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Permittivity Measurement

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37.1 Introduction

Dielectric materials possess relatively few free charge carriers. Most of the charge carriers are bound and cannot participate in conduction. However, these bound charges can be displaced by applying an external electric field. In such cases, the atom or molecule forms an electric dipole that maintains an electric field. Consequently, each volume element of the material behaves as an electric dipole. The dipole field tends to oppose the applied field. Dielectric materials that exhibit nonzero distribution of such bound charge separations are said to be polarized. The volume density of polarization \( \vec{P} \) describes the volume density of those electric dipoles. When a material is linear and isotropic in nature, the polarization density is related to applied electric field intensity, \( \vec{E} \), as follows:

\[
\vec{P} = \varepsilon_0 \chi_e \vec{E}
\]  (37.1)

where

\( \varepsilon_0 (=8.854 \times 10^{-12} \text{ F m}^{-1}) \) is the permittivity of free space

\( \chi_e \) is called the electric susceptibility of the material

The electric flux density, or displacement, \( \vec{D} \) is defined as follows:

\[
\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}
\]  (37.2)

where

\( \varepsilon \) is called the permittivity of the material

\( \varepsilon_r \) is its relative permittivity or dielectric constant

Electric flux density is expressed in coulombs per meter (C m\(^{-1}\)).
Equation 37.2 represents a relation between the electric flux density and electric field intensity in frequency domain. It will hold well in time domain only if the permittivity is independent of frequency. A material is called dispersive if its characteristics are frequency dependent. The product of Equation 37.2 in frequency domain will be replaced by a convolution integral for the time-domain fields.

Assuming that the fields are time harmonic as \( e^{j\omega t} \), the generalized Ampere's law can be expressed in phasor form as follows:

\[
\nabla \times \vec{H} = \vec{j} + j\omega \vec{D} \tag{37.3}
\]

where

- \( \vec{H} \) is the magnetic field intensity in \( \text{A m}^{-2} \)
- \( \vec{j} \) is the current-source density in \( \text{A m}^{-2} \)
- \( \vec{D} \) is the conduction current density in \( \text{A m}^{-2} \)
- the last term represents the displacement current density
- \( \vec{j} \) will be zero for a source-free region

The conduction current density is related to the electric field intensity through Ohm's law as follows:

\[
\vec{J} = \sigma \vec{E} \tag{37.4}
\]

where \( \sigma \) is the conductivity of material in \( \text{S m}^{-1} \).

From Equations 37.2 through 37.4, one obtains

\[
\nabla \times \vec{H} = \vec{j}^c + \sigma \vec{E} + j\omega \varepsilon \vec{E} \tag{37.5}
\]

Conduction current represents the loss of power. There is another source of loss in dielectric materials. When a time-harmonic electric field is applied, the dipoles flip back and forth constantly. Because the charge carriers have finite mass, the field must do work to move them and they might not respond instantaneously. There is means that the polarization vector will lag behind the applied electric field. This factor shows up at high frequencies. Therefore, Equation 37.5 is modified as follows:

\[
\nabla \times \vec{H} = \vec{j}^c + \sigma \vec{E} + \omega k^* \vec{E} + j\omega \varepsilon \vec{E} = \vec{j}^c + j\omega \varepsilon \vec{E} \tag{37.6}
\]

The complex relative permittivity of a material is defined as follows:

\[
\varepsilon_r^* = \frac{\varepsilon_r}{\varepsilon_0} = 1 - j \left( \frac{\sigma + \omega k^*}{\omega} \right) = \varepsilon_r' - j\varepsilon_r'' = \varepsilon_r(1 - j \tan \delta) \tag{37.7}
\]

where \( \varepsilon_r' \) and \( \varepsilon_r'' \) represent real and imaginary parts of the complex relative permittivity. The imaginary part is zero for a lossless material. The term \( \tan \delta \) is called the loss tangent. It represents the tangent of angle between the displacement phasor and total current, as shown in Figure 37.1. Thus, it will be close to zero for a low-loss material.

