3.1 Introduction

The primary purpose of this chapter is to present dynamic methods for analyzing bridge structures when subjected to earthquake loads. Basic concepts and assumptions used in typical dynamic analysis are presented first. Various approaches to bridge dynamics are then discussed. Several examples are presented to illustrate their practical applications.

3.1.1 Static versus Dynamic Analysis

The main objectives of a structural analysis are to evaluate structural behavior under various loads and to provide the information necessary for design such as forces, moments, and deformations. Structural analysis can be classified as “static” or “dynamic, whereas statics deals with time-independent loading, dynamics considers any load where the magnitude, direction, and position vary with time. Typical dynamic loads for a bridge structure include vehicular motions and wave actions such as winds, stream flow, and earthquakes.

3.1.2 Characteristics of Earthquake Ground Motions

An earthquake is a natural ground movement caused by various phenomena including global tectonic processes, volcanism, landslides, rock-bursts, and explosions. The global tectonic processes are continually producing mountain ranges and ocean trenches at the earth’s surface and causing earthquakes.
This section briefly discusses the earthquake input for seismic bridge analysis. Detailed discussions of ground motions are presented in Chapter 1.

The ground motion is represented by the time history or seismograph in terms of acceleration, velocity, and displacement for a specific location during an earthquake. Time history plots contain complete information about the earthquake motions in the three orthogonal directions (two horizontal and one vertical) at the strong-motion instrument location. Acceleration is usually recorded by strong-motion accelerograph and the velocities and displacements are determined by numerical integration. The accelerations recorded at locations that are approximately same distance away from the epicenter may differ significantly in duration, frequency content, and amplitude because of the different local soil conditions. Figure 3.1 shows several time histories of recent earthquakes.

From a structural engineering view, the most important characteristics of an earthquake are the peak ground acceleration (PGA), duration, and frequency content. The PGA is the maximum acceleration and represents the intensity of a ground motion. Although the ground velocity may be a more significant measure of intensity than the acceleration, it is not often measured directly, but determined using supplementary calculations (Clough and Penzien, 1993). The duration is the length of time between the first and the last peak exceeding a specified strong motion level. The longer the duration of a strong motion, the more energy is imparted to a structure. Because the elastic strain energy absorbed by a structure is very limited, a longer strong earthquake has a greater possibility to enforce a structure into inelastic range. The frequency content can be represented by the number of zero crossings per second in the accelerogram. It is well understood that when the frequency of a regular disturbing force is the same as the natural vibration frequency of a structure (resonance), the oscillation of structure can be greatly magnified and effects of damping become minimal. Although earthquake motions are never as regular as a sinusoidal waveform, there is usually a period that dominates the response.

Because it is impossible to measure detailed ground motions for all structure sites, the rock motions or ground motions are estimated at a fault and then propagated to the earth surface using a computer

![Ground motions recorded during 1989 Loma Prieta and 1994 Northridge earthquakes.](image-url)

**FIGURE 3.1** Ground motions recorded during 1989 Loma Prieta and 1994 Northridge earthquakes.
program considering the local soil conditions. Two guidelines (Caltrans, 1996a and 1996b) developed by the California Department of Transportation provide the methods to develop seismic ground motions for bridges.

3.1.3 Dynamic Analysis Methods for Seismic Bridge Design

Depending on the seismic zone, geometry, and importance of the bridge, the following analysis methods may be used for seismic bridge design:

- The single-mode method (single-mode spectral and uniform load analysis) (AASHTO, 2011) assumes that seismic load can be considered as an equivalent static horizontal force applied to an individual frame in either the longitudinal or transverse direction. The equivalent static force is based on the natural period of a single degree of freedom (SDOF) and code-specified response spectra. Engineers should recognize that the single-mode method (sometimes referred to as equivalent static analysis) is best suited for structures with well-balanced spans with equally distributed stiffness.

- Multi-mode spectral analysis assumes that member forces, moments, and displacements because of seismic load can be estimated by combining the responses of individual modes using the methods such as Complete Quadratic Combination (CQC) method and the Square Root of the Sum of the Squares (SRSS) method. The CQC method is adequate for most bridge systems (Wilson et al., 1981; Wilson, 2009; Menun and Kiureghian, 1998) and the SRSS method is best suited for combining responses of well-separated modes.

- The multiple support response spectrum (MSRS) method provides response spectra and the peak displacements at individual support degrees of freedom (DOF) by accurately accounting for the spatial variability of ground motions including the effects of incoherence, wave passage, and spatially varying site response. This method can be used for multiply supported long structures (Kiureghian et al., 1997).

- Time history method is a numerical step-by-step integration of equation of motions. It is usually required for critical/important or geometrically complex bridges. Inelastic analysis provides a more realistic measure of structural behavior when compared to an elastic analysis.

Selection of the analysis method for a specific bridge structure should not purely be based on performing structural analysis, but be based on the effective design decisions (Powell, 1997). The detailed discussions of above methods are presented in the following sections.

3.2 Single Degree of Freedom System

The familiar spring-mass system represents the simplest dynamic model and is shown in Figure 3.2a. When the idealized, undamped structures are excited by either moving the support or by displacing the mass in one direction, the mass oscillates about the equilibrium state forever without coming to rest. But, real structures do come to rest after a period of time because of a phenomenon called damping. To incorporate the effect of the damping, a massless viscous damper is always included in the dynamic model as shown in Figure 3.2b.

In a dynamic analysis, the number of displacements required to define the displaced positions of all the masses relative to their original positions is called the number of DOF. When a structural system can be idealized with a single-mass concentrated at one location and moved only in one direction, this dynamic system is called SDOF system. Some of the structures such as a water tank supported by a single column, one-story frame structure, and two-span bridges supported by a single column could be idealized as SDOF models (Figure 3.3).

In the SDOF system shown in Figure 3.3c, mass of the bridge superstructure is the mass of the dynamic system. The stiffness of the dynamic system is the stiffness of the column against the sidesway and viscous damper of the system is the internal energy absorption of the bridge structure.
The response of a structure depends on its mass, stiffness, damping, and applied load or displacement. The structure could be excited by applying an external force $p(t)$ on its mass or by a ground motion $u(t)$ at its supports. In this chapter, since the seismic loading is induced by exciting the support, we focus mainly on the equations of motion of a SDOF system subjected to a ground excitation.

The displacement of the ground motion $u_g$, the total displacement of the single mass $u_t$, and the relative displacement between the mass and ground $u$ (Figure 3.4) are related by

$$u_t = u + u_g \tag{3.1}$$

By applying the Newton’s law and D’Alembert’s principle of dynamic equilibrium, it can be shown that

$$f_t + f_D + f_s = 0 \tag{3.2}$$

**FIGURE 3.2** Idealized dynamic model (a) undamped SDOF system; (b) damped SDOF system.

**FIGURE 3.3** Examples of SDOF structures (a) water tank supported by single column; (b) one-story frame building; (c) two-span bridge supported by single column.

### 3.2.1 Equation of Motion

The response of a structure depends on its mass, stiffness, damping, and applied load or displacement. The structure could be excited by applying an external force $p(t)$ on its mass or by a ground motion $u(t)$ at its supports. In this chapter, since the seismic loading is induced by exciting the support, we focus mainly on the equations of motion of a SDOF system subjected to a ground excitation.

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By applying the Newton’s law and D’Alembert’s principle of dynamic equilibrium, it can be shown that

$$f_t + f_D + f_s = 0 \tag{3.2}$$
where $f_I$ is the inertial force of the single mass and related to the acceleration of the mass by $f_I = m\ddot{u}_i$; $f_D$ is the damping force on the mass and related to the velocity across the viscous damper by $f_D = c\dot{u}$; $f_S$ is the elastic force exerted on the mass and related to the relative displacement between the mass and the ground by $f_S = ku$, where $k$ is the spring constant; $c$ is the damping ratio; and $m$ is the mass of the dynamic system.

