F15: Arising from his work on non-commutative algebra, Hamilton considered cycles passing through all the vertices of a dodecahedron. He subsequently invented a game, called the icosian game (see Figure 1.3.4), in which the player was challenged to find such cycles on a solid dodecahedron, satisfying certain extra conditions.

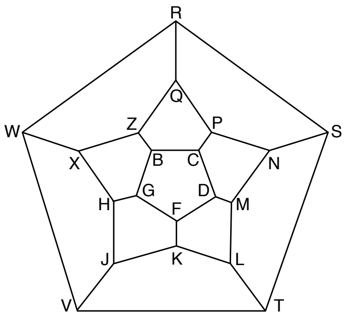

Figure 1.3.4: Hamilton's icosian game.

F16: In 1884, P. G. Tait asserted that every 3-valent polyhedron has a hamiltonian cycle. This assertion was subsequently disproved by W. T. Tutte [Tu46] in 1946 (see Figure 1.3.5).

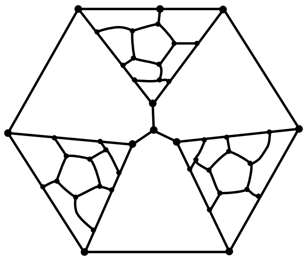

Figure 1.3.5: Tutte's 3-valent non-hamiltonian polyhedron.

F17: Sufficient conditions for a graph to be hamiltonian were later obtained by G. A. Dirac [Di52], O. Ore [Or60], J. A. Bondy and V. Chvátal [BoCh76], and others.

F18: Hamiltonian digraphs have also been investigated, by A. Ghouila-Houri (1960), H. Meyniel (1973), and others.

1.3.2 Trees

The concept of a tree, a connected graph without cycles, appeared implicitly in the work of Gustav Kirchhoff (1824–87), who employed graph-theoretical ideas in the calculation of currents in electrical networks. Later, trees were used by Arthur Cayley (1821–95), James Joseph Sylvester (1806–97), Georg Pólya (1887–1985), and others, in connection with the enumeration of certain chemical molecules.

Counting Trees

Enumeration techniques involving trees first arose in connection with a problem in the differential calculus, but they soon came to be fundamental tools in the counting of chemical molecules, as well as providing a fascinating topic of interest in their own right. Enumeration of various kinds of graphs is discussed in §6.3.

FACTS [BiLlWi98, Chapter 3] [PóRe87]

F19: While working on a problem inspired by some work of Sylvester on "differential transformation and the reversion of serieses", Cayley [Ca:1857] was led to the enumeration of rooted trees.

F20: Cayley's method was to take a rooted tree and remove its root, thereby obtaining a number of smaller rooted trees (see Figure 1.3.6).

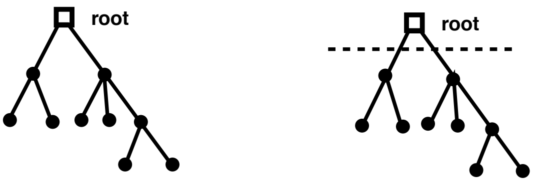

Figure 1.3.6: Splitting a rooted tree.

Letting A_n be the number of rooted trees with n branches, Cayley proved that the generating function

$$1 + A_1 x + A_2 x^2 + A_3 x^3 + \dots$$

is equal to the product

$$(1 - x)^{-1} \cdot (1 - x^2)^{-A_1} \cdot (1 - x^3)^{-A_2} \cdot \dots$$

Using this equality, he was able to calculate the first few numbers A_n, one at a time.

F21: Around 1870, Sylvester and C. Jordan independently defined the *center/bicenter* and the *centroid/bicentroid* of a tree.

F22: In 1874, Cayley [Ca:1874] found a method for solving the more difficult problem of counting unrooted trees. This method, which he applied to chemical molecules, consisted essentially of starting at the center or centroid of the tree or molecule and working outwards.

F23: In 1889, Cayley [Ca:1889] presented his n^{n-2} formula for the number of labeled trees with n vertices. He explained why the formula holds when $n = 6$, but he did not give a proof in general. The first accepted proof was given by H. Prüfer [Pr18]: his method was to establish a one-to-one correspondence between such labeled trees and sequences of length $n - 2$ formed from the numbers $1, 2, \dots, n$.

