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Stochastic Reservoir Analysis

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Stochastic Reservoir Analysis

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26.1 Introduction

Reservoirs are essential elements of water resource systems. Many water supply systems depend on river flow as their main water source. Due to the inherent natural variability in river flow, direct withdrawal does not guarantee a stable supply over time, and fluctuations in river flow will reflect directly into fluctuations in yield. Storage reservoirs have therefore been used to absorb such fluctuations and guarantee a certain level of yield. The main function of storage reservoirs is to store excess water at some point in time to be used at other times when the supply is low. Due to the stochastic nature of river flows, however, a reservoir of infinite storage volume would be needed to achieve perfect regulation of flow and guarantee a yield equal to the mean flow. A finite reservoir, on the other hand, can only guarantee a yield that is less than the mean flow. It thus becomes a matter of economy to choose a reservoir of suitable size that would guarantee a certain yield with a given reliability. However, due to the limited capacity of a finite reservoir, both spillage of excess water and emptying of the reservoir are likely to occur. Of particular importance is the emptying case for which the reservoir fails to deliver the specified yield.

The elements involved in the hydrologic design of a water supply reservoir consist of the reservoir size $V$, the demand or draft $D$, and the probability of failure $f$. Probability of failure is usually expressed in terms of its complement, the probability of no failure, that is, $1 - f$, commonly known as the reliability of the reservoir. Design relationships known as reservoir storage–yield–reliability (S–R–Y) relationships have therefore been developed to describe the relationship between the size of a reservoir, its yield, and reliability.

Evaluation of storage, spillage, and yield for a reservoir with a given size and input sequence was investigated by Rippl [32] using a number of graphical and arithmetical techniques. Hazen [11], Sudler [34], and Barnes [2] contributed to the introduction of the use of synthetic data in reservoir studies. Hurst [14] used rescaled range analysis for sizing reservoirs with long-term storage. According to Klemeš [15], modern stochastic reservoir theory was introduced by Kritskiy and Menkel [18,19] and Savarenskiy [33], only to be independently reinvented in the mid-1950s by Moran [25]. Further significant contributions to stochastic reservoir analysis were made by Gani [4], Langbein [20], Lloyd [21], and Klemeš [16].

Several methods have been suggested in the literature to solve the stochastic reservoir equation to obtain the distribution of storage as well as S–R–Y relationships. Monte Carlo simulation (behavior analysis) is one of the widely used methods, which involves the generation of synthetic inflow data of the desired probability distribution and applying the mass balance equation at a suitable time interval for a large number of time steps. Alternatively, a class of stochastic methods involves finding the equilibrium probability distribution of the storage function given a certain reservoir size and yield for
an inflow with a given probability distribution, which facilitate the calculation of different types of probabilities, including those of reservoir failure and spilling. This problem is commonly referred to as the “solution of the stochastic reservoir equation.” Due to the complexity of the equilibrium probability distribution of the storage function, exact solutions are only available for a limited, yet very flexible, class of inflow distributions, which include the exponential distribution [26] and a subset of the gamma distribution with integral shape parameters [30]. Numerical solution methods based on the discretization of the stochastic reservoir equation have been suggested in the literature [16,29]. A number of approximate methods have also been proposed, such as the diffusion approximation for normally distributed inflows [3]. A comprehensive review of different methods for solving the stochastic reservoir equation can also be found in Lloyd [22].

26.2 Reservoir Simulation

For a given reservoir of size \( V \) receiving a certain inflow \( I \) with an outflow \( Q \), the distribution of reservoir storage \( S \) can be obtained through the application of the mass conservation principle, which states that the rate of change of reservoir storage is equal to the difference between the rate of inflow \( (i) \) and the rate of outflow \( (q) \) from the reservoir. The resulting equation is the well-known continuity equation that takes the form

\[
\frac{dS}{dt} = i - q
\]  

(26.1)

Selecting a time step that is suitable for the type of analysis, the evolution of reservoir storage can thus be calculated as

\[
S_{t+1} = S_t + I_t - Q_t
\]  

(26.2)

where

- \( I_t \) is the total inflow to the reservoir between time steps \( t \) and \( t+1 \)
- \( Q_t \) is the total outflow from the reservoir in the same period

The total outflow \( Q_t \) usually comprises the draft \( D \) (which may be constant or time dependent) and any losses from the reservoir such as evaporation and seepage losses.