Substituting these expressions for $f_I$, $f_D$, and $f_S$ into Equation 3.2 gives:

$$m\ddot{u}_i + c\dot{u} + ku = 0 \quad (3.3)$$

The equation of motion for a SDOF system subjected to a ground motion can then be obtained by substituting the Equation 3.1 with Equation 3.3, and is given by

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g \quad (3.4)$$

### 3.2.2 Characteristics of Free Vibration

To determine the characteristics of the oscillations such as the time to complete one cycle of oscillation ($T_s$) and number of oscillation cycles per second ($\omega_s$), we first look at the free vibration of a dynamic system. Free vibration is typically initiated by disturbing the structure from its equilibrium state by an external force or displacement. Once the system is disturbed, the system vibrates without any external input. Thus, the equation of motion for free vibration can be obtained by setting $\ddot{u}_g$ to zero in Equation 3.4 and is given by

$$m\ddot{u} + c\dot{u} + ku = 0 \quad (3.5)$$

Dividing the Equation 3.5 by its mass $m$ will result in

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0 \quad (3.6)$$

$$\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2u = 0 \quad (3.7)$$

where $\omega_n = \sqrt{k/m}$ the natural circular frequency of vibration or the undamped frequency; $\xi = \frac{c}{\omega_n}$ the damping ratio; $c_{cr} = 2m\omega_n = 2\sqrt{k/m} = \frac{2k}{\omega_n}$ the critical damping coefficient.
Figure 3.5a shows the response of a typical idealized, undamped SDOF system. The time required for the SDOF system to complete one cycle of vibration is called natural period of vibration \( T_n \) of the system and is given by

\[
T_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}
\]  

(3.8)

Furthermore, the natural cyclic frequency of vibration \( f_n \) is given by

\[
f_n = \frac{\omega_n}{2\pi} = 2\pi\sqrt{\frac{k}{m}}
\]  

(3.9)

Figure 3.5b shows the response of a typical damped SDOF structure. The circular frequency of the vibration or damped vibration frequency of the SDOF structure, \( \omega_d \), is given by

\[
\omega_d = \sqrt{\omega_n^2 - \xi^2 \omega_n^2}
\]

The damped period of vibration \( T_d \) of the system is given by

\[
T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\xi^2}}\sqrt{\frac{m}{k}}
\]  

(3.10)

When \( \xi = 1 \) or \( c = c_c \) the structure returns to its equilibrium position without oscillating and is referred to as critically damped structure. When \( \xi > 1 \) or \( c > c_c \), the structure is “overdamped” and comes to rest without oscillating, but at a slower rate. When \( \xi < 1 \) or \( c < c_c \), the structure is “underdamped” and oscillates about its equilibrium state with progressively decreasing amplitude. Figure 3.6 shows the response of SDOF structures with different damping ratios.

FIGURE 3.5 Typical response of an idealized SDOF system (a) undamped; (b) damped.
For structures such as buildings, bridges, dams, and offshore structures, the damping ratio is <0.15 and thus can be categorized as underdamped structures. The basic dynamic properties estimated using damped or undamped assumptions are approximately the same. For example, when $\xi = 0.10$, $\omega_n = 0.995\omega_n$ and $T_n = 1.01T_n$.

Damping dissipates the energy out of a structure in opening and closing of micro cracks in concrete, stressing of nonstructural elements, and friction at connection of steel members. Thus, damping coefficient accounts for all energy dissipating mechanisms of the structure and can only be estimated by experimental methods. Two seemingly identical structures may have slightly different material properties and may dissipate energy at different rates. Since damping does not play an important quantitative role except for resonant responses in structural responses, it is common to use an average damping ratios based on types of construction materials. Relative damping ratio for common types of structures such as welded metal of 2%–4%, bolted metal structures of 4%–7%, prestressed concrete structures of 2%–5%, reinforced concrete structures of 4%–7%, and wooden structures of 5%–10% are recommended by Chmielewski et al. (1996).

**3.2.3 Response to Earthquake Ground Motion**

A typical excitation of an earth movement is shown in Figure 3.7. The basic equation of motion of a SDOF system is expressed in Equation 3.4. Since the excitation force $\dot{m}\ddot{u}_g$ cannot be described by simple mathematical expression, closed form solutions for Equation 3.4 are not available. Thus, the entire ground excitation needs to be treated as a superposition of short-duration impulses to evaluate the response of the structure to the ground excitation. An impulse is defined as the product of the force times duration. For example, the impulse of the force at time $\tau$ during the time interval $d\tau$ equals $-\dot{m}\ddot{u}_g(\tau)d\tau$ and is represented by the shaded area in Figure 3.7. The total response of the structure for the earthquake motion can then be obtained by integrating all responses of the increment impulses. This approach is sometimes referred to as “Time History Analysis.” Various solution techniques are available in technical literature on structural dynamics (Clough and Penzien, 1993; Chopra, 2007).

In seismic structural design, designers are interested in the maximum or extreme values of the response of a structure as discussed in the following sections. Once the dynamic characteristics ($T_n$ and $\omega_n$) of the structure are determined, the maximum displacement, moment, and shear on the SDOF system can easily be estimated using basic principles of mechanics.
3.2.4 Response Spectra

The response spectrum is defined as a relationship of the peak values of a response quantity (acceleration, velocity, or displacement) with a structural dynamic characteristic (natural period or frequency). The core concept of the response spectrum in earthquake engineering provides a much more convenient and meaningful measure of effects of an earthquake than any other single quantity. It represents the peak response of all possible SDOF systems to a particular ground motion.

3.2.4.1 Elastic Response Spectrum

The elastic response spectrum is the response spectrum of an elastic structural system and can be obtained by the following steps (Chopra, 2007):

1. Define the ground acceleration time history (typically at a 0.02 second interval).
2. Select the natural period $T_n$ and damping ratio $\xi$ of an elastic SDOF system.
3. Compute the deformation response $u(t)$ using any numerical method.
4. Determine $u_o$, the peak value of $u(t)$.
5. Calculate the spectral ordinates by $D = u_o, V = 2\pi D/T_n$ and $A = (2\pi/T_n)^2 D$.
6. Repeat steps 2 and 5 for a range of $T_n$ and $\xi$ values for all possible cases.
7. Construct results graphically to produce three separate spectra as shown in Figure 3.8 or a combined tripartite plot as shown in Figure 3.9.

It is noted that although three spectra (displacement, velocity, and acceleration) for a specific ground motion contain the same information, each of them provides a physically meaningful quantity. The displacement spectrum presents the peak displacement. The velocity spectrum is related directly to the peak strain energy stored in the system. The acceleration spectrum is related directly to the peak value of the equivalent static force and base shear.

A response spectrum (Figure 3.9) can be divided into three ranges of periods (Chopra, 2007):

- Acceleration-sensitive region (very short-period region): A structure with a very short period is extremely stiff and expected to deform very little. Its mass moves rigidly with the ground and its peak acceleration approximately equals to the ground acceleration.
- Velocity-sensitive region (intermediate period region): A structure with an intermediate period responds greatly to the ground velocity than other ground motion parameters.
- Displacement-sensitive region (very long-period region): A structure with a very long period is extremely flexible and expected to remain stationary while the ground moves. Its peak deformation is closer to the ground displacement. The structural response is most directly related to ground displacement.
FIGURE 3.8 Example of response spectra (5% critical damping) for 1989 Loma Prieta earthquake.

Response spectra: PSV, PSA & SD
Fourier amplitude spectrum: FS
Damping values: 0, 2, 5, 10, 20%

FIGURE 3.9 Tripartite plot—response spectra (Northridge earthquake, Arleta—Rordhoff Ave. Fire Station).
3.2.4.2 Elastic Design Spectrum

Since seismic bridge design is intended to resist future earthquakes, use of a response spectrum obtained from a particular past earthquake motion is inappropriate. In addition, jagged spectrum values over small ranges would require an unreasonable accuracy in the determination of the structure period (Lindeburg, 1998). It is also impossible to predict a jagged response spectrum in all its details for a ground motion that may occur in the future. To overcome these shortcomings, the elastic design spectrum, a smoothened idealized response spectrum is usually developed to represent the envelopes of ground motions recorded at the site during past earthquakes. The development of elastic design spectrum is based on statistical analysis of the response spectra for the ensemble of ground motions. Figure 3.10 shows a set of elastic design spectrum in Caltrans Seismic Design Criteria (SDC) (Caltrans, 2013). Figure 3.11 shows project-specific acceleration response spectra for the California Sonoma Creek Bridge.