F24: In a fundamental paper of 1937, Pólya [Pó37] combined the classical idea of a generating function with that of a permutation to obtain a powerful theorem that enabled him to enumerate certain types of configuration under the action of a group of symmetries. Some of Pólya's work was anticipated by J. H. Redfield [Re27], but Redfield's paper was obscurely written and had no influence on the development of the subject.

F25: Later results on the enumeration of trees were derived by R. Otter [Ot48] and others. The field of graphical enumeration (see [HaPa73]) was subsequently further developed by F. Harary [Ha55], R. C. Read [Re63], and others.

Chemical Trees

By 1850 it was already known that chemical elements combine in fixed proportions. Chemical formulas such as CH_4 (methane) and C_2H_5OH (ethanol) were known, but it was not understood how the elements combine to form such substances. Around this time, chemical ideas of valency began to be established, particularly when Alexander Crum Brown presented his graphic formulae for representing molecules. Figure 1.3.7 presents his representation of ethanol, the usual drawing, and the corresponding tree graph.

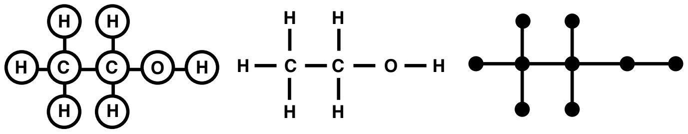

Figure 1.3.7: Representations of ethanol.

FACTS [BiLlWi98, Chapter 4]

F26: Crum Brown's graphic notation explained for the first time the phenomenon of isomerism, whereby there exist pairs of molecules (isomers) with the same chemical formula but different chemical properties. Figure 1.3.8 shows isomers with chemical formula C_4H_{10}.

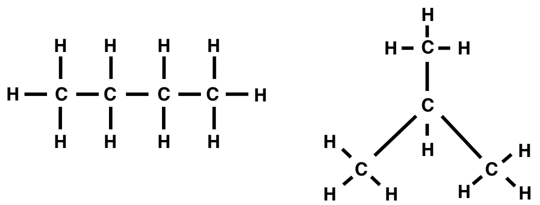

Figure 1.3.8: Two isomers: butane and isobutane.

F27: Cayley [Ca:1874] used tree-counting methods to enumerate paraffins (alkanes) with up to 11 carbon atoms, as well as various other families of molecules; the followiing table gives the number of isomers of alkanes for $n = 1, \ldots, 8$.

Formula	CH_4	C_2H_6	C_3H_8	C_4H_{10}	C_5H_{12}	C_6H_{14}	C_7H_{16}	C_8H_{18}
Number	1	1	1	2	3	5	9	18

F28: W. K. Clifford and Sylvester believed that a connection could be made between chemical atoms and binary quantics in invariant theory, a topic to which Cayley and Sylvester had made significant contributions. In 1878, Sylvester [Sy:1877–8] wrote a short note in *Nature* about this supposed connection, remarking that:

> Every invariant and covariant thus becomes expressible by a *graph* precisely identical with a Kekuléan diagram or chemicograph.

This was the first appearance of the word *graph* in the graph-theoretic sense.

F29: In 1878, Sylvester [Sy:1878] wrote a lengthy article on the graphic approach to chemical molecules and invariant theory in the first volume of the *American Journal of Mathematics*, which he had recently founded.

F30: Little progress was made on the enumeration of isomers until the 1920s and 1930s. A. C. Lunn and J. K. Senior [LuSe29] recognized the importance of permutation groups for this area, and Pólya's above-mentioned paper solved the counting problem for several families of molecules.

1.3.3 Topological Graphs

Euler's polyhedron formula [Eu:1750] was the foundation for topological graph theory, since it holds also for planar graphs. It was later extended to surfaces other than the sphere. In 1930, a fundamental characterization of graphs imbeddable in the sphere was given by Kazimierz Kuratowski (1896–1980), and recent work – notably by Neil Robertson, Paul Seymour, and others – has extended these results to the higher order surfaces.

Euler's Polyhedron Formula

The Greeks were familiar with the five regular solids, but there is no evidence that they knew the simple connection between the numbers V of vertices, E of edges, and F of faces of a polyhedron:

$$V - E + F = 2$$

In the 17th century, René Descartes studied polyhedra, and he obtained results from which Euler's formula could later be derived. However, since Descartes had no concept of an edge, he was unable to make such a deduction.