Dispersion characteristics of a large class of materials can be represented by the following empirical equation of Cole-Cole:

\[
\varepsilon_r^* = \varepsilon_r + \frac{\varepsilon_b - \varepsilon_r}{1 + (j\omega \tau)^{1-\alpha}} \tag{37.8}
\]
FIGURE 37.1 A phasor diagram representing displacement and loss currents.

where

\[ \varepsilon_\infty \text{ and } \varepsilon_s \text{ are the relative permittivities of material at infinite and zero frequencies, respectively} \]
\[ \omega \text{ is the signal frequency in radians per second} \]
\[ \tau \text{ is the characteristic relaxation time in seconds} \]

For \( \alpha \) equal to zero, Equation 37.8 reduces to the Debye equation. Dispersion parameters for a few liquids are given in Table 37.1.

Complex permittivity of a material is determined using lumped circuits at low frequencies, and distributed circuits or free-space reflection and transmission of waves at high frequencies. Capacitance and dissipation factor of a lumped capacitor are measured using a bridge or a resonant circuit. The complex permittivity is calculated from these data. Complex permittivities for some substances are presented in Table 37.2.

At high frequencies, the sample is placed inside a transmission line or a resonant cavity. Propagation constants of the transmission line or resonant frequency and quality factor of the cavity resonator are used to calculate the complex permittivity. Propagation characteristics of electromagnetic waves are influenced by the complex permittivity of that medium. Therefore, a material can be characterized by monitoring the reflected and transmitted wave characteristics as well.

<table>
<thead>
<tr>
<th>Substance</th>
<th>( \varepsilon_\infty )</th>
<th>( \varepsilon_s )</th>
<th>( \alpha )</th>
<th>( \tau ) (ps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>5</td>
<td>78</td>
<td>0</td>
<td>8.0789</td>
</tr>
<tr>
<td>Methanol</td>
<td>5.7</td>
<td>33.1</td>
<td>0</td>
<td>53.0516</td>
</tr>
<tr>
<td>Ethanol</td>
<td>4.2</td>
<td>24</td>
<td>0</td>
<td>127.8545</td>
</tr>
<tr>
<td>Acetone</td>
<td>1.9</td>
<td>21.2</td>
<td>0</td>
<td>3.3423</td>
</tr>
<tr>
<td>Ethylene glycol</td>
<td>3</td>
<td>37</td>
<td>0.23</td>
<td>79.5775</td>
</tr>
<tr>
<td>Propanol</td>
<td>3.2</td>
<td>19</td>
<td>0</td>
<td>291.7841</td>
</tr>
<tr>
<td>Butanol</td>
<td>2.95</td>
<td>17.1</td>
<td>0.08</td>
<td>477.4648</td>
</tr>
<tr>
<td>Chlorobenzene</td>
<td>2.35</td>
<td>5.63</td>
<td>0.04</td>
<td>10.2920</td>
</tr>
</tbody>
</table>
37.2 Measurement of Complex Permittivity at Low Frequencies

A parallel-plate capacitor is used to determine the complex permittivity of dielectric sheets [1,2]. For a separation $d$ between the plates of area $A$ in vacuum, the capacitance is given by

$$C_0 = \frac{8.854 A}{d} \text{ pF}$$

(37.9)

where all dimensions are measured in meters. If the two plates have different areas, then the smaller one is used to determine $C_0$. Further, it is assumed that the field distribution is uniform and perpendicular to the plates. Obviously, the fringing fields along the edges do not satisfy this condition. The guard electrodes, as shown in Figure 37.2, are used to ensure that the field distribution is close to the assumed condition. For best results, the width of the guard electrode must be at least 2$d$, and the unguarded plate must extend to outer edge of the guard electrode. Further, the gap between the guarded and guard electrodes must be as small as possible.

The radius of guarded electrode is $r_2$, and the inner radius of guard electrode is $r_1$. It is assumed that $R - r_2 \geq 2d$. The area $A$ for this parallel-plate capacitor is $\pi r^2$, where $r$ is defined as follows:

$$r = r_1 + \Delta$$

(37.10)

$$\Delta = \frac{1}{2} (r_2 - r_1) - \frac{2d}{\pi} \ln \left( \cosh \frac{\pi (r_2 - r_1)}{4d} \right) = \frac{1}{2} (r_2 - r_1) - 1.4659 d \ln \left( \cosh \frac{0.7854 (r_2 - r_1)}{d} \right)$$

(37.11)

![Figure 37.2](image-url) Geometry of a guarded capacitor.