Engineers should recognize the conceptual differences between a response spectrum and a design spectrum (Chopra, 2007). A response spectrum is only the peak response of all possible SDOF systems because of a particular ground motion, whereas a design spectrum is a specified level of seismic design forces or deformations and is the envelope of two different elastic design spectra. The elastic design spectrum provides a basis for determining the design force and deformation for elastic SDOF systems.

3.2.4.3 Inelastic Response Spectrum

A bridge structure may experience inelastic behavior during a major earthquake. The typical elastic and elasto-plastic responses of an idealized SDOF to severe earthquake motions are shown in Figure 3.12. The input seismic energy received by a bridge structure is dissipated by both viscous damping and yielding (localized inelastic deformation converting into heat and other irreversible forms of energy). Both viscous damping and yielding reduce the response of inelastic structures compared to elastic structures. Viscous damping represents the internal friction loss of a structure when deformed and is approximately a constant because it depends mainly on structural materials. Yielding, however, varies depending on structural materials, structural configurations, and loading patterns and histories. Damping has negligible effects on the response of structures for the long- and short-period systems and is most effective in reducing response of structures for intermediate-period systems.

In seismic bridge design, a main objective is to ensure that a structure is capable of deforming in a ductile manner when subjected to a larger earthquake loading. It is desirable to consider inelastic response of a bridge system to a major earthquake. Although a nonlinear inelastic dynamic analysis is not difficult in concept, it requires careful structural modeling and intensive computing effort (Powell, 1997). To consider inelastic seismic behavior of a structure without performing a true nonlinear inelastic analysis, the ductility-factor method can be used to obtaining the inelastic response spectra from the elastic response spectra. The ductility of a structure is usually referred as the displacement ductility factor \( \mu \) defined by (Figure 3.13)

\[
\mu = \frac{\Delta_u}{\Delta_y} \tag{3.11}
\]

where \( \Delta_u \) is ultimate displacement capacity and \( \Delta_y \) is yield displacement.

A simplest approach to develop the inelastic design spectrum is scale the elastic design spectrum down by some function of the available ductility of a structural system

\[
\text{ARS}_{\text{inelastic}} = \frac{\text{ARS}_{\text{elastic}}}{f(\mu)} \tag{3.12}
\]

\[
f(\mu) = \begin{cases} 
1 & \text{For } T_n \leq 0.03 \text{ s} \\
2\mu - 1 & \text{For } 0.03 \text{ s} < T_n \leq 0.5 \text{ s} \\
\mu & \text{For } T_n > 0.5 \text{ s}
\end{cases} \tag{3.13}
\]
Note: The 5 lowest curves (corresponding to 0.1g–0.5g PGA at $V_{S30} = 760$ m/s) are based on a vertical strike-slip surface rupture. The highest 3 curves (corresponding to 0.6–0.8g PGA) are based on a 45-degree dipping reverse surface rupture. Where the 0.5g PGA strike-slip curve exceeds the reverse fault curves, the strike-slip curve is used.

FIGURE 3.10 Typical Caltrans elastic design response spectra.

FIGURE 3.11 Acceleration response spectra for Sonoma Creek Bridge.
FIGURE 3.12  Response of a SDOF to earthquake ground motion (a) elastic system; (b) inelastic system.

FIGURE 3.13  Lateral load—displacement relations.
For very short period \((T_n \leq 0.03\ s)\) in the acceleration-sensitive region, the elastic displacement demand \(\Delta_{ed}\) is less than displacement capacity \(\Delta_u\) (see Figure 3.13). The reduction factor \(f(\mu) = 1\) implies that the structure should be designed and remained at elastic to avoid excessive inelastic deformation. For intermediate period \((0.03\ s < T_n \leq 0.5\ s)\) in the velocity sensitive region, elastic displacement demand \(\Delta_{ed}\) may be greater or less than displacement capacity \(\Delta_u\) and reduction factor is based on the equal energy concept. For the very-long period \((T_n > 0.5\ s)\) in the displacement-sensitive region, the reduction factor is based on the equal displacement concept.

### 3.2.5 Example of a Single Degree of Freedom System

Given: A SDOF bridge structure is shown in Figure 3.14. To simplify the problem, the bridge is assumed to move only in longitudinal direction. The total resistance against the longitudinal motion comes in the form of friction at bearings and this could be considered as a damper. Assume the following properties for the structure: damping ratio \(\xi = 0.05\); area of superstructure \(A = 38.43\ ft^2\ (3.57\ m^2)\); moment of column \(I_c = 12\ ft^4\ (0.1036\ m^4)\); elastic modulus of column \(E = 3,000\ ksi\ (20,700\ MPa)\); material density \(\rho = 150\ lb/ft^3\ (2,400\ kg/m^3)\); length of column \(L_c = 30\ ft\ (9.14\ m)\); and length of the superstructure \(L_s = 120\ ft\ (36.6\ m)\).

The acceleration response curve of the structure is given in the Figure 3.11. Determine (1) natural period of the structure, (2) damped period of the structure, (3) maximum displacement of the superstructure, and (4) maximum moment in the column.

Solution:

**Stiffness:**

\[
k = \frac{12EI_c}{L_c^3} = \frac{12(3,000)(12)(12^2)}{30^3} = 2,304\ \text{kip/ft}\ (33,624,346\ \text{N/m})
\]

**Mass:**

\[
m = \frac{AL_c\rho}{g} = \frac{(38.43)(120)(0.15)}{32.174} = 20.878\ \text{kip/ft/s}^2\ (313,768\ \text{kg/s}^2)
\]

Natural circular frequency \(\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2,304}{20.878}} = 10.35\ \text{rad/s}\)

Natural cyclic frequency \(f_n = \frac{\omega_n}{2\pi} = \frac{10.35}{2\pi} = 1.65\ \text{cycles/s}\)

Natural period of the structure \(T_n = \frac{1}{f_n} = \frac{1}{1.65} = 0.606\ s\)

**FIGURE 3.14** SDOF bridge example (a) two-span bridge schematic diagram; (b) single-column bent; (c) idealized equivalent model for longitudinal response.
The damped circular frequency is given by,
\[ \omega_d = \omega_n \sqrt{1 - \xi^2} = 10.36 \sqrt{1 - 0.05^2} = 10.33 \text{ rad/sec} \]

The damped period of the structure is given by
\[ T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{10.33} = 0.608 \text{ sec} \]

From the ARS curve, for a period of 0.606 seconds, the maximum acceleration of the structure will be \( 0.9 \text{ g} = 0.9 \times 32.174 = 28.96 \text{ ft/s}^2 (11.10 \text{ m/s}^2) \).

Then, the force acting on the mass \( F = m(28.96) = (20.878)(28.96) = 604.63 \text{ kip} (2.69 \text{ MN}) \)

The maximum displacement \( u_{\text{max}} = \frac{F}{k} = \frac{604.42}{2,304} = 0.262 \text{ ft} (0.079 \text{ m}) \)

The maximum moment in the column \( M = \frac{FL}{2} = \frac{(604.63)(30)}{2} = 9,069.45 \text{ kip-ft} (12.30 \text{ MN-m}) \)

### 3.3 Multi-Degree of Freedom System

The SDOF approach may not be applicable for complex structures such as multilevel frame structure and bridges with several supports. To predict the response of a complex structure, the structure is discretized with several members of lumped masses. As the number of lumped masses increases, number of displacements required to define the displaced positions of all masses increases. The response of a multi degree of freedom (MDOF) system is discussed in this section.