FACTS [BiLlWi98, Chapter 5] [Cr99] [BeWi09]

F31: The first appearance of the polyhedron formula appeared in a letter, dated 14 November 1750, from Euler to C. Goldbach. Denoting the number of faces, solid angles (vertices) and joints (edges) by $\mathcal{H}$, $\mathcal{S}$, and $\mathcal{A}$, he wrote:

- In every solid enclosed by plane faces the aggregate of the number of faces and the number of solid angles exceeds by two the number of edges, or $\mathcal{H}+\mathcal{S} = \mathcal{A}+2$.

F32: Euler was unable to prove his formula. In 1752 he attempted a proof by dissection, but it was deficient. The first valid proof was given by A.-M. Legendre [Le:1794] in 1794, using metrical properties of spherical polygons.

F33: In 1813, A.-L. Cauchy [Ca:1813] obtained a proof of Euler's formula by stereographically projecting the polyhedron onto a plane and considering a triangulation of the resulting planar graph.

F34: Around the same time, S.-A.-J. Lhuilier [Lh:1811] gave a topological proof that there are only five regular convex polyhedra, and he anticipated the idea of duality by noting that four of them occur in reciprocal pairs. He also found three types of polyhedra for which Euler's formula fails – those with indentations in their faces, those with an interior cavity, and ring-shaped polyhedra drawn on a torus (that is, polyhedra with a 'tunnel' through them). For such ring-shaped polyhedra, Lhuilier derived the formula

$$V - E + F = 0$$

and extended his discussion to prove that, if g is the number of tunnels in a surface on which a polyhedral map is drawn, then

$$V - E + F = 2 - 2g$$

The number g is now called the ***genus of the surface***, and the value of the quantity $2 - 2g$ is called the ***Euler characteristic***. (See §7.1.)

F35: In 1861–2, Listing [Li:1861–2] wrote *Der Census räumliche Complexe*, an extensive investigation into complexes, and studied how their topological properties affect the generalization above of Euler's formula. This work proved to be influential in the subsequent development of topology. In particular, H. Poincaré took up Listing's ideas in his papers of 1895–1904 that laid the foundations for algebraic topology.

F36: Poincaré's work was instantly successful, and it appeared in an article by M. Dehn and P. Heegaard [DeHe07] on analysis situs (topology) in the ten-volume *Encyklopädie der Mathematischen Wissenschaften*. His ideas were further developed by O. Veblen [Ve22] in a series of colloquium lectures on analysis situs for the American Mathematical Society in 1916.

Planar Graphs

The study of planar graphs originated in two recreational problems involving the complete graph K_5 and the complete bipartite graph $K_{3,3}$. These graphs (shown in Figure 1.3.9) are the main obstructions to planarity, as was subsequently demonstrated by Kuratowski.

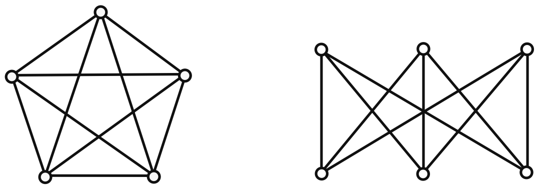

Figure 1.3.9: The Kuratowski graphs K_5 and $K_{3,3}$.

FACTS [BiLlWi98, Chapter 8]

F37: Around the year 1840, A. F. Möbius presented the following puzzle to his students:

> There was once a king with five sons. In his will he stated that after his death the sons should divide the kingdom into five regions so that the boundary of each region should have a frontier line in common with each of the other four regions. Can the terms of the will be satisfied?

This question asks whether one can draw five mutually neighboring regions in the plane. The connection with graph theory can be seen from its dual version, later formulated by H. Tietze:

> The king further stated that the sons should join the five capital cities of his kingdom by roads so that no two roads intersect. Can this be done?

In this dual formulation, the problem is that of deciding whether the graph K_5 is planar.

F38: An old problem, whose origins are obscure, is the *utilities problem*, or *gas–water–electricity problem*, mentioned by H. Dudeney [Du13] in the *Strand Magazine* of 1913:

> The puzzle is to lay on water, gas, and electricity, from W, G, and E, to each of the three houses, A, B, and C, without any pipe crossing another (see Figure 1.3.10).