### Table 37.2 Complex Permittivity of Some Substances at Room Temperature

<table>
<thead>
<tr>
<th>Substance</th>
<th>60 Hz</th>
<th>1 MHz</th>
<th>10 GHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nylon</td>
<td>3.60 − j0.06</td>
<td>3.14 − j0.07</td>
<td>2.80 − j0.03</td>
</tr>
<tr>
<td>Plexiglas</td>
<td>3.45 − j0.22</td>
<td>2.76 − j0.04</td>
<td>2.5 − j0.02</td>
</tr>
<tr>
<td>Polyethylene</td>
<td>2.26 − j0.0005</td>
<td>2.26 − j0.0005</td>
<td>2.26 − j0.0011</td>
</tr>
<tr>
<td>Polystyrene</td>
<td>2.55 − j0.0077</td>
<td>2.55 − j0.0077</td>
<td>2.54 − j0.0008</td>
</tr>
<tr>
<td>Styrofoam</td>
<td>1.03 − j0.0002</td>
<td>1.03 − j0.0002</td>
<td>1.03 − j0.0001</td>
</tr>
<tr>
<td>Teflon</td>
<td>2.1 − j0.01</td>
<td>2.1 − j0.01</td>
<td>2.1 − j0.0008</td>
</tr>
<tr>
<td>Glass (lead barium)</td>
<td>6.78 − j0.11</td>
<td>6.73 − j0.06</td>
<td>6.64 − j0.31</td>
</tr>
</tbody>
</table>
Permittivity Measurement

Using the Debye model (\( \alpha = 0 \) in Equation 37.8), an equivalent circuit for a dielectric-filled parallel-plate capacitor can be drawn as shown in Figure 37.3. If a step voltage \( V \) is applied to it, then the current \( I \) can be found as follows [2]:

\[
I = \varepsilon_\infty C_0 V \delta(t) + \frac{V C_0 (\varepsilon_0 - \varepsilon_\infty)}{\tau} \exp\left(-\frac{t}{\tau}\right)
\]  

(37.12)

where \( \tau = RC_0(\varepsilon_0 - \varepsilon_\infty) \).

The first term in Equation 37.12 represents the charging current of capacitor \( \varepsilon_\infty C_0 \) in the upper branch. The second term is not measured because it disappears instantaneously. In practice, it needs to be bypassed at short times to protect the detector from overloading or burning. The time constant, \( \tau \), is determined following the decay characteristics of this current. Further, the resistance \( R \) can be found after extrapolating this current-time curve to \( t = 0 \). The discharging current characteristics are used to remove \( V \) at \( t = 0 \).

A typical circuit arrangement for the characterization of dielectric materials using a step voltage is shown in Figure 37.4. A standard resistor \( R_1 \) of either 10 G\( \Omega \) or 1 T\( \Omega \) is connected between the guarded electrode and the load resistor \( R_2 \). A feedback circuit is used that forces the voltage drop across \( R_1 \) equal in magnitude but opposite in polarity to that of across \( R_2 \). It works as follows. Suppose that the node between capacitor and \( R_1 \) has a voltage \( V_1 \) with respect to ground. It is amplified but reversed in polarity by the amplifier. Therefore, the current through \( R_1 \) will change. The process continues until the input to the amplifier is zero. The junction between \( R_1 \) and the capacitor will then be at the ground potential. Thus, the meter \( M \) measures voltage across \( R_2 \) that is negative of the voltage across \( R_1 \). Since \( R_1 \) is known, the current through it can be calculated. The current also flows through the sample. \( S_2 \) is used to switch from charging to discharging mode, while \( S_1 \) is used to provide a path for surge currents.
Capacitance and dissipation factor of the dielectric-loaded parallel-plate capacitor are used in the medium frequency range to determine the complex permittivity of materials. A substitution method is generally employed in a Schering bridge circuit for this measurement.