#### 3.3.1 Equation of Motion

The equation of motion of a MDOF system is similar to the SDOF system, but the stiffness \( k \), mass \( m \), and damping \( c \) are matrices. The equation of motion to a MDOF system under ground motion can be written as

\[ [M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = -[M]\{B\}\ddot{u}_g \]  

(3.14)

The stiffness matrix \([K]\) can be obtained from standard static displacement-based analysis models and may have off-diagonal terms. The mass matrix \([M]\) because of the negligible effect of mass coupling can best be expressed in the form of tributary lumped masses to the corresponding displacement DOFs, resulting in a diagonal or uncoupled mass matrix. The damping matrix \([C]\) accounts for all the energy dissipating mechanisms in the structure and may have off-diagonal terms. The vector \([B]\) is a displacement transformation vector that has values 0 and 1 to define DOFs to which the earthquake loads are applied.

#### 3.3.2 Free Vibration and Vibration Modes

To better understand the response of MDOF systems, we look at the undamped, free vibration of \( N \) degrees of freedom (\( N \)-DOF) system first.
3.3.2.1 Undamped Free Vibration

By setting $[C]$ and $\ddot{u}_k$ to zero in the Equation 3.14, the equation of motion of undamped, free vibration of $N$-DOF system can be shown as

$$[M]\{\ddot{u}\} + [K]\{u\} = 0$$

(3.15)

where $[M]$ and $[K]$ are $n \times n$ square matrices.

Equation 3.15 could then be rearranged to

$$[[K] - \omega^2_n[M]]\{\phi_n\} = 0$$

(3.16)

where $\{\phi_n\}$ is the deflected shape matrix. Solution to this equation can be obtained by setting

$$[[K] - \omega^2_n[M]] = 0$$

(3.17)

The roots or eigenvalues of the Equation 3.17 will be the $N$ natural frequencies of the dynamic system. Once the natural frequencies ($\omega_n$) are estimated, Equation 3.16 can be solved for the corresponding $N$ independent, deflected shape matrices (or eigenvectors), $\{\phi_n\}$. In other words, a vibrating system with $N$-DOFs will have $N$ natural frequencies (usually arranged in sequence from smallest to largest), corresponding $N$ natural periods $T_n$, and $N$ natural mode shapes $\{\phi_n\}$. These eigenvectors are sometimes referred to as natural modes of vibration or natural mode shapes of vibration. It is important to recognize that the eigenvectors or mode shapes represent only the deflected shape corresponding to the natural frequency, not the actual deflection magnitude.

The $N$ eigenvectors can be assembled in a single $n \times n$ square matrix $[\Phi]$, modal matrix, where each column represents the coefficients associated with the natural mode. One of the important aspects of these mode shapes is that they are orthogonal to each other. Stated mathematically

If $\omega_n \neq \omega$, $\{\Phi_n\}^T [K] \{\phi_n\} = 0$ and $\{\Phi_n\}^T [M] \{\phi_n\} = 0$

(3.18)

$$[K^*] = [\Phi]^T [K] [\Phi]$$

(3.19)

$$[M^*] = [\Phi]^T [M] [\Phi]$$

(3.20)

where $[K]$ and $[M]$ have off diagonal elements, whereas $[K^*]$ and $[M^*]$ are diagonal matrices.

3.3.2.2 Damped-Free Vibration

When damping of the MDOF system is included, the free vibration response of the damped system will be given by

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = 0$$

(3.21)

The displacements are first expressed in terms of natural mode shapes, and later they are multiplied by the transformed natural mode matrix to obtain the following expression:

$$[M^*] \{\ddot{Y}\} + [C^*] \{\dot{Y}\} + [K^*] \{Y\} = 0$$

(3.22)

where, $[M^*]$ and $[K^*]$ are diagonal matrices given by Equations 3.19 and 3.20 and

$$[C^*] = [\Phi]^T [C] [\Phi]$$

(3.23)
Although $[M^*]$ and $[K^*]$ are diagonal matrices, $[C^*]$ may have off diagonal terms. When $[C^*]$ has off diagonal terms, the damping matrix is referred to as nonclassical or nonproportional damping matrix. When $[C^*]$ is diagonal, it is referred to as classical or proportional damping matrix. Classical damping is an appropriate idealization when similar damping mechanisms are distributed throughout the structure. Nonclassical damping idealization is appropriate for the analysis when the damping mechanisms differ considerably within a structural system.

Since most bridge structures have predominantly one type of construction material, bridge structures could be idealized as classical damping structural system. Thus, the damping matrix of Equation 3.22 will be a diagonal matrix for most bridge structures. And, the equation of $n$th mode shape or generalized $n$th modal equation is given by

$$\ddot{Y}_n + 2\xi_n \omega_n \dot{Y}_n + \omega_n^2 Y_n = 0 \quad (3.24)$$

Equation 3.24 is similar to the Equation 3.7 of a SDOF system. And, the vibration properties of each mode can be determined by solving the Equation 3.24.

### 3.3.2.3 Rayleigh Damping

The damping of a structure is related to the amount of energy dissipated during its motion. It could be assumed that a portion of the energy lost because of the deformations and thus damping could be idealized as proportion to the stiffness of the structure. Another mechanism of energy dissipation could be attributed to the mass of the structure and thus damping is idealized as proportion to the mass of the structure. In Rayleigh damping it is assumed that the damping is proportional to the mass and stiffness of the structure.

$$[C] = a_o [M] + a_1 [K] \quad (3.25)$$

The generalized damping of $n$th mode, is then given by

$$C_n = a_o M_n + a_1 K_n \quad (3.26)$$

$$C_n = a_o M_n + a_1 \omega_n^2 M_n \quad (3.27)$$

$$\xi_n = \frac{C_n}{2M_n \omega_n} \quad (3.28)$$

$$\xi_n = \frac{a_o}{2} \frac{1}{\omega_n} + \frac{a_1}{2} \omega_n \quad (3.29)$$

Figure 3.15 shows the Rayleigh damping variation with natural frequency. The coefficients $a_o$ and $a_1$ can be determined from specified damping ratios at two independent dominant modes (say $i$th and $j$th modes). Expressing Equation 3.29 for these two modes will lead to the following equations:

$$\xi_i = \frac{a_o}{2} \frac{1}{\omega_i} + \frac{a_1}{2} \omega_i \quad (3.30)$$

$$\xi_j = \frac{a_o}{2} \frac{1}{\omega_j} + \frac{a_1}{2} \omega_j \quad (3.31)$$
When the damping ratio at both $i$th and $j$th modes is the same and equals $\xi$, it can be shown that

$$a_{ij} = \xi (\omega_i + \omega_j)$$

It is important to note that the damping ratio at a mode between the $i$th and $j$th mode is less than $\xi$. And, in practical problems the specified damping ratios should be chosen to ensure reasonable values in all the mode shapes that lie between the $i$th and $j$th mode shapes.

### 3.3.3 Modal Analysis and Modal Participation Factor

In previous sections, we have discussed the basic vibration properties of a MDOF system. Now, we will look at the response of a MDOF system to earthquake ground motion. The basic equation of motion of the MDOF for an earthquake ground motion is given by the Equation 3.14, which is repeated here:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = -[M]\{B\}\ddot{g}$$

The displacement is first expressed in terms of natural mode shapes, and later it is multiplied by the transformed natural mode matrix to obtain the following expression:

$$[M^*]\{\ddot{y}\} + [C^*]\{\dot{y}\} + [K^*]\{y\} = -[\Phi]^* [M] [B]\ddot{g}$$

And, the equation of $n$th mode shape is given by

$$M_n^* \dddot{y}_n + 2\xi_n \omega_n M_n^* \ddot{y}_n + \omega^2 M_n^* Y_n = L_n \dddot{g}$$

where

$$M_n^* = \{\phi_n\}^T [M] \{\phi_n\}$$

$$L_n = -\{\phi_n\}^T [M] [B]$$

**FIGURE 3.15** Rayleigh damping variation with natural frequency.
The $L_n$ is referred to as the “modal participation factor” of $n$th mode.

By dividing the Equation 3.34 by $M_n$, the generalized modal equation of $n$th mode becomes

$$\ddot{Y}_n + 2\xi_n \omega_n \dot{Y}_n + \omega_n^2 Y_n = \left( \frac{L_n}{M_n} \right) \ddot{u}_g$$

(3.37)

Equation 3.34 is similar to the equation motion of a SDOF system and thus $Y_n$ can be determined by using similar methods described for SDOF systems. Once $Y_n$ is established, the displacement because of the $n$th mode will be given by $u_n(t) = \phi_n Y_n(t)$. The total displacement because of combination of all mode shapes can then be determined by summing up all displacements for each mode and is given by

$$u(t) = \sum \phi_n Y_n(t)$$

(3.38)

This approach is sometimes referred to as classical mode superposition method. Similar to the estimation of the total displacement, the element forces can also be estimated by adding the element forces for each mode shapes.