This problem is that of deciding whether $K_{3,3}$ is planar.

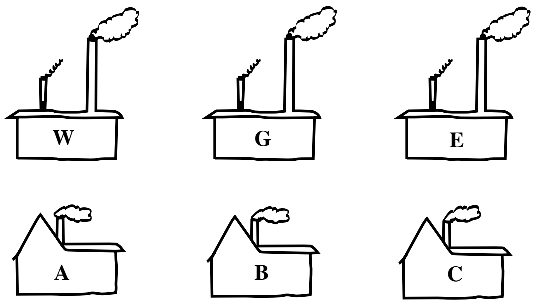

Figure 1.3.10: The gas–water–electricity problem.

F39: In 1930 Kuratowski [Ku30] published a celebrated paper proving that every nonplanar graph has a subgraph homeomorphic to K_5 or $K_{3,3}$; this result was obtained independently by O. Frink and P. A. Smith.

F40: In 1931 H. Whitney [Wh31] discovered an abstract definition of duality that is purely combinatorial and agrees with the geometrical definition of duality for planar graphs. He proved that, with this general definition of duality, a graph is planar if and only if it has an abstract dual. Related results were obtained by S. MacLane and others.

F41: In 1935 Whitney [Wh35] generalized the idea of independence in graphs and vector spaces to the concept of a matroid. The dual of a matroid extends and clarifies the duality of planar graphs, and Tutte [Tu59] used these ideas in the late 1950s to obtain a Kuratowski-type criterion for a matroid to arise from a graph (see §6.6).

Graphs on Higher Surfaces

A graph drawn without crossings on a plane corresponds (by stereographic projection) to a graph similarly drawn on the surface of a sphere. This leads to the idea of graphs drawn on surfaces other than the sphere. The initial work in this area was carried out, in the context of coloring maps, by Percy Heawood (1861–1955) and Lothar Heffter (1862–1962) for orientable surfaces, and by Heinrich Tietze (1880–1964) for non-orientable surfaces, but the basic problems in the area were not solved until Gerhard Ringel and Ted Youngs solved the Heawood conjecture in the 1960s and Neil Robertson and Paul Seymour generalized Kuratowski's theorem to other surfaces in the 1980s.

FACTS [BiLlWi98, Chapter 7; Ri74]

F42: In 1890, Heawood [He:1890] presented an imbedding of the complete graph K_7 on a torus. He also derived a formula for the genus of a surface on which a given complete graph can be imbedded, but his attempted proof of this formula was deficient.

F43: In 1891, L. Heffter [He:1891] investigated the imbedding of complete graphs on orientable surfaces other than the sphere and the torus, and he proved that Heawood's formula is correct for orientable surfaces of low genus and certain other surfaces.

F44: In 1910, H. Tietze [Ti10] extended Heffter's considerations to certain non-orientable surfaces, such as the Möbius band and the projective plane, and stated a corresponding Heawood formula. He was unable to prove it for the Klein bottle, but this case was settled in 1934 by P. Franklin [Fr34], who found that it was an exception to the formula. In 1935, I. N. Kagno [Ka35] proved the formula for surfaces of non-orientable genus 3, 4, and 6.

F45: The Heawood formula for general non-orientable surfaces was proved in 1952 by Ringel. The proof for orientable surfaces proved to be much more difficult, involving 300 pages of consideration of 12 separate cases. Most of these were settled in the mid-1960s, and the proof was completed in 1968 by Ringel and Youngs [RiYo68], using W. Gustin's [Gu63] combinatorial inspiration in 1963 of a *current graph*. Since then, the transformation by J. L. Gross [Gr74] of numerous types of specialized combinatorial current graphs into a unified topological object, with its dualization to a *voltage graph* (see §7.4), has led to simpler solutions (see Gross and T. W. Tucker [GrTu74]).

F46: In a sequence of papers in the 1980s of great mathematical depth, Robertson and Seymour [RoSe85] proved that, for each orientable genus g, the set of "forbidden subgraphs" is finite (see §7.7). However, apart from the sphere, the number of forbidden subgraphs runs into hundreds, even for the torus. For non-orientable surfaces, there is a similar result, and in 1979 H. H. Glover, J. P. Huneke, and C. S. Wang [GlHuWa79] obtained a set of 103 forbidden subgraphs for the projective plane.