In the Schering bridge shown in Figure 37.5, assume that the capacitor \( C_d \) is disconnected for the time being and the capacitor \( C_s \) contains the dielectric sample. In the case of a lossy dielectric sample, it can be modeled as an ideal capacitor \( C_s \) in series with a resistor \( R_s \). The bridge is balanced by adjusting \( C_d \) and \( R_s \). An analysis of this circuit under the balanced condition produces the following relations:

\[
R_x = \frac{C_dr_c}{C_T} \tag{37.13}
\]

and

\[
C_x = \frac{C_TR_d}{R_c} \tag{37.14}
\]

Quality factor \( Q \) of a series RC circuit is defined as the tangent of its phase angle, while the inverse of \( Q \) is known as the dissipation factor \( D \). Hence,

\[
Q = \frac{X_s}{R_x} = \frac{1}{\omega C_x R_x} = \frac{1}{D} \tag{37.15}
\]

For a fixed \( R_d \), the capacitor \( C_d \) can be calibrated directly in terms of dissipation factor. Similarly, the resistor \( R_c \) can be used to determine \( C_s \). However, an adjustable resistor limits the frequency range. A substitution method is preferred for precision measurement of \( C_s \) at higher frequencies. In this technique, a calibrated precision capacitor \( C_{p} \) is connected in parallel with \( C_s \) as shown in Figure 37.5, and the bridge is balanced. Assume that the settings of two capacitors at this condition are \( C_{d1} \) and \( C_{v1} \). The capacitor \( C_s \) is then removed, and the bridge is balanced again. Let the new settings of these capacitors be \( C_{d2} \) and \( C_{v2} \), respectively. Equivalent circuit parameters of the dielectric-loaded capacitor \( C_s \) are then found as follows:

\[
C_s = C_{v2} - C_{v1} \tag{37.16}
\]
Permittivity Measurement

\[ D_x = \frac{C_{x1}}{C_x} \delta D \]  
\[ \delta D = \omega R_d (C_{d1} - C_{d2}) \]  

(37.17)

where

Complex permittivity of the specimen is calculated from these data as follows:

\[ \varepsilon'_r = \frac{C_x}{C_0} \]  
\[ \varepsilon''_r = \frac{C_s D_x}{C_0} \]  

(37.18)  
(37.19)

Thus far, a series RC circuit equivalent model is used for the dielectric-loaded capacitor. As illustrated in Figure 37.6, an equivalent parallel RC model can also be obtained for it. The following equations can be used to switch back and forth between these two equivalent models:

\[ G_p = \frac{R_c}{R^2 + (1/\omega^2 C^2)} = \frac{1}{R_c} \left(1 + \left(1 + Q^2\right) \right) \]  
\[ C_p = \frac{C_s}{1 + (\omega R_s C_s)^2} = \frac{C_s}{1 + D^2} \]  
\[ R_s = \frac{G_p}{G_p + \omega^2 C_p} = \frac{1}{G_p} \left(1 + Q^2\right) \]  
\[ C_s = \frac{G_p^2 + \omega^2 C_p^2}{\omega^2 C_p} = C_p(1 + D^2) \]  

(37.20)  
(37.21)  
(37.22)  
(37.23)

and

\[ Q = \frac{1}{D} = \frac{\omega C_p}{G_p} = \frac{1}{\omega R_s C_s} \]  

(37.24)

Proper shielding and grounding arrangements are needed for a reliable measurement, especially at higher frequencies. Grounding and edge capacitances of the sample holder need to be taken into account for improved accuracy. Further, a guard point needs to be obtained that may require balancing in some cases. An inductive-ratio-arm capacitance bridge can be another alternative to consider for such application [1].

![Diagram](image)

**FIGURE 37.6** Series and parallel equivalent circuits of a dielectric-loaded capacitor.
37.3 Measurement of Complex Permittivity Using Distributed Circuits

Measurement techniques based on the lumped circuits are limited up to the lower end of the very high frequency (VHF) band. Characterization of materials at microwave frequencies requires the distributed circuits. A number of techniques have been developed on the basis of wave refraction and transmission characteristics inside a transmission line or in free space. Some other methods employ a resonant cavity that is loaded with the sample. Cavity parameters are measured, and the material characteristics are deduced from that. A number of these techniques, described in [3,4], can be used for a sheet material. These techniques require cutting a piece of sample to be placed inside a transmission line or a cavity. In case of liquid or powder samples, a so-called modified infinite sample method can be used. In this technique, a waveguide termination is filled completely with the sample, as shown in Figure 37.7. Since a tapered termination is embedded in the sample, the wave incident on it will be dissipated with negligible reflection, and it will look like the sample is extending to infinity. The impedance at its input port will depend on the electric properties of filling sample. Its voltage standing wave ratio (VSWR) \( S \) and location of first minimum \( d \) from the load plane are measured using a slotted line. The complex permittivity of sample is then calculated as follows [5]:

\[
\varepsilon'_r = \left(\frac{\lambda}{\lambda_c}\right)^2 + \frac{[1-(\lambda/\lambda_c)^2] \times [S^2 \sec^4(\beta d) - (1-S^2)^2 \tan^2(\beta d)]}{[1 + S^2 \tan^2(\beta d)]^2}
\]  