### 3.3.4 Example of a MDOF System

Given: The bridge shown in Figure 3.16 is a three-span continuous frame structure. Details of the bridge are as follows: Span lengths are 60 ft + 80.4 ft + 60 ft (18.3 m + 24.5 m + 18.3 m); column length is 30.17 ft (9.5 m); area of superstructure is 60.1 ft$^2$ (5.58 m$^2$); moment of inertia of superstructure is 9,356 ft$^4$ (70.77 m$^4$); moment of inertia of column is (25.25 ft$^4$) 0.218 m$^4$; modulus of elasticity of concrete is 3,000 ksi (20,700 MPa). Determine the vibration modes and frequencies of the bridge.

Solution: As shown in Figure 3.16b, c, and d, five DOF are available for this structure. Stiffness and mass matrices are estimated separately and the results are given here.

\[
[K] = \begin{bmatrix}
126318588 & 0 & 0 & 0 & 0 \\
0 & 1975642681 & -1194370500 & -1520122814 & -14643288630 \\
0 & -1194370500 & 1975642681 & 14643288630 & 1520122814 \\
0 & -1520122814 & 14643288630 & 479327648712 & 119586857143 \\
0 & -14643288630 & 1520122814 & 119586857143 & 479327648712 \\
\end{bmatrix}
\]

\[
[M] = \begin{bmatrix}
81872 & 0 & 0 & 0 & 0 \\
0 & 286827 & 0 & 0 & 0 \\
0 & 0 & 286827 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \text{ Kg}
\]

Condensation procedure will eliminate the rotational DOF and will result in three DOF. (Condensation procedure is done separately and the result is given here.) The equation of motion of free vibration of the structure is

\[
[M] \ddot{u} + [K] u = 0
\]

Substituting condensed stiffness and mass matrices with the above equation gives
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Above equation can be rearranged in the following form:

\[
\frac{1}{\omega^2} [M]^{-1} [K] \{\phi\} = \{\phi\}
\]

Substitution of appropriate values in the above expression gives the following:
By assuming different vibration modes, natural frequencies of the structure can be estimated. Substitution of vibration mode \( \{100\}^T \) will result in the first natural frequency.

\[
\frac{1}{\omega_n^2} \begin{bmatrix}
154.39 & 0 & 0 \\
0 & 5292.9 & -4238.2 \\
0 & -4238.2 & 5292.9 \\
\end{bmatrix}
\begin{bmatrix}
\phi_{1n} \\
\phi_{2n} \\
\phi_{3n} \\
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
\]

Thus, \( \omega_1^2 = 154.39 \) and \( \omega_n = 12.43 \) rad/s.

By substituting the vibration modes of \( \{0 \ 1 \ 1\}^T \) and \( \{0 \ 1 \ -1\}^T \) in the above expression, other two natural frequencies are estimated as 32.48 rad/s and 97.63 rad/s.

### 3.3.5 Multiple-Support Excitation

So far we have assumed that all supports of a structural system undergo the same ground motion. This assumption is valid for structure with foundation supports close to each other. However, for long-span bridge structures, supports may be widely spaced. As described in Section 3.1.2, earth motion at a location depends on localized soil layer and distance from the epicenter. Thus, bridge structures with supports lie far from each other may experience different earth excitation. For example, Figure 3.17 shows the predicted earthquake motions at Pier-W3 and Pier-W6 of the San Francisco Oakland Bay Bridge (SFOBB) of California. The distance between Pier-W3 and Pier-W6 of the SFOBB is approximately 1411 m (Figure 3.17b). These excitations are predicted by the California Department of Transportation by considering the soil and rock properties in the vicinity of SFOBB and expected earth movements at San Andreas Fault and Hayward fault (Caltrans, 1997). Note that the earth motion at Pier-W3 and Pier-W6 are very different. Furthermore, Figures 3.17c, d, and e indicates that the earth motion not only varies with the location, but also varies with direction. Thus, to evaluate the response of long, multiply supported, and complicated bridge structures, use of the actual earthquake excitation at each support is recommended.

The equation of motion of a multi-support excitation would be similar to the Equation 3.14, but the only difference is now that \( \{B\} \ddot{u}_k \) is replaced by a displacement array \( \{\ddot{u}_k\} \). And, the equation of motion multi-support system becomes

\[
[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = -[M]\{\ddot{u}_k\}
\]

(3.39)

where, \( \{\ddot{u}_k\} \) has the acceleration at each support locations and has zero value at nonsupport locations. By using uncoupling procedure described in the previous sections, the modal equation of \( n \)th mode can be written as

\[
\ddot{Y}_n + 2\xi_n \omega_n \dot{Y}_n + \omega_n^2 Y_n = -\sum_{k=1}^{N_e} \frac{L_n}{M_n} \ddot{u}_k
\]

(3.40)

where \( N_e \) is the total number of externally excited supports.
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FIGURE 3.17 San Francisco Oakland Bay Bridge (SFOBB) West Spans. (a) Vicinity map; (b) general plan elevation; (c) longitudinal motion at rock level; (d) transverse motion at rock level; (e) vertical motion at rock level; (f) displacement response at top of Pier W3.
The deformation response of the \( n \)th mode can then be determined as described in Section 3.3.3. Once the displacement response of the structure for all the mode shapes are estimated, the total dynamic response can be obtained by combining the displacements.

### 3.3.6 Time History Analysis

When the structure enters the nonlinear range, or has nonclassical damping properties, modal analysis cannot be used. A numerical integration method sometimes referred to as time history analysis, is required to get more accurate responses of the structure.

In a time history analysis, the time scale is divided into a series of smaller steps, \( d\tau \). Let us say the response at \( i \)th time interval has already determined and is denoted by \( u_i, \dot{u}_i, \ddot{u}_i \). Then, the response of the system at \( i \)th time interval will satisfy the equation of motion (Equation 3.39).

\[
[M] \{ \ddot{u}_i \} + [C] \{ \dot{u}_i \} + [K] \{ u_i \} = -[M] \{ \ddot{u}_y \}
\]  

(3.41)

The time stepping method enables us to step ahead and determine the responses \( u_{i+1}, \dot{u}_{i+1}, \ddot{u}_{i+1} \) at \( i + 1 \)th time interval by satisfying the Equation 3.39. Thus, the equation of motion at \( i + 1 \)th time interval will be

\[
[M] \{ \ddot{u}_{i+1} \} + [C] \{ \dot{u}_{i+1} \} + [K] \{ u_{i+1} \} = -[M] \{ \ddot{u}_{y(i+1)} \}
\]  

(3.42)

Equation 3.42 needs to be solved before proceeding to the next time step. By stepping through all the time steps, the actual response of the structure can be determined at all time instants.

#### 3.3.6.1 Example on Time History Analysis

The Pier W3 of San Francisco Oakland Bay Bridge West Spans was modeled using ADINA (1995) Program and nonlinear analysis was performed using the displacement time histories. The displacement time histories in three directions are applied at the bottom of the Pier W3 and the response of the Pier W3 was studied to estimate the demand on the Pier W3. One of the results, the displacement response at top of the Pier W3 is shown in Figure 3.17f.

### 3.4 Response Spectrum Analysis

Response spectrum analysis is an approximate method of dynamic analysis that gives the maximum response (acceleration, velocity, or displacement) of a SDOF system with the same damping ratio, but with different natural frequencies, responding to a specified seismic excitation. Structural models with \( n \) DOF can be transformed to \( n \) single-degree systems and response spectra principles can be applied to systems with many DOF. For most ordinary bridges a complete time history is not required. Because the design is generally based on the maximum earthquake response and response spectrum analysis is probably the most common method used in the design offices to determine the maximum structural response because of transient loading. In this section, we will discuss basic procedures of response spectrum analysis for bridge structures.