1.3.4 Graph Colorings

Early work on colorings concerned the coloring of the countries of a map and, in particular, the celebrated four-color problem. This was first posed by Francis Guthrie in 1852, and a celebrated (incorrect) "proof" by Alfred Bray Kempe appeared in 1879. The four-color theorem was eventually proved by Kenneth Appel and Wolfgang Haken in 1976, building on the earlier work of Kempe, George Birkhoff, Heinrich Heesch, and others, and a simpler proof was subsequently produced by Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas [1997]. Meanwhile, attention had turned to the dual problem of coloring the vertices of a planar graph and of graphs in general.

There was also a parallel development in the coloring of the edges of a graph, starting with a result of Tait [1880], and leading to a fundamental theorem of V. G. Vizing in 1964. As mentioned earlier, the corresponding problem of coloring maps on other surfaces was settled by Ringel and Youngs in 1968.

The Four-Color Problem

Many developments in graph theory can be traced back to attempts to solve the celebrated four-color problem on the coloring of maps.

FACTS [BiLlWi98, Chapter 6] [Wi02]

F47: The earliest known mention of the four-color problem occurs in a letter from A. De Morgan to Hamilton, dated 23 October 1852. De Morgan described how a student had asked him whether every map can be colored with just four colors in such a way that neighbouring countries are colored differently. The student later identified himself as Frederick Guthrie, giving credit for the problem to his brother Francis, who formulated it while coloring the counties of a map of England. Hamilton was not interested in the problem.

F48: De Morgan wrote to various friends, outlining the problem and trying to describe where the difficulty lies. On 10 April 1860, the problem appeared in print, in an unsigned book review in the *Athenaeum*, written by De Morgan. This review was read in the U.S. by C. S. Peirce, who developed a life-long interest in the problem. An earlier printed reference, signed by "F.G.", appeared in the *Athenaeum* in 1854 [McK12].

F49: On 13 June 1878, at a meeting of the London Mathematical Society, Cayley asked whether the problem had been solved. Shortly after, he published a short note describing where the difficulty might lie, and he showed that it is sufficient to restrict one's attention to trivalent maps.

F50: In 1879, Kempe [Ke:1879], a former Cambridge student of Cayley, published a purported proof of the four-color theorem in the *American Journal of Mathematics*, which had recently been founded by Sylvester. Kempe showed that every map must contain a country with at most five neighbours, and he showed how any coloring of the rest of the map can be extended to include such a country. His solution included a new technique, now known as a ***Kempe-chain*** argument, in which the colors in a two-colored section of the map are interchanged. Kempe's proof for a map that contains a digon, triangle, or quadrilateral was correct, but his argument for the pentagon (where he used two simultaneous color-interchanges) was fallacious.

F51: In 1880, Tait [Ta:1878–80] presented "improved proofs" of the four-color theorem, all of them fallacious. Other people interested in the four-color problem at this time were C. L. Dodgson (Lewis Carroll), F. Temple (Bishop of London), and the Victorian educator J. M. Wilson.

F52: In 1890, Heawood [He:1890] published a paper in the *Quarterly Journal of Pure and Applied Mathematics*, pointing out the error in Kempe's proof, salvaging enough to deduce the five-color theorem, and generalizing the problem in various ways, such as for other surfaces (see §1.1.3). Heawood subsequently published another six papers on the problem, the last while he was in his 90th year. Kempe admitted his error, but he was unable to put it right.

F53: During the first half of the 20th century two ideas emerged, each of which finds its origin in Kempe's paper. The first is that of an ***unavoidable set*** — a set of configurations, at least one of which must appear in any map. Unavoidable sets were produced by P. Wernicke [We:1904] (see Figure 1.3.11), by P. Franklin, and by H. Lebesgue.

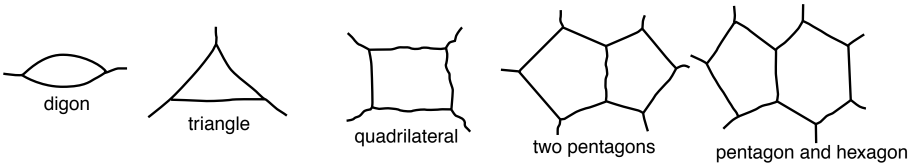

Figure 1.3.11: Wernicke's unavoidable set.