(37.25)

and

\[
\varepsilon''_r = \frac{[1-(\lambda/\lambda_c)^2] \times [2S(1-S^2) \sec^4(\beta d) \tan(\beta d)]}{[1 + S^2 \tan^2(\beta d)]^2}
\]

(37.26)

where

- \( \lambda \) is the free-space wavelength
- \( \lambda_c \) is the cutoff wavelength for the mode of propagation in empty guide
- \( \beta \) is the propagation constant in the feeding guide

It is assumed that the waveguide supports TE\(_{10}\) mode only.

37.3.1 Resonant Cavity Method

A cavity resonator can be used to determine the complex permittivity of materials at microwave frequencies. If a cavity can be filled completely with the sample, then the following procedure can be used.

**FIGURE 37.7** A waveguide termination filled with liquid or powder sample.
Permittivity Measurement

Measure the resonant frequency $f_1$ and the quality factor $Q_1$ of an empty cavity. Next, fill that cavity with the sample material and measure its new resonant frequency $f_2$ and quality factor $Q_2$. The dielectric parameters of the sample are then calculated from the following formulae [3]:

$$\varepsilon_r = \left( \frac{f_1}{f_2} \right)^2$$  \hspace{1cm} (37.27)

and

$$\tan \delta = \frac{1}{Q_2} - \frac{1}{Q_1 \sqrt{f_2}}$$  \hspace{1cm} (37.28)

On the other hand, a cavity perturbation technique will be useful for smaller samples [4]. If the sample is available in a circular cylindrical form, then it may be placed inside a TE$_{101}$ rectangular cavity through the center of its broad face where the electric field is the maximum. Its resonant frequency and quality factor with and without the sample are then measured. Complex permittivity of the sample is calculated as follows:

$$\varepsilon'_r = 1 + \frac{1}{2} \frac{f_1 - f_2}{f_2} \frac{V}{v}$$  \hspace{1cm} (37.29)

and

$$\varepsilon''_r = \frac{V}{4v} \frac{Q_2 - Q_1}{Q_1 Q_2}$$  \hspace{1cm} (37.30)

where $V$ and $v$ are cavity and sample volumes, respectively.

Similarly, for a small spherical sample of radius $r$ that is placed in a uniform field at the center of the rectangular cavity, the dielectric parameters are as follows:

$$\varepsilon'_r = \frac{a b d}{8 \pi r^3} \frac{f_1 - f_2}{f_2}$$  \hspace{1cm} (37.31)

and

$$\varepsilon''_r = \frac{a b d}{16 \pi r^3} \left( \frac{Q_2 - Q_1}{Q_1 Q_2} \right)$$  \hspace{1cm} (37.32)

where $a$, $b$, and $d$ are the width, height, and length of the rectangular cavity, respectively. For best accuracy in cavity perturbation method, the shift in frequency $(f_1 - f_2)$ must be very small.

37.3.2 Free-Space Method for Measurement of Complex Permittivity

When a plane electromagnetic wave is incident on a dielectric interface, its reflection and transmission depend on the contrast in the dielectric parameters. Many researchers have used it for determining the complex permittivity of dielectric materials placed in free space. An automatic network analyzer and phase-corrected horn antennas can be used for such measurements [6]. The system is calibrated using the TRL (through, reflect, and line) technique. A time-domain gating is used to minimize the error due to multiple reflections. The sample of thickness $d$ is placed in front of a conducting
plane, and its reflection coefficient $S_{11}$ is measured. A theoretical expression for this reflection coefficient is found as follows:

$$S_{11} = \frac{jZ_d \tan(\beta d) - 1}{jZ_d \tan(\beta d) + 1}$$

(37.33)

where

$$Z_d = \frac{1}{\sqrt{\varepsilon_r^*}}$$

(37.34)

$$\beta_d = \frac{2\pi}{\lambda} \sqrt{\varepsilon_r^*}$$

(37.35)

$\lambda$ is the free-space wavelength of electromagnetic signal.

Equation 37.33 is solved for $\varepsilon_r^*$ after substituting the measured $S_{11}$. Since it represents a nonlinear relation, an iterative numerical procedure can be used.

### 37.3.3 Electric Probe Method for Determining the Dielectric Properties of Materials

Coaxial line-excited short electric probes have been used for the in situ determination of dielectric properties. The procedure requires an accurate measurement of the input impedance of the in situ probe and its mathematical model that is dependent on the electric characteristics of the surrounding medium. The procedure is complex, and therefore, there are a number of models reported in the literature for simplification. In one of these techniques, it is assumed that the electric probe is immersed in the medium under test and a coaxial line of inner and outer radii $a$ and $b$, respectively, feeds this probe from behind the ground plane, as illustrated in Figure 37.8 [7]. The medium has conductivity $\sigma \text{ S/m}$ and the dielectric constant $\varepsilon_r$. Further, the opening $(b - a)$ is much smaller than the probe radius $a$, and the probe length $\ell$ is smaller than a quarter wavelength. The so-called induced EMF method provides an expression for the input impedance $Z_{in}$ of this probe as follows:

$$Z_{in} = \frac{j\omega \mu_0}{4\pi \beta_d \sin(\beta_d \ell)} \int_{0}^{\ell} \sin[\beta_d(\ell - z)] f(z)dz$$

(37.36)

![A coaxial line-excited short electric probe on the ground plane.](image)

**FIGURE 37.8** A coaxial line-excited short electric probe on the ground plane.
where

\[
f(z) = e^{-j\beta_0 r_1} + e^{-j\beta_0 r_2} - 2\cos(\beta_0 l) \cdot e^{-j\beta_0 r}
\]

(37.37)

\[r_1 = \sqrt{(l - z)^2 + a^2}
\]

(37.38)

\[r_2 = \sqrt{(l + z)^2 + a^2}
\]

(37.39)

\[r = \sqrt{z^2 + a^2}
\]

(37.40)

\[\frac{\omega\mu_0}{\beta_0} = \sqrt{\frac{\mu_0}{\epsilon^*}} = \eta = \text{Intrinsic impedance of the medium}
\]

(37.41)

\[\epsilon^* = \epsilon_0 \left( \epsilon_r - j \frac{\sigma}{\omega\epsilon_0} \right)
\]

(37.42)

\[\beta_0^2 = \omega^2 \mu_0 \epsilon^*
\]

(37.43)

Since the input impedance \(Z_{in}\) is known precisely after measurement, Equation 37.36 can be solved numerically for complex \(\beta_0\) using a root-finding routine. The material characteristics are then determined from Equations 37.42 and 37.43.

### 37.3.4 Nondestructive Method for Measuring the Complex Permittivity of Materials

Most of the techniques described thus far require cutting and placing a part of sample in the test fixture. Sometimes, it may not be permissible to do so. Further, the dielectric parameters can change in that process. It is especially important in the case of a biological specimen to perform in vivo measurements. In one such technique, an open-ended coaxial line is placed in close contact with the sample, and its input reflection coefficient is measured using an automatic network analyzer [8,9]. As recommended by the manufacturers, the network analyzer is calibrated initially using an open-circuit, a short-circuit, and a matched load. The reference plane is then moved to the measuring end of the coaxial line using short circuit.

Assume that \(a\) and \(b\) are inner and outer radii of the coaxial line, respectively. \(\omega\) is the angular frequency; \(\mu_0\) and \(\epsilon_0\) are the permeability and permittivity of the free space, respectively; \(k = \omega\sqrt{\mu_0\epsilon_0}\) is the wavenumber in material medium. Admittance of the coaxial aperture in contact with material medium is as follows:

\[Y_L = \frac{2}{\int_a^b E_p(\rho', 0)\,d\rho'} - \frac{2\pi}{\sqrt{(\mu_0/\epsilon_0\epsilon_1)\ln(b/a)}}
\]

(37.44)

where \(E_p(\rho', 0)\) is radial electric field intensity over the aperture. It is evaluated from the following integral equation:

\[
\frac{1}{\pi\rho} + j\omega\epsilon_0\epsilon_0 \int_a^b E_p(\rho', 0)K_1(\rho, \rho')\rho'\,d\rho' = \frac{j\omega\epsilon_0\epsilon_0}{\pi} \int_a^b E_p(\rho', 0)\rho'\,d\rho' \int_0^\pi \cos(\phi) \frac{\exp(-jkr)}{r} \,d\phi'
\]

(37.45)
where

\[ r = \sqrt{\rho^2 + \rho'^2 - 2\rho \rho' \cos(\phi')} \]  \hspace{1cm} (37.46)

\[ K_c(\rho, \rho') = j \sum_{n=0}^{\infty} \frac{\phi_n(\rho)\phi_n(\rho')}{A_n^2} \]  \hspace{1cm} (37.47)

\[ \phi_n = Y_0(\gamma_n a)J_1(\gamma_n \rho) - J_0(\gamma_n a)Y_1(\gamma_n \rho) \]  \hspace{1cm} (37.48)

\[ \beta_n = \begin{cases} \sqrt{k_1^2 - \gamma_n^2} & \text{if } k_1 > \gamma_n \\ -j\sqrt{\gamma_n^2 - k_1^2} & \text{if } k_1 < \gamma_n \end{cases} \]  \hspace{1cm} (37.49)

\[ A_n^2 = \frac{2}{\pi^2 \gamma_n^2} \left[ \frac{J_0^2(\gamma_n a)}{J_0^2(\gamma_n b)} - 1 \right] \]  \hspace{1cm} (37.50)

The eigenvalues \( \lambda_n \) are solutions to the following characteristic equation:

\[ J_0(\gamma_n b)Y_0(\gamma_n a) = J_0(\gamma_n a)Y_0(\gamma_n b) \]  \hspace{1cm} (37.51)

where

\( J_n \) and \( Y_n \) are Bessel functions of the first and second kind of order \( n \), respectively

\( \varepsilon_1 \) is the dielectric constant of the insulator

\( k_1 \) is wavenumber inside the coaxial line

Equation 37.45 is solved numerically using the method of moments. A numerical root-finding procedure, such as Muller's method, is used to solve Equation 37.44 for the complex wavenumber \( k \). Complex permittivity, in turn, is determined from the following relation:

\[ \varepsilon' = \frac{k^2}{\omega^2 \mu_0 \varepsilon_0} \]  \hspace{1cm} (37.52)

A relatively much simpler method may be used if the coaxial aperture of the probe is electrically small such that there is negligible radiation. The technique requires four standard media to calibrate the system. For example, water, open circuit, short circuit, and methanol are used to find calibration constants. Further, it does not require precise value of reflection coefficient (or input impedance), and therefore, there is no need to calibrate the automatic network analyzer to manufacturer's standards. The procedure may be summarized as follows.

With open end of the probe in free space, its reflection parameter \( R_1 \) is recorded. The probe is then dipped in distilled water and then in methanol, and the corresponding reflection parameters are recorded as \( R_2 \) and \( R_4 \). With a conductive foil tightly pressed on the aperture, the reflection parameter
Permittivity Measurement 37-13

\( R_3 \) is recorded. Note that the magnitudes of reflection parameters may be larger than unity sometimes because this setup is not calibrated for measuring the reflection coefficients. Now, a coefficient \( \chi \) is determined to find the equivalent admittance parameters at the opening as follows:

\[
\chi = \frac{(1 + \Lambda) e_{\text{m}}^* - \frac{4}{\Lambda} e_{\text{m}}^* - \frac{1}{\Lambda} e_{\text{m}}^*}{e_{\text{m}}^* + \frac{4}{\Lambda} e_{\text{m}}^* - (1 + \Lambda) e_{\text{m}}^*}
\]  

(37.53)

where \( e_{\text{m}}^\ast \), \( e_{\text{w}}^\ast \), and \( e_{\text{w}}^\ast \) are the complex relative permittivities of free space (~1), water, and methanol, respectively, at the measurement frequency, and

\[
\Lambda = \frac{(R_4 - R_1) \cdot (R_3 - R_1)}{(R_4 - R_1) \cdot (R_1 - R_3)}
\]  

(37.54)