#### 3.4.1 Single-Mode Spectral Analysis

The single-mode spectral analysis is based on the assumption that earthquake design forces for structures respond predominantly in the first mode of vibration. This method is most suitable to regular linear elastic bridges to compute the forces and deformations, but not applicable for irregular bridges (unbalanced spans, unequal stiffness in the columns, etc.) because higher modes of vibration affects the
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distribution of the forces and resulting displacements significantly. This method can be applied to both continuous and noncontinuous bridge superstructures in either the longitudinal or transverse direction. Foundation flexibility at the abutments can be included in the analysis.

The single-mode analysis is based on Rayleigh’s energy method—an approximate method, which assumes a vibration shape for a structure. The natural period of the structure is then calculated by equating the maximum potential and kinetic energies associated with the assumed shape. The inertial forces \( p_e(x) \) are calculated using the natural period and the design forces and displacements are then computed using static analysis. The detailed procedure can be described in the following steps:

1. Apply uniform loading \( p_0 \) over the length of the structure and compute the corresponding static displacements \( u_s(x) \). The structure deflection under earthquake loading, \( u_t(x, t) \) is then approximated by the shape function, \( u_s(x) \), multiplied by the generalized amplitude function, \( u(t) \), which satisfies the geometric boundary conditions of the structural system. This dynamic deflection is shown as

\[
u(x, t) = u_s(x) \cdot u(t)
\]

2. Calculate the generalized parameters \( \alpha \), \( \beta \), and \( \gamma \) using the following equations:

\[
\alpha = \int u_s(x) \, dx \\
\beta = \int w(x) \cdot u_s(x) \, dx \\
\gamma = \int w(x) \left[ u_s(x) \right]^2 \, dx
\]

where \( w(x) \) is the weight of the dead load of the bridge superstructure and tributary substructure.

3. Calculate the period \( T_n \):

\[
T_n = 2\pi \sqrt{\frac{\gamma}{p_e g x}}
\]

where \( g \) is acceleration of gravity (ft/s\(^2\)) and \( p_e \) a uniform load arbitrarily set equal to 1.0 (kip/ft).

4. Calculate the static loading \( p_s(x) \), which approximates the inertial effects associated with the displacement \( u_s(x) \) using the ARS curve or following equation (AASHTO, 2011 and 2012):

\[
p_s(x) = \frac{\beta C_{sm}}{\gamma} \cdot w(x) \cdot u_s(x)
\]

\[
C_{sm} = \frac{1.2AS}{T_n^{2/3}}
\]

where \( C_{sm} \) is the dimensionless elastic seismic response coefficient; \( A \) the acceleration coefficient from the acceleration coefficient map; \( S \) the dimensionless soil coefficient based on the soil profile type; \( T_n \) the period of the structure as determined above; \( p_e(x) \) is the intensity of the equivalent static seismic loading applied to represent the primary mode of vibration (kip/ft).

5. Apply the calculated loading \( p_s(x) \) to the structure as shown in the Figure 3.18 and compute the structure deflections and member forces.

This method is an iterative procedure, and the previous calculations are used as input parameters for the new iteration leading to a new period and deflected shape. The process is continued until the assumed shape matches the fundamental mode shape.
FIGURE 3.18 Single-mode spectral analysis method.
3.4.2 Uniform Load Method

The uniform load method is essentially an equivalent static method that uses the uniform lateral load to compute the effect of seismic loads. For simple bridge structures with relatively straight alignment, small skew, balanced stiffness, relative light substructure, and with no hinges, uniform load method may be applied to analyze the structure for seismic loads. This method is not suitable for bridges with stiff substructures such as pier walls. This method assumes continuity of the structure and distributes earthquake force to all elements of the bridge and is based on the fundamental mode of vibration in either longitudinal or transverse direction (AASHTO, 2011 and 2012). The period of vibration is taken as that of an equivalent single mass-spring oscillator. The maximum displacement that occurs under the arbitrary uniform load is used to calculate the stiffness of the equivalent spring. The seismic elastic response coefficient $C_{sm}$ or the ARS curve is then used to calculate the equivalent uniform seismic load using, which the displacements and forces are calculated. The following steps outline the uniform load method:

1. Idealize the structure into a simplified model and apply a uniform horizontal load ($P_o$) over the length of the bridge as shown in Figure 3.19. It has units of force/unit length and may be arbitrarily set equal to 1 kip/ft.
2. Calculate the static displacements $u_s(x)$ under the uniform load $P_o$ using static analysis.
3. Calculate the maximum displacement $u_{s,\text{MAX}}$ and adjust it to 1 ft adjusting the uniform load $P_o$.
4. Calculate bridge lateral stiffness $K$ using the following equation:

$$K = \frac{P_o L}{u_{s,\text{MAX}}}$$

(3.50)

where $L$ is total length of the bridge (ft); and $u_{s,\text{MAX}}$ maximum displacement (ft).
5. Calculate the total weight $W$ of the structure including structural elements and other relevant loads such as pier walls, abutments, columns, and footings by

![Figure 3.19](image-url) Structure idealization and deflected shape for uniform load method. (a) Structure idealization; (b) deflected shape with maximum displacement of 1 ft.
where \( w(x) \) is nominal, unfactored dead load of the bridge superstructure, and tributary substructure.

6. Calculate the period of the structure \( T_n \) using the following equation:

\[
T_n = \frac{2\pi \sqrt{\frac{W}{gK}}}{gK}
\]

where \( g \) is acceleration of gravity (ft/s^2).

7. Calculate the equivalent static earthquake force \( p_e \) using the ARS curve or using the following equation:

\[
p_e = \frac{C_{sm} W}{L}
\]

8. Calculate the structure deflections and member forces by applying \( p_e \) to the structure.

### 3.4.3 Multi-Mode Spectral Analysis

#### 3.4.3.1 Basic Concept

The multi-mode spectral analysis method is more sophisticated than single-mode spectral analysis and is very effective in analyzing the response of more complex linear elastic structures to an earthquake excitation. This method is appropriate for structures with irregular geometry, mass, or stiffness. These irregularities induce coupling in three orthogonal directions within each mode of vibration. Also, for these bridges, several modes of vibration contribute to the complete response of the structure. A multimode spectral analysis is usually done by modeling the bridge structure consisting of three-dimensional frame elements with structural mass lumped at various locations to represent the vibration modes of the components. Usually, five elements per span are sufficient to represent the first three modes of vibration. A general rule of thumb is to capture \( i \)th mode of vibration, the span should have at least \((2i-1)\) elements. For long-span structures many more elements should be used to capture all the contributing modes of vibration. To obtain a reasonable response, the number of modes should be equal to at least three times the number of spans. This analysis is usually performed with a dynamic analysis computer programs such as ADINA (2011), SAP2000 (2011), and ANSYS (2010). For bridges with outrigger bents, C-bents, and single-column bents, rotational moment of inertia of the superstructure should be included. Discontinuities at the hinges and abutments should be included in the model. The columns and piers should have intermediate nodes at quarter points in addition to the nodes at the ends of the columns.

#### 3.4.3.2 Analysis Procedure

Using the programs mentioned in Section 3.4.4.1, frequencies, mode shapes, member forces, and joint displacements can be computed. The following steps summarize the equations used in the multi-mode spectral analysis (AASHTO, 2011 and 2012).

1. Calculate the dimensionless mode shapes \( \{\phi_i\} \) and corresponding frequencies \( \omega_i \) by

\[
\left[ [K] - \omega_i^2 [M] \right] \{u\} = 0
\]

\[
W = \int w(x) \, dx
\]
where

\[
   u_i = \sum_{j=1}^{n} \phi_j y_j = \Phi y_i
\]

(3.55)

\( y_j \) = modal amplitude of \( j \)th mode; \( \phi_j \) = shape factor of \( j \)th mode; \( \Phi \) = mode-shape matrix.