The second is that of a ***reducible configuration*** — a configuration of countries with the property that any coloring of the rest of the map can be extended to the configuration: no such configuration can appear in any counter-example to the four-color theorem. Birkhoff [Bi13] showed that the arrangement of four pentagons in Figure 1.3.12 (known as the ***Birkhoff diamond***) is a reducible configuration.

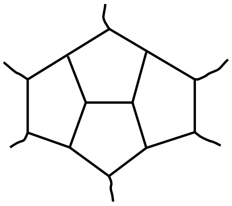

Figure 1.3.12: The Birkhoff diamond.

F54: In 1912, Birkhoff [Bi12] investigated the number of ways of coloring a given map with k colors, and he showed that this is always a polynomial in k, now called the *chromatic polynomial* of the map.

F55: In 1922, Franklin [Fr22] presented further unavoidable sets and reducible configurations, and he deduced that the four-color theorem is true for all maps with up to 25 countries. This number was later increased several times by other authors.

F56: Around 1950 Heesch started to search for an unavoidable set of reducible configurations. Over the next few years, Heesch [He69] produced thousands of reducible configurations.

F57: In 1976, Appel and Haken [ApHa77, ApHaKo77], with the assistance of J. Koch, obtained an unavoidable set of 1482 reducible configurations, thereby proving the four-color theorem. Their solution required substantial use of a computer to test the configurations for reducibility.

F58: Around 1994, Robertson, Sanders, Seymour, and Thomas [RoSaSeTh97] produced a more systematic proof. Using a computer to assist with both the unavoidable set and the reducible configuration parts of the solution, they systematized the Appel–Haken approach, and they obtained an unavoidable set of 633 reducible configurations.

Other Graph Coloring Problems

Arising from work on the four-color problem, progress was being made on other graph problems involving the coloring of edges or vertices.

FACTS [BiLlWi98, Chapter 6] [FiWi77] [JeTo95]

F59: In his 1879 paper on the coloring of maps, Kempe [Ke:1879] outlined the dual problem of coloring the vertices of a planar graph in such a way that adjacent vertices are colored differently. This dual approach to map-coloring was later taken up by H. Whitney in a fundamental paper of 1932 and by most subsequent workers on the four-color problem.

F60: In 1880, Tait [Ta:1878–80] proved that the four-color theorem is equivalent to the statement that the edges of every trivalent map can be colored with three colors in such a way that each color appears at every vertex.

F61: In 1916, D. König [Kö16] proved that the edges of any bipartite graph with maximum degree d can be colored with d colors. (See §11.3.)

F62: The idea of coloring the vertices of a graph so that adjacent vertices are colored differently developed a life of its own in the 1930s, mainly through the work of Whitney, who wrote his Ph.D. thesis on the coloring of graphs.

F63: In 1941, L. Brooks [Br41] proved that the chromatic number of any simple graph with maximum degree d is at most $d + 1$, with equality only for odd cycles and odd complete graphs. (See §5.1.)

F64: In the 1950s, substantial progress on vertex-colorings was made by G. A. Dirac, who introduced the idea of a *critical graph*.

F65: In 1964, V. G. Vizing [Vi64] proved that the edges of any simple graph with maximum degree d can always be colored with $d + 1$ colors. In the following year, Vizing produced many further results on edge-colorings.

F66: The concepts of the chromatic number and edge-chromatic number of a graph have been generalized by a number of writers — for example, M. Behzad and others introduced total colorings in the 1960s, and P. Erdős and others introduced list colorings.

Factorization

A graph is *k-regular* if each of its vertices has degree k. Such graphs can sometimes be split into regular subgraphs, each with the same vertex-set as the original graph. A *k-factor* in a graph is a k-regular subgraph that contains all the vertices of the original graph. Fundamental work on factors in graphs was carried out by Julius Petersen [1839–1910] and W. T. Tutte [1914–2002]. (See §5.4.)