A dimensionless equivalent admittance parameter \( y_n \) of the aperture with the \( n \)th medium is found as follows:

\[
y_n = e_m^\ast + \chi e_m^\ast
\]  

(37.55)

Now with the test-sample terminating the coaxial aperture, reflection parameter \( R_3 \) is recorded, and it is used to find the coefficient \( \xi \) as follows:

\[
\xi = \frac{(R_3 - R_4) \cdot (R_1 - R_3)}{(R_3 - R_2) \cdot (R_3 - R_1)}
\]  

(37.56)

Using Equation 37.55, the admittance parameters \( y_1 \) and \( y_2 \) for the free space and water, respectively, are computed. The admittance parameter \( y_3 \) for the sample is then found as follows:

\[
y_3 = \frac{y_1 - y_2 \xi}{1 - \xi}
\]  

(37.57)

The complex relative permittivity \( e_{\text{m}}^\ast \) of the sample can now be determined via Equation 37.55 as follows:

\[
e_{\text{m}}^\ast = \frac{-1 \pm \sqrt{1 + 4y_3 \chi}}{2\chi}
\]  

(37.58)

Only one of these two values found for \( e_{\text{m}}^\ast \) will have positive real part and negative imaginary part. The other solution is not physically possible and, hence, can be thrown.

Note: The relevant ASTM standards and a list of manufacturers are included at the end of the text.

Defining Terms

Conduction current density: Current per unit area caused by conduction of charge carriers.

Dielectric constant: A dimensionless constant that represents the permittivity of a material relative to the permittivity of free space.

Displacement current density: It represents the time rate of change of electric flux density.

Dissipation factor: It is the inverse of the quality factor.

Electric dipole: A pair of equal and opposite electric charges separated by a small distance.

Electric flux density: A fundamental electric field quantity that is related to volume density of free charges. It is also known as the electric displacement.
**Electric polarization density**: The average electric dipole moment per unit volume.

**Electric susceptibility**: A dimensionless parameter that relates the polarization density in a material with electric field intensity.

**Frequency domain field**: A phasor quantity (a complex function in general) that depends on space coordinates. The time dependency is assumed to be sinusoidal.

**Isotropic material**: A material in which the electric polarization has the same direction as the applied electric field.

**Loss tangent**: A ratio of the imaginary part to the real part of the complex permittivity of a material.

**Quality factor**: A dimensionless quantity that represents the time average energy stored in an electric circuit relative to energy dissipated in one period.

**Reflection coefficient**: Defined as a ratio of reflected phasor voltage to that of incident phasor voltage at a point in the circuit.

**Relaxation time**: It represents the time taken by a charge placed inside a material volume to decay to about 37% of its initial value.

**Time-domain field**: A field expressed as a function of time. It is a real function that is dependent on time and space coordinates.

**Voltage standing wave ratio (VSWR)**: Defined as a ratio of maximum voltage to the minimum voltage on a transmission line.

### Related ASTM Standards

<table>
<thead>
<tr>
<th>Standard</th>
<th>Material Type</th>
<th>Frequency Range</th>
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</thead>
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<td>Microwaves</td>
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<tr>
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<td>Solid insulating materials</td>
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<tr>
<td>D5493-06</td>
<td>Geotextiles</td>
<td></td>
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<tr>
<td>D7449/D7449M-08</td>
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<td>D150-11</td>
<td>Uses lumped impedance standards</td>
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<tr>
<td>D2149-97</td>
<td>Solids</td>
<td>50 Hz–10 MHz</td>
</tr>
</tbody>
</table>

### References


**Relevant Websites**

http://www.mtt.org
http://ewh.ieee.org/soc/deis
http://iopscience.iop.org
http://www.nist.gov
http://www.npl.co.uk

**Selected Test-Instrument Manufacturers**

2. QuadTech ([www.quadtech.com](http://www.quadtech.com)): LD-3 dielectric cell for liquids, solids, or powdered material. Requires an LCR meter.
3. Scientifica ([www.scientifica.com](http://www.scientifica.com)): Liquid dielectric constant meter Model 871; measurement is performed at 10 kHz.
4. Brookhaven ([www.brookhaveninstruments.com](http://www.brookhaveninstruments.com)): B1-870 dielectric constant meter, used for measurements on liquids at 10 kHz.