The periods for \( i \)th mode can then be calculated by

\[
   T_i = \frac{2\pi}{\omega_i} \quad (i=1,2,\ldots,n)
\]

(3.56)

2. Determining the maximum absolute mode amplitude for the entire time history is given by

\[
   Y_i(t)_{\text{max}} = \frac{T_i^2 S_i(\xi, T_i)}{4\pi^2} \left\{ \Phi_i \right\}^T [M] [B] u_i
\]

(3.57)

where \( S_i(\xi, T_i) = g C_m \) is the acceleration response spectral value; \( C_m \) is the elastic seismic response coefficient for mode \( m = 1,2,\ldots,n \); \( A \) is the acceleration coefficient from the acceleration coefficient map; \( S \) the dimensionless soil coefficient based on the soil profile type; \( T_n \) the period of the \( n \)th mode of vibration.

3. Calculate the value of any response quantity \( Z(t) \) (shear, moment, displacement) using the following equation:

\[
   Z(t) = \sum_{i=1}^{n} A_i Y_i(t)
\]

(3.58)

where coefficients \( A_i \) are functions of mode shape matrix \( \Phi \) and force displacement relationships.

4. Compute the maximum value of \( Z(t) \) during an earthquake using the mode combination methods described in Section 3.4.3.3.

### 3.4.3.3 Modal Combination Rules

Mode combination method is a very useful tool for analyzing bridges with large number of DOF. In a linear structural system maximum response can be estimated by mode combination after calculating natural frequencies and mode shapes of the structure using free vibration analysis. The maximum response cannot be computed by adding the maximum response of each mode because different modes attain their maximum values at different times. The absolute sum (AS) of the individual modal contributions provides an upper bound that is generally very conservative and not recommended for design. The following are several commonly used mode combination methods to compute the maximum total response. The variable \( Z \) represents the maximum value of some response quantity (displacement, moment, shear, etc.), \( Z_i \) is the peak value of that quantity in the \( i \)th mode and \( N \) the total number of contributing modes.

1. **AS:** The AS is sum of the modal contributions

   \[
   Z = \sum_{i=1}^{N} |Z_i|
   \]

   (3.59)

2. **Square root of sum of squares (SRSS) or root mean square (RMS) method.** This method computes the maximum by taking the SRSS of the modal contributions.

   \[
   Z = \left[ \sum_{i=1}^{N} Z_i^2 \right]^{1/2}
   \]

   (3.60)
3. CQC: Cross correlation between all modes are considered.

\[
Z = \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} Z_i \rho_{ij} Z_j \right]^{1/2}
\]  
(3.61)

\[
\rho_{ij} = \frac{8 \sqrt{\xi_i \xi_j} \left( \xi_i + r \xi_j \right)}{(1 - r^2) + 4 \xi_i \xi_j r \left( 1 + r^2 \right) + 4 \left( \xi_i^2 + \xi_j^2 \right) r^2}
\]  
(3.62)

where

\[
r = \frac{\omega_i}{\omega_j}
\]  
(3.63)

4. CQC with three components (CQC3):

\[
Z = \left[ Z_0^2 + a^2 Z_{90}^2 - (1-a^2)(Z_0^2 - Z_{90}^2) \sin^2 \theta \right. \\
\left. + 2(1-a^2)Z_{0-90}^2 \sin \theta \cos \theta + Z_z^2 \right]^{1/2}
\]  
(3.64)

\[
Z_0^2 = \sum_{i} \sum_{j} Z_{0i} \rho_{ij} Z_{0j}
\]  
(3.65)

\[
Z_{90}^2 = \sum_{i} \sum_{j} f_{90i} \rho_{ij} f_{90j}
\]  
(3.66)

\[
Z_{0-90}^2 = \sum_{i} \sum_{j} f_{0i} f_{90j}
\]  
(3.67)

\[
Z_z^2 = \sum_{i} \sum_{j} f_z \rho_{ij} f_z
\]  
(3.68)

where \( Z_0 \) and \( Z_{90} \) are the modal values produced by 100% of the lateral spectrum applied between 0 and 90°, respectively; \( Z_z \) the modal response from the vertical spectrum that can be different from the lateral spectrum; \( \theta \) an arbitrary angle between one computation axis and one of directions of basic horizontal input spectra; and \( a \) the ratio (between 0 and 1.0) of two basic horizontal input spectra.

It is important to note that for equal spectra \( a = 1.0 \), the value \( F \) is not a function of \( \theta \) and the selection of the analysis reference system is arbitrary, that is

\[
Z = \sqrt{Z_0^2 + Z_{90}^2 + Z_z^2}
\]  
(3.69)

5. Percentage combination (AASHTO, 2011; Caltrans, 2013).

100% of the prescribed seismic effects in one direction plus 30% or 40% of the prescribed seismic effects applied in the perpendicular direction. For examples, AASHTO Guide Specifications for LRFD Seismic Bridge Design (AASHTO, 2011) and Caltrans SDC (Caltrans, 2013) specify the following load cases:

a. Seismic load case 1: 100% transverse + 30% longitudinal + 30% vertical
b. Seismic load case 2: 30% transverse + 100% longitudinal + 30% vertical
c. Seismic load case 3: 30% transverse + 30% longitudinal + 100% vertical
3.4.3.4 Remarks

- Theoretically all mode shapes must be included to calculate the response but fewer mode shapes can be used when the corresponding mass participation is over 85% of the total structure mass. In general, the factors considered to determine the number of modes required for the mode combination are dependent on structural characteristics of the bridge, spatial distribution, and frequency content of the earthquake loading.
- AS is the most conservative method. It assumes that the maximum modal values for all modes occur at the same time.
- SRSS is a very common approach and is suitable for structures with well-spaced modes. It assumes that all of the maximum modal values are statistically independent. For three-dimensional structures in which a large number of frequencies are almost identical, this assumption is not justified (Wilson, 2009).
- CQC is based on random vibration theories and has been accepted by most experts in earthquake engineering (Wilson et al., 1981). For an undamped structure, the CQC is identical to the SRSS. For structures with closely spaced dominant mode shapes the CQC method is precise, whereas the SRSS estimates inaccurate results. Closely spaced modes are those within 10% of each other in terms of natural frequency.
- CQC3 combines the effects of three orthogonal spectra (Menun and Kiureghian, 1998). For $a = 1$, the CQC3 reduces to the SRSS. For three-dimensional response spectra analyses, the CQC3 method should be used if a value of $a < 1.0$ can be justified. It will produce realistic results that are not a function of the user-selected reference system. At the present time, no specific guidelines have been suggested for the value of $a$.
- Percentage rule (100/30) is the simplest and recommended by the AASHTO (2011) and Caltrans (2013). However, it is empirical and can underestimate the design forces in certain members and produces a member design that is relatively weak in one direction.

3.4.4 Multiple Support Response Spectrum Method

Records from recent earthquakes indicate that seismic ground motions can significantly vary at different support locations for multiply supported long structures. When different ground motions are applied at various support points of a bridge structure, the total response can be calculated by superposition of response because of independent support input. This analysis involves combination of dynamic response from single input and pseudo-static response resulting from the motion of the supports relative to each other. The combination effects of dynamic and pseudo-static forces because of multiple support excitations on a bridge depend on the structural configuration of the bridge and ground motion characteristics. Kiureghian et al. (1997) presented a comprehensive study on the MSRS method based on fundamental principles of stationary random vibration theory for seismic analysis of multiply supported structures that accounts for the effects of variability between the support motions. Using MSRS combination rule, the response of a linear structural system subjected to multiple support excitations can be computed directly in terms of conventional response spectra at the support DOF and a coherency function describing the spatial variability of the ground motion. This method accounts for the three important effects of ground motion spatial variability, namely, the incoherence, the wave passage, and the site response effect. These three components of ground motion spatial variability can strongly influence the response of multiply supported bridges and may amplify or de-amplify the response by one order of magnitude. Two important limitations of this method are nonlinearities in the bridge structural components and/or connections and the effects of soil structure interaction. This method is an efficient, accurate, and versatile solution and requires less computational time than a true time history analysis. Following are the steps that describe the MSRS analysis procedure.
1. Determine the necessity of variable support motion analysis: Three factors that influence the response of the structure under multiple support excitations are the distance between the supports of the structure, rate of variability of the local soil conditions, and the stiffness of the structure. The first factor, the distance between the supports influences the incoherence and wave passage effects. The second factor, the rate of variability of the local conditions influences the site response. The third factor, the stiffness of the superstructure plays an important role in determining the necessity of variable support motion analysis. Stiff structures such as box girder bridges may generate large internal forces under variable support motion, whereas flexible structures such as suspension bridges easily conform to the variable support motion.