FACTS [BiLlWi98, Chapter 10]

F67: In 1891, Petersen [Pe:1891] wrote a fundamental paper on the factorization of regular graphs, arising from a problem in the theory of invariants. In this paper he proved that if k is even, then any k-regular graph can be split into 2-factors. He also proved that any 3-regular graph possesses a 1-factor, provided that it has not more than two "leaves"; a leaf is a subgraph joined to the rest of the graph by a single edge.

F68: In 1898, Petersen [Pe:1898] produced a trivalent graph with no leaves, now called the ***Petersen graph*** (see Figure 1.3.13), which cannot be split into three 1-factors; it can, however, be split into a 1-factor (the spokes) and a 2-factor (the pentagon and pentagram).

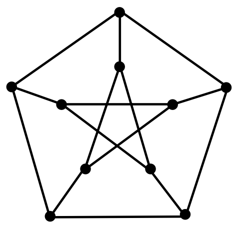

Figure 1.3.13: The Petersen graph.

F69: In 1947, Tutte [Tu47] produced a characterization of graphs that contain a 1-factor. Five years later he extended his result to a characterization of graphs that contain a k-factor, for any k.

1.3.5 Graph Algorithms

Graph theory algorithms can be traced back to the 19th century, when Fleury gave a systematic method for tracing an eulerian graph and G. Tarry showed how to escape from a maze (see §4.2). The 20th century saw algorithmic solutions to such problems as the minimum connector problem, the shortest and longest path problems, and the *Chinese Postman Problem* (see §4.3), as well as to a number of problems arising in operational research. In each of these problems we are given a network, or weighted graph, to each edge (and/or vertex) of which has been assigned a number, such as its length or the time taken to traverse it.

FACTS [Da82] [LLRS85] [LoPl86]

F70: The *Traveling Salesman Problem*, in which a salesman wishes to make a cyclic tour of a number of cities in minimum time or distance, appeared in rudimentary form in 1831. It reappeared in mathematical circles in the early 1930s, at Princeton, and was later popularized at the RAND Corporation. This led to a fundamental paper of G. B. Dantzig, D. R. Fulkerson, and S. M. Johnson [DaFuJo54] that included the solution of a traveling salesman problem with 49 cities. In the 1980s a problem with 2392 cities was settled by Padberg and Rinaldi [PaRi87]. (See §4.6.)

F71: The greedy algorithm for the *minimum connector problem*, in which one seeks a minimum-length spanning tree in a weighted graph, can be traced back to O. Boruvka [Bo26] and was later rediscovered by J. B. Kruskal [Kr56]. A related algorithm, due to V. Jarník (1931), was rediscovered by R. C. Prim (1957). (See §10.1.)

F72: Graph algorithms were developed by D. R. Fulkerson and G. B. Dantzig [FuDa55] for finding the maximum flow of a commodity between two nodes in a capacitated network, and by R. E. Gomory and T. C. Hu [GoHu61] for determining *maximum flows* in multi-terminal networks.

F73: Finding a longest path, or critical path, in an activity network dates from the 1940s and 1950s, with *PERT* (Program Evaluation and Review Technique) used by the U.S. Navy for problems involving the building of submarines and CPM (*Critical Path Method*) developed by the Du Pont de Nemours Company to minimize the total cost of a project. (See §3.2.)

F74: There are several efficient algorithms for finding the shortest path in a given network, of which the best known is due to E. W. Dijkstra [Di59]. (See §10.1.)

F75: The Chinese postman problem, for finding the shortest route that covers each edge of a given weighted graph, was originated by Meigu Guan (Mei-Ku Kwan) [Gu60] in 1960. (See §4.3.)

F76: In matching and assignment problems one wishes to assign people as appropriately as possible to jobs for which they are qualified. This work developed from work of König and from a celebrated result on matching due to Philip Hall [Ha35], later known as the "marriage theorem" [HaVa50]. These investigations led to the subject of polyhedral combinatorics and were combined with the newly emerging study of linear programming. (See §11.3.)

F77: By the late 1960s it became clear that some problems seemed to be more difficult than others, and Edmonds [Ed65] discussed problems for which a polynomial-time algorithm exists. Cook [Co71], Karp [Ka72], and others later developed the concept of NP-completeness. The assignment, transportation, and minimum spanning-tree problems are all in the *polynomial-time class* P, while the traveling salesman and Hamiltonian cycle problems are NP-hard. It is not known whether P = NP. Further information can be found in [GaJo79].

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