Because a reservoir has a finite size \( V \), storage in Equation 26.2 earlier is constrained by two conditions. First, if the calculated storage \( S^*_{t+1} \) exceeds the reservoir size \( V \), water is spilled and the stored volume cannot exceed the reservoir size \( V \). The volume of spilled water in this case is equal to the difference between the calculated storage \( S^*_{t+1} \) and the reservoir size \( V \). Similarly, if the calculated storage \( S^*_{t+1} \) is negative, the reservoir becomes empty and the outflow is limited by the amount of available water, and a deficit equal to \(-S^*_{t+1}\) occurs. These two conditions can be put in equation form as follows:

\[
S_{t+1} = \min (S^*_{t+1}, V), \quad \text{spill} = S^*_{t+1} - S_{t+1}
\]  

(26.3)

\[
S_{t+1} = \max (S^*_{t+1}, 0), \quad \text{deficit} = S_{t+1} - S^*_{t+1}
\]  

(26.4)

Equations 26.3 and 26.4 can actually be combined into one equation in the form

\[
S_{t+1} = \max \left[ 0, \min (S^*_{t+1}, V) \right], \quad \Delta Q = S_{t+1} - S^*_{t+1}
\]  

(26.5)

where

- a positive \( \Delta Q \) indicates spillage
- a negative \( \Delta Q \) indicates deficit
Using the aforementioned equations offers a very flexible way of simulating the evolution of reservoir storage, as it allows different types of inflows and outflows to be accounted for, including outflows that depend on the state of the reservoir storage, such as evaporation and seepage losses as well as a variable draft from the reservoir.

The inflow in Equation 26.2 earlier may be a historical time series, but more often, it will be a synthetically generated inflow sequence (or a large number of sequences) with a length that is suitable to get reliable estimates of the distribution of storage and various reservoir performance indicators, such as the probability of failure or that of overtopping. The procedure in this case is commonly known as Monte Carlo simulation or as behavior analysis. Koutsoyiannis [17] shows that the number of simulation steps required to obtain an estimate of a given probability \( \beta \) with an acceptable error of \( \pm \varepsilon \beta \) and confidence \( \alpha \) with an annual time step is given by

\[
 n = \left( \frac{z_{(1+\alpha)/2}}{\varepsilon} \right)^2 \left( \frac{1-\beta}{\beta} \right) \tag{26.6}
\]

where \( z_{(1+\alpha)/2} \) is the standard normal variate corresponding to a cumulative probability of \( (1+\alpha)/2 \), so if a 90% confidence is required, then \( z_{(1+\alpha)/2} = z_{0.95} = 1.64 \). For example, if the probability of failure is \( \beta = 0.1 \), and the error tolerance \( \varepsilon = 0.01 \), the required number of steps for 90% confidence in \( \beta \) would be equal to \( n = 242,064 \) simulation steps. It should be noted, however, that due to the stochastic nature of the problem, there would still be a 10% chance that the error in the obtained value of \( \beta \) exceeds the designated 0.01 tolerance. Smaller error tolerance and/or higher confidence will require much longer sequences, as is clear from Equation 26.6. Pretto et al. [31] discuss the convergence properties of reservoir storage analysis using Monte Carlo simulation.

As a simplified example, consider a reservoir of size \( V = 4 \) units with an exponential flow of mean of 2 units and a draft \( m = 1.8 \) units. Simulation results using \( 1 \times 10^6 \) simulation steps indicate that the probability of reservoir emptying is equal to 0.212896, while the probability of spilling is equal to 0.191747. These quantities will be compared later with those obtained using other solution methods. The simulated distribution of the storage is shown in Figure 26.1.

### 26.3 Standard Reservoir

Gomide [6] introduced the standard reservoir convention, as he noted that reservoirs with different characteristics give the same results if the ratios of their parameters to the standard deviation of the inflow are the same. A standardized reservoir size \( C \) is equal to the actual reservoir size \( V \) divided by the standard deviation of the inflow \( \sigma \) as in Equation 26.7:

\[
 C = \frac{V}{\sigma} \tag{26.7}
\]

The corresponding standardized net mean inflow to the reservoir is known as the drift \( \varepsilon \), which was first introduced by Pegram [29] and is defined as in Equation 26.8:

\[
 \varepsilon = \frac{\mu - D}{\sigma} \tag{26.8}
\]

where
- \( \mu \) is the mean of inflow
- \( D \) is the draft
The net inflow to the standard reservoir is the standardized net inflow given by

\[ N_I D_t = -\sigma \]  

(26.9)

We note that adopting the standardized reservoir convention is economic, since results for reservoirs with proportional values of the quantities \( V, D, \mu, \) and \( \sigma \) can all be expressed in terms of \( C \) and \( \epsilon \) only with standardized inflow. As an example, the results for a standard reservoir of size \( C = 2 \) and drift \( \epsilon = 0.2 \) having a standard normal inflow would be the same for a reservoir of size \( V = 20 \times 10^9 \) m\(^3\), having a draft \( D = 23 \times 10^9 \) m\(^3\), with a normal inflow with mean \( \mu = 25 \times 10^9 \) m\(^3\), and standard deviation \( \sigma = 10 \times 10^9 \) m\(^3\). The same results apply for a reservoir of size \( V = 10 \times 10^6 \) m\(^3\), having a draft \( D = 14 \times 10^6 \) m\(^3\), with a normal inflow with mean \( \mu = 15 \times 10^6 \) m\(^3\), standard deviation \( \sigma = 5 \times 10^6 \) m\(^3\), and so forth.

### 26.4 Stochastic Reservoir Equation

As seen in Section 26.2, the conservation of mass principle was used to derive Equations 26.2 through 26.5 that describe the evolution of storage in time. As a result of the application of these equations, the stochastic distribution of the inflow is converted into a stochastic distribution of the storage. However, the distribution of storage is obtained in this case through a sample of a limited size, and the number of simulation steps needed to attain a reasonable accuracy is usually large as detailed in Section 26.2 earlier. From a statistical point of view, Equation 26.2 represents storage \( S_i \) as a function of the random variable \( I_i \). The distribution of a function of a random variable is a well-known problem in probability theory, although the conditions imposed by Equations 26.3 through 26.5 often prevent an explicit closed-form solution of the distribution of storage, except in a few cases as will be shown later.

Basic principles of the probability theory can be used to directly derive the distribution of storage given that of the inflow for the case of a fixed draft and independent inflow. Consider a stochastic reservoir of size \( C \) and net inflow \( N \) that is independent in time. At any time step \( i \), the reservoir can be in
any of three distinct states: empty \( (S_i = 0) \), full \( (S_i = C) \), and partially full \( (0^+ < S_i < C^-) \), where \( 0^+ \) and \( C^- \) are used to indicate the continuous part of the distribution between 0 and \( C \), excluding the \( S_i = 0 \) and \( S_i = C \) states, each of which has its own discrete probability mass.

The probability distribution of storage \( S_{i+1} \) in the following time step \( i+1 \) can simply be expressed \([10]\) using the well-known total probability rule as

\[
P(S_{i+1} \leq s) = P(S_{i+1} < s \cap S_i = 0) + P(S_{i+1} < s \cap 0 < S_i < C) + P(S_{i+1} < s \cap S_i = C) - P(N \leq s)P(S_i = 0) + \int_{u=0^+}^{u=C^-} P(N \leq s-u)P(S_i = u)du + P(N \leq s-C)P(S_i = C)
\]  

(26.10)

The middle term in the RHS of Equation 26.10 is a convolution of the net inflow and the storage distribution functions. In essence, Equation 26.10 can be viewed as an application of the "conservation of probability" principle, in analogy to the "conservation of mass" principle.

In terms of the distribution functions \( G(s) \) of the storage and \( F(n) \) of the net inflow, Equation 26.10 becomes \([10,29]\)

\[
G_{i+1}(s) = F(s) \cdot G_i(0) + \int_{0^+}^{C^-} F(s-u) \cdot g_i(u)du + F(s-C) \left[1 - G_i(C^-)\right]
\]  

(26.11)

The sequence \( G_i(s), G_{i+1}(s), \ldots \) converges to a limiting "equilibrium" distribution \( G(s) \) of the storage in the reservoir \([22,29]\) and the subscripts can be dropped. Because of the special form of Equation 26.11, where the distribution function of storage \( G(s) \), as well as its derivative \( g(s) \), appears on both sides of the equation, a general analytical solution is not available. It is important to note that, unlike the simulation approach, this approach does not require the generation of lengthy synthetic data to obtain the distribution of storage, since all that is needed is the distribution function of the net inflow.

As stated earlier, Equation 26.11 is valid only for reservoirs with fixed draft and independent net inflow, but extensions to accommodate variable draft and inflows having first-order Markov dependence have also been tackled in the literature \([29]\). Although this approach has some limitations compared to the very flexible reservoir simulation approach, it can offer tools for rapid sizing and assessment of reservoirs in the planning phase, prior to detailed simulation studies. At the same time, results obtained using this approach can provide benchmark tests for validating detailed simulation models.

### 26.5 Solution of the Stochastic Reservoir Equation

#### 26.5.1 Prabhu’s Exact Solution

Prabhu \([30]\) gives the only available complete exact solution of the stochastic reservoir equation in the case of independent gamma-distributed inflow with integer shape parameters, of which exponentially distributed inflow is a special case, which had been tackled earlier by Moran \([26,27]\). The class of gamma inflows with integral shape parameter, also known as the “Erlangian” family, have a probability density function given by

\[
dG(x) = \frac{\mu^p}{(p-1)!} e^{-\mu x} x^{p-1} dx, \quad 0 < x < \infty
\]  

(26.12)

where \( \mu > 0 \) and \( p = 1, 2, 3, \ldots \) are the distribution parameters. The value of \( p = 1 \) corresponds to the exponential distribution, while integer values of \( p > 1 \) correspond to gamma distributions with integral
shape parameter. The mean of the distribution $\mu$ is equal to $p/\mu$, while the variance $\sigma^2$ is given by $p/\mu^2$. The coefficient of variation and the skewness coefficient are given by $C_v=1/\sqrt{p}$ and $\gamma=2/\sqrt{p}$, respectively.

For this family of distributions, the stationary distribution of the reservoir storage for a reservoir of size $K$ with a draft $m$ was studied by Prabhu [30]. The reservoir size $K$ in Prabhu’s work is the gross reservoir size that is equal to $V+m$, that is, the reservoir size plus the draft. This convention assumes that a reservoir first receives the full inflow before the draft is initiated, which is the Moran model convention [22,29]. To keep with the “simultaneous reservoir” convention used throughout this chapter, Prabhu’s equations are written here in terms of the net reservoir size $V$, that is, replacing $K$ with $V+m$:

$$F(z) = \begin{cases} 
1 - e^{\mu(V-z)} \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{N+1} \frac{(-\lambda)^q}{(qp+r)!} \frac{[V-qm-z]^{qp+r}}{qp+r!}, & -m \leq z < V-(N+1)m \\
1 - e^{\mu(V-z)} \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{N} \frac{(-\lambda)^q}{(pq+r)!} \frac{[V-qm-z]^{pq+r}}{pq+r!}, & V-(s+1)m \leq z < V-sm, \\
1 - e^{\mu(V-z)} \frac{\alpha_r}{r!} (V-z)^r, & V-m \leq z < \infty
\end{cases} \quad (26.13)$$

where

$$V = N + m + U, \quad 0 < U < m \quad (26.14)$$

that is, $N$ is the integral part of $(V/m)$ and $U$ is the remaining part, which is a fraction of $m$. If $V$ is a proper multiple of $m$, then $U$ would be equal to zero.

The parameter $\lambda$ in Equation 26.13 is given by

$$\lambda = (-1)^{p-1} \mu^p e^{-\mu m} \quad (26.15)$$

Also, the values $\alpha_r$ in Equation 26.13, where $r=0, 1, ..., p-1$, can be obtained by solving the $p$ linear equations given by

$$\alpha_r - \lambda \sum_{s=0}^{p-1} d_r \alpha_s = (-\mu)^r e^{-\mu(V+m)} \sum_{s=0}^{p-1} \frac{[\mu(V+m)]^s}{s!}, \quad r=0,1,...,p-1 \quad (26.16)$$

where

$$d_r = (-1)^{p+r-1} \sum_{q=0}^{N} \frac{(-\lambda)^q}{(pq+r)!} \int_{qm}^{\lambda} \frac{[t-qm]^{pq+r}(t+m)^{p+r-1}}{(pq+s)!(p-r-1)!} dt, \quad r,s=0,1,...,p-1 \quad (26.17)$$

The probabilities of emptying and filling, respectively, based on Equation 26.13, are thus given by

$$P_0 = F(0) = 1 - e^{\mu(V-z)} \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{N} \frac{(-\lambda)^q}{(pq+r)!} \frac{[V-qm]^{pq+r}}{(pq+r)!} \quad (26.18)$$
Hamed [9] shows that Equation 26.13 gives the full distribution of storage in the reservoir, including the distribution of both deficit and spillage. The reader is referred to Hamed [9] for further discussion and examples.

For Prabhu’s solution, a standardized reservoir size $C$ is equal to the actual reservoir size $V$ divided by the standard deviation of the inflow as in Equation 26.20:

$$
C = \frac{V}{\sigma_x} = \frac{V}{\left(\frac{\mu}{\sqrt{p}}\right)} = \frac{\mu V}{\sqrt{p}}
$$

(26.20)

The corresponding standardized net mean inflow to the reservoir, or the drift, is defined as in Equation 26.21:

$$
\varepsilon = \frac{\mu_x - m}{\sigma_x} = \frac{\mu - m}{\sqrt{p} / \mu} = \frac{p - m \mu}{\sqrt{p}}
$$

(26.21)

Now the solution can be obtained in terms of $C$, drift $\varepsilon$, and shape parameter $p$.

The computer program P1 in Appendix 26.A is a MATLAB® implementation of Prabhu’s solution in the case of $p = 1$, that is, exponentially distributed inflow. The case of general $p$ requires special calculations and the Symbolic Math toolbox in MATLAB is required to correctly perform the calculations. The reader is referred to Hamed [9] for important details of implementation. Consider the example given in Section 26.2. The application of Prabhu’s exact solution gives an emptying probability of 0.211634 and a spilling probability of 0.191868, which are very close to those obtained by simulation. Furthermore, Figure 26.2 shows the PDF of the storage distribution as given by Prabhu’s solution, which is similar to that obtained through simulation. Prabhu’s exact solution clearly identifies the jump in the PDF near storage equal to 2.2.

![FIGURE 26.2](image_url)

FIGURE 26.2  The PDF of storage in a reservoir of size $V = 4$ units with an exponential flow of mean $\mu = 2$ units and a draft $m = 1.8$ units using Prabhu’s exact solution.
26.5.2 Integral Equation Solution

The integral equation of the stochastic reservoir as discussed earlier is given as

\[ G(s) = G(0) \cdot F(s) + \int_0^C g(u)F(s-u) \cdot du + \left[ 1 - G(C^-) \right] F(s-C), \quad 0 \leq s \leq C \] (26.22)

where

- \( G(.) \) and \( g(.) \) are, respectively, the cumulative distribution function and the probability density function of storage
- \( F(.) \) is the cumulative distribution function of the net inflow to the reservoir
- \( C \) is the reservoir size

The following simpler equation can be obtained through integration by parts [29]:

\[ G(s) = F(s-C) + \int_0^C f(s-u) \cdot G(u) \cdot du \] (26.23)

If a reservoir of size \( C \) is divided into \( k \) equal slices, each of width \( h = C/k \), \( G(s) \) can be approximated by \( H(s) \) at \( s = ih \), \( i = 0, 1, \ldots, k \). The integral on the right-hand side of Equation 26.23 can be evaluated by using a suitable quadrature formula, such as the trapezoidal or Simpson's rule, in terms of the unknown \( G(u) \). If the weights of the quadrature formula are given by \( w_j \), \( j = 0, 1, \ldots, k \), then Equation 26.23 can be discretized as

\[ H(ih) = F(ih-c) + \sum_{j=0}^{k} w_j \cdot f[(i-j)h] \cdot H(jh) \] (26.24)

Equation 26.24 gives a system of \( k + 1 \) simultaneous equations, which can be solved to obtain the values of \( H(ih) \) using a suitable method. The emptying probability is thus given by \( H(0) \), while the filling probability is given by \( 1 - H(c) \). Once the values of \( H(ih) \) are obtained, the expression in Equation 26.24 works as an interpolation formula that employs all the information in the kernel to obtain the storage distribution function at any intermediate point [29] as

\[ H(s) = F(s-c) + \sum_{j=0}^{k} w_j \cdot f[(s-jh)H(jh) \] (26.25)

Other solutions of different forms of Equation 26.22 have also been suggested in the literature [22] using different schemes of reservoir discretization.

The computer program P2 is a MATLAB implementation of the integral equations solution of the stochastic reservoir equation for a reservoir with exponentially distributed inflow as an example.

Consider the example given in Section 26.5.1 earlier. The application of the integral equation solution with \( k = 1000 \) stages of the reservoir gives an emptying probability of 0.212986 and a spilling probability of 0.191611, which are very close to those obtained by Prabhu's exact solution and by simulation.

26.5.3 Difference Equations Solution

The first step in this solution method is the discretization of the reservoir into \( k \) states of equal size \( h \). With the addition of the two states of full and empty, a total of \( k + 2 \) states are obtained for the reservoir.
Pegram [29] found that \( k \leq 30 \) gave a reasonable accuracy in most cases, with standardized reservoir sizes up to 8 and standardized net mean inflow up to 1. Using this discretization, the elements \( q_{ij} \) of the \( k+2 \) square transition matrix \( Q \) between states can be easily calculated based on the cumulative distribution function of the input \( F(u) \) as

\[
q_{i,j} = \begin{cases} 
F[0], & i=0 \\
F[ih] - F[(i-1)h] - F[c], & i=1,2,\ldots,k \\
1 - F[c], & i=k+1
\end{cases}
\]

\[
q_{i,k+1} = \begin{cases} 
F[-c], & i=0 \\
F[ih - c] - F[(i-1)h - c], & i=1,2,\ldots,k \\
1 - F[0], & i=k+1
\end{cases}
\]

\[
q_{i,k} = \begin{cases} 
F[(-j + \frac{1}{2})h], & i=0 \\
F[(i - j + \frac{1}{2})h] - F[(i - j - \frac{1}{2})h], & i=1,2,\ldots,k \\
1 - F[c - (j - \frac{1}{2})h], & i=k+1
\end{cases}
\]

\[ (26.26) \]

It is to be noted that each column of \( Q \) should sum to unity, which is the total probability for each state. Based on this transition probability matrix, the \((k+2)\)-element vector \( v(t+1) \) that gives the probabilities of the storage states at time \( t+1 \) is related to \( v(t) \) as [27]

\[
v(t+1) = Qv(t)
\]

\[ (26.27) \]

The equilibrium storage distribution vector \( \pi \) can thus be obtained as

\[
\pi = \lim_{t \to \infty} v(t)
\]

\[ (26.28) \]

It is to be noted that the first and last elements of \( v(t) \) represent the probabilities of emptying and filling, respectively. Several methods have been proposed to solve Equation 26.28 such as the power method that starts with \( v(0) \) to be any initial vector with elements that sum to unity and then successively apply Equation 26.27 for 20–30 iterations to reach the stable equilibrium [29].

The same result can be obtained by raising \( Q \) to a large power using linear algebra methods. Other methods involve the solution of the homogeneous system of equations \((I - Q^T)\pi = 0\), the calculation of the eigenvectors of \((I - Q^T)\), and the use of generalized inverses [12,13].

The computer program P3 is a MATLAB implementation of the difference equation solution of the stochastic reservoir equation for a reservoir with exponentially distributed inflow as an example. Consider the example given in Section 26.5.1 earlier. The application of the difference equation solution with \( k = 200 \) states of the reservoir gives an emptying probability of 0.212986 and a spilling probability of 0.191611, which are very close to those obtained by Prabhu’s exact solution and by simulation.

### 26.6 Probabilistic Approach

The solutions of the stochastic reservoir equations covered in the previous section apply only for independent inflow. Pegram [29] outlines a method for Markovian inflow, that is, inflow with first-order autocorrelation between successive values. Hamed [10] suggested a probabilistic method to calculate the
probabilities of filling and emptying of reservoirs with dependent Gaussian inflow. First, let us consider the distribution of the critical period. The critical period \((T_{c0})\) of a reservoir is defined as the period of time it takes a full reservoir to completely empty without reverting to the full state [28]. According to Oğuz and Bayazit [28], the critical period concept was first introduced in reservoir analysis by Hall et al. [8] and further extended to various models such as normal and lognormal zero-, first-, and second-order Markov models as well as normal and lognormal fractional Gaussian noise [1]. Further study of the properties of the critical period has been performed by [7] as well as Loucks et al. [23], Oğuz and Bayazit [28], and Montaseri and Adeloye [24]. Because of the condition in the definition of the critical period that the reservoir goes from the full state to the empty state without reverting to the full state, it is straightforward to derive the distribution of the critical period for Gaussian inflow in terms of a number of multivariate normal integrals. Recent advances in computing power and numerical integration methods make this task less demanding than it used to be earlier, thus offering an alternative way of calculating reservoir reliability as will be shown.

Consider an over-year storage reservoir of size \(C\) with normally distributed net inflow \(I = [I_1, I_2, I_3, \ldots]\) and an annual time step. Let \(f_T^{(1)}\) be the probability that a full reservoir becomes empty in exactly \(n\) time steps without reverting to the full state (as per the definition of the critical drawdown period). For \(n = 1\) (i.e., the reservoir empties in exactly one step), the required probability is equal to the probability that \(I_1\), the net inflow in the same time step, is less than \(-C\):

\[
f_T^{(1)} = P(I_1 < -C)
\]

The aforementioned probability involves only a univariate normal distribution integral.

For \(n = 2\) (i.e., the reservoir empties in exactly two steps), we have two conditions. The first condition is that the net inflow in the first time step must be greater than \(-C\) (so as not to empty the reservoir in one time step), but less than zero (so as not to revert to the full state, as per the definition of the critical drawdown period). For \(n = 1\) (i.e., the reservoir empties in exactly one step), the required probability is equal to the probability that \(I_1\), the net inflow in the same time step, is less than \(-C\):

\[
f_T^{(2)} = P(-C < I_1 < 0, I_1 + I_2 < -C)
\]

In this case, a bivariate normal integral is involved.

In general, for the reservoir to empty in \(n\) steps, the cumulative sums of the first \(n - 1\) time steps must each lie between zero and \(-C\), while the sum of net inflow for all \(n\) time steps must be less than \(-C\):

\[
f_T^{(n)} = P\left[\bigcap_{i=1}^{n-1}(-C < \sum_{j=1}^{i} I_j < 0), \sum_{j=1}^{n} I_j < -C\right]
\]

For the Gaussian net inflow considered here, the aforementioned probabilities can be calculated by the numerical integration of the standard multivariate normal density with \(n\) variates, which is given by

\[
f_I(I) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-1/2 (I-\beta)^T \Sigma^{-1} (I-\beta)}
\]

where

- \(I\) is the net inflow vector \([I_1, I_2, \ldots, I_n]\)
- \(\beta\) is the mean of the net inflow
- \(\Sigma\) is the \(n \times n\) covariance matrix of the net inflow
The integration is performed in the region specified by the conditions stated earlier, which can be put in a linear inequality matrix form as follows:

\[ a < A \mathbf{i} < b \]  

(26.33)

where

\[ a = [-C, -C, \ldots, -C, -\infty]^T \]

\[ b = [0, 0, \ldots, 0, -C]^T \]

\( A \) is a lower-triangle matrix of ones. Numerical methods for the calculation of the multivariate normal probability with the integration region specified as a set of linear inequalities as in Equation 26.33 earlier are given by Genz [5].

Oğuz and Bayazit [28] noted that the probabilities \( f_{T}^{(n)} \) obtained earlier need to be normalized, by dividing by their sum, to obtain proper probabilities that sum to unity, but they did not comment on the meaning of their sum. However, careful examination indicates that critical period events with all possible lengths \( n = 1, 2, \ldots, \infty \) constitute a group of mutually exclusive and collectively exhaustive events. The probability \( f_{T}^{(n)} \) represents the intersection of each of these events with the event of the reservoir being empty. Therefore, it can be conjectured that the probability of the reservoir being empty should be equal to the sum of \( f_{T}^{(n)} \), that is,

\[ G(0) = \sum_{n=1}^{\infty} f_{T}^{(n)} \]  

(26.34)

It is also worth noting that storage evolution is a Markov process [22], since the storage at any time step is fully determined through the storage at the previous time step only. Although properties of the critical period (which is a special type of first-passage times as seen earlier) do not appear to have been considered in the literature related to Markov processes, the relationship in Equation 26.34 earlier has been verified by Hamed [10]. The relationship in Equation 26.34 is a very useful link between the distribution of the critical period and the classical probability of the reservoir being empty, thus offering an alternative way of computing reservoir reliability.

The sum in Equation 26.34 is an infinite sum, but the distribution decays rapidly, and thus, one can set a reasonable limit for the inclusion of terms in the sum. For example, the calculation could be stopped when a term \( f_{T}^{(n)} \) becomes smaller than, say, 1/1000 of the largest value attained (the mode of the distribution). The main advantage of this method is that it can accommodate any arbitrary correlation structure represented by the matrix \( \Sigma \) in Equation 26.32, as long as \( \Sigma \) is positive definite to qualify as a valid covariance matrix. Calculations can be made based on the autocorrelation function only without the need for actually generating synthetic data. Another advantage of this approach is that estimates of the error in calculating the multivariate normal probabilities can also be obtained from the computer code.

The probability of spilling (or the reservoir being full), \( G(C) \), can be obtained in a similar manner by considering the distribution of the period of transition from an empty to a full reservoir without reverting to the empty state. This period could be termed the “direct filling period” as mentioned earlier. It is easy to show that the integration region in this case is defined as in Equation 26.7 earlier, but with \( a = [0, 0, \ldots, 0, C]^T \), \( b = [C, C, \ldots, C, \infty]^T \), and \( A \) is a lower-triangle matrix of ones, as before.

The computer program P4 is a MATLAB implementation of this probabilistic method to calculate the probabilities of emptying and filling of a reservoir having a Gaussian inflow using the proposed method. The code uses the multivariate normal numerical integration function “qscmvnv.m,” which is available at (http://www.math.wsu.edu/faculty/genz/homepage). As an example, consider a
reservoir with standard size $C=4$, drift $m=0.1$, and standard Gaussian inflow with first-order auto-
correlation coefficient $r=0.6$. The emptying probability is calculated as 0.186321 and the spilling prob-
ability as 0.282422. Monte Carlo simulation results give these probabilities as 0.185447 and 0.282248,
respectively.

26.7 Summary and Conclusions

Reservoirs are important elements of many water resource systems. The reliability of storage reservoirs
as well as the distribution of storage, spillage, and deficit can be obtained through various methods.
The first method covered in this chapter is Monte Carlo simulation, which is the most flexible method,
allowing different components of inflow and outflow with possibly variable values. Convergence of the
method, however, should be carefully assessed and long sequences of inflow are usually needed to have
accurate results. Direct calculation of the distribution of storage in a reservoir from the distribution of
inflow through the stochastic reservoir equation is another option. An exact solution for the special case
of the gamma inflow with integer shape parameter, which is known as Prabhu’s solution, is presented.
Two numerical solution methods of the stochastic reservoir equation, namely, the integral equations
solution and the difference equations solution, have also been outlined. Finally, a probabilistic approach
to calculating the probabilities filling and emptying of a reservoir fed with dependent inflow is covered.

Appendix 26.A

The following programs are written in MATLAB language. The text should be saved as a plain text
file with the same function name and extension “m.” For example, Program P1 should be saved as
“Prabhuexp.m.”

Program P1

This program calculates the probabilities of filling and emptying of a reservoir of size $c$ with draft $m$
and exponential inflow with parameter $\mu$. The program also calculates and plots the PDF and CDF of the
storage distribution.

```matlab
function [pe pf] = Prabhuexp(c, m, mu)
    n = floor(c/m);
    u = c-n*m;
    lambda = mu*exp(-mu*m);
    d = 0;
    for q = 0:n
        y = (c-q*m)^q/(factorial(q+1));
        d = d + (-lambda)^q*y;
    end;
    alpha = exp(-mu*(c+m))/(1-lambda*d);
    pe = 1;
    for q = 0:n
        pe = pe-exp(mu*c)*alpha*(-lambda)^q*(c-q*m)^q/factorial(q);
    end;
    pf = alpha;
    fprintf(‘\nEmpty =%10.9f, Full =%10.9f \n\n’, pe, pf);
%CDF and PDF
nx = 100;
x = [];`
f = []; F = []; if abs(u) > 0.0000001; z = linspace(-m, -m+u, ceil((nx+1)*u/m)); [G g] = getd1(z, c, alpha, mu, lambda, m, n+1); x = [x z]; f = [f g]; F = [F G]; end; for s = n:-1:1 z = linspace(c-(s+1)*m, c-s*m, nx+1); if z(end) > 0 && z(1) < 0; idd = find(z<0, 1, 'last'); z = [z(1:idd) 0 z(idd+1:end)]; end; [G g] = getd1(z, c, alpha, mu, lambda, m, s); x = [x z]; f = [f g]; F = [F G]; end; for s = 0:-1:-1 z = linspace(c-(s+1)*m, c-s*m, nx+1); if z(end) > 0 && z(1) < 0; idd = find(z<0, 1, 'last'); z = [z(1:idd) 0 z(idd+1:end)]; end; [G g] = getd2(z, c, alpha, mu); x = [x z]; f = [f g]; F = [F G]; end; subplot(1,2,1); plot(x,f,'-','linewidth',1.6); grid on; hold on; plot([0 0],[0 f(find(x>0,1,'first'))],'-r','linewidth',2); plot([c c],[0 f(find(x>c,1,'first'))],'-r','linewidth',2); set(gca,'xlim',[-m c+m]); hold off; subplot(1,2,2); idx = find(x>=0 & x<=c); plot([0 x(idx) c],[0 F(idx) 1],'-','linewidth',1.6); hold on; plot(x,F,'-','linewidth',2); plot([0 0],[0 F(find(x>=0,1,'first'))],'-r','linewidth',2); plot([c c],[0 F(find(x>c,1,'first'))],'-r','linewidth',2); axis([-m c+m 0 1]); grid on; hold off; function [G g] = getd1(z, c, alpha, mu, lambda, m, s) sm1 = 1; sm2 = mu; for q = 1:s sm1 = sm1*(-lambda)^(c-(q-1)*m-z)/(factorial(q)); sm2 = sm2