2. Determine the frequency response function for each support location. Programs such as SHAKE (Idriss et al., 1991) can be used to develop these functions using borehole data and time-domain site response analysis. Response spectra plots, peak ground displacements in three orthogonal directions for each support location, and a coherency function for each pair of DOF are required to perform the MSRS analysis. The comprehensive report by Kiureghian et al. (1997) provides all the formulas required to account for the effect of nonlinearity in the soil behavior and the site frequency involving the depth of the bedrock.

3. Calculate the structural properties such as effective modal frequencies, damping ratios, influence coefficients, and effective modal participation factors ($\omega_i$, $\xi_i$, $\alpha_k$, and $b_{ik}$) are to be computed externally and provided as input.

4. Determine the response spectra plots, peak ground displacements in three directions, and a coherency function for each pair of support DOF required to perform MSRS analysis. The three components of coherency function are incoherence, wave passage effect, and site response effect. Analysis by array of recordings is used to determine incoherence component. The models for this empirical method are widely available (Abrahamson et al., 1991). Parameters such as shear wave velocity, the direction of propagation of seismic waves, and the angle of incidence are used to calculate the wave passage effect. Frequency response function determined in the previous steps is used to calculate the site response component.

3.4.5 Remarks

It should be understood clearly that the response spectrum method is an approximate method used to estimate maximum peak values of displacements and forces (Wilson, 2009) and must be used very carefully. It is strictly limited to linear elastic analysis where damping properties can only be estimated with a low degree of confidence. The use of nonlinear spectra has very little theoretical background, and this approach should not be applied in the analysis of complex three-dimensional structures (Wilson, 2009).

3.5 Inelastic Dynamic Analysis

3.5.1 Equations of Motion

Inelastic dynamic analysis is usually performed for the safety evaluation of important bridges to determine the inelastic response of bridges when subjected to design earthquake ground motions. Inelastic dynamic analysis provides a realistic measure of response because the inelastic model accounts for the redistribution of internal actions because of the nonlinear force displacement behavior of the components (I&A, 1992; ATC, 1996; Bathe, 1996; Priestly et al., 1996; Buchholdt, 1997; Paz and Leigh, 2004). Inelastic dynamic analysis considers nonlinear damping, stiffness, load deformation behavior of members including soil, and mass properties. A step-by-step integration procedure is the most powerful method used for nonlinear dynamic analysis. One important assumption of this procedure is that the acceleration varies linearly, whereas the properties of the system such as damping and stiffness remain
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constant during the time interval. Using this procedure, a nonlinear system is approximated as series of linear systems and the response is calculated for a series of small equal intervals of time $\Delta t$ and equilibrium is established at the beginning and end of each interval.

The accuracy of this procedure depends on the length of the time increment $\Delta t$. This time increment should be small enough to consider rate of change of loading $p(t)$, nonlinear damping and stiffness properties, and the natural period of the vibration. A SDOF system and its characteristics are shown in Figure 3.20. The characteristics include spring and damping forces, forces acting on mass of the system, and arbitrary applied loading. The force equilibrium can be shown as

$$f_i(t) + f_d(t) + f_s(t) = p(t) \quad (3.70)$$

and the incremental equations of motion for time $t$ can be shown as

$$m \Delta \ddot{u}(t) + c(t) \Delta \dot{u}(t) + k(t) \Delta u(t) = \Delta p(t) \quad (3.71)$$

Current damping $f_d(t)$, elastic forces $f_s(t)$ are then computed using the initial velocity $\dot{u}(t)$, displacement values $u(t)$, nonlinear properties of the system, damping $c(t)$, and stiffness $k(t)$ for that interval.

**FIGURE 3.20** Definition of nonlinear dynamic system. (a) Basis SDOF structure; (b) force equilibrium; (c) nonlinear damping; (d) nonlinear stiffness; (e) applied load.
New structural properties are calculated at the beginning of the each time increment based on the current deformed state. The complete response is then calculated by using the displacement and velocity values computed at the end of each time step as the initial conditions for the next time interval and repeating until the desired time.

### 3.5.2 Modeling Considerations

A bridge structural model should have sufficient DOF and proper selection of linear/nonlinear elements such that a realistic response can be obtained. Nonlinear analysis is usually preceded by a linear analysis as a part of a complete analysis procedure to capture the physical and mechanical interactions of seismic input and structure response. Output from the linear response solution is then used to predict which nonlinearities will affect the response significantly and model them appropriately. In other words, engineers can justify the effect of each nonlinear element introduced at the appropriate locations and establish the confidence in the nonlinear analysis. Although discretizing the model, engineers should be aware of the tradeoffs between the accuracy, computational time, and use of the information such as the regions of significant geometric and material nonlinearities. Nonlinear elements should have material behavior to simulate the hysteresis relations under reverse cyclic loading observed in the experiments. A detailed discussion is presented in Chapter 12 of Volume I—Fundamentals.

The general issues in modeling of bridge structures include geometry, stiffness, mass distribution, and boundary conditions. In general, abutments, superstructure, bent caps, columns and pier walls, expansion joints, and foundation springs are the elements included in the structural model. The mass distribution in a structural model depends on the number of elements used to represent the bridge components. The model must be able to simulate the vibration modes of all components contributing to the seismic response of the structure.

**Superstructure:** Superstructures are usually modeled using linear elastic three-dimensional beam elements. Detailed models may require nonlinear beam elements.

**Columns and pier walls:** Plastic hinge zones in columns and pier walls are usually modeled using nonlinear beam-column (frame) elements having response properties with a yield surface described by the axial load and biaxial bending. Outside plastic hinge zone are modeled using linear elastic beam-column elements. Some characteristics of the column behavior include, initial stiffness degradation because of concrete cracking, flexural yielding at the fixed end of the column, strain hardening, pinching at the point of load reversal. Shear actions can be modeled using either linear or nonlinear load deformation relationships for columns. For both columns and pier walls, torsion can be modeled with linear elastic properties. For out-of-plane loading, flexural response of a pier wall is similar to that of columns where as for in-plane loading, the nonlinear behavior is usually shear action. Effective cross-section properties including area, moment of inertia, and torsional constant should be used.

**Cap Beam:** For multi-column bent bridges, the cap beam should be modeled as an elastic beam element. The torsional constant of the cap beam $J$ should be modified by an amplification factor with a minimum value of in order of 100 (Aviram et al., 2008).

**Expansion joints:** Expansion joints can be modeled using gap elements that simulate the nonlinear behavior of the joint. The variables include initial gap, shear capacity of the joint, and nonlinear load deformation characteristics of the gap.

**Abutments:** Abutments are modeled using nonlinear spring and gap elements to represent the backfill soil material, abutment back wall, the piles, bearings, and gaps at the seat.

**Foundations:** Foundations are typically modeled using nonlinear spring elements to represent the translational and rotational stiffness of the foundations to simulate the expected behavior during a design earthquake.
3.6 Summary

This chapter has presented the basic principles and methods of dynamic analysis for seismic design of bridges. Response spectrum analysis—the SDOF or equivalent SDOF-based equivalent static analysis is efficient, convenient, and most frequently used for ordinary bridges with simple configurations. Elastic dynamic analysis is required for bridges with complex configurations. A MSRS analysis developed by Kiureghian et al. (1997) using a lumped-mass beam element mode may be used in lieu of an elastic time-history analysis.

Inelastic response spectrum analysis is a useful concept, but the current approaches apply only to SDOF structures. An actual nonlinear dynamic time-history analysis may be necessary for some important and complex bridges, but linearized dynamic analysis (dynamic secant stiffness analysis), and inelastic static analysis (static push-over analysis) (Chapters 5 and 6) are the best possible alternatives (Powell, 1997) for the most bridges.

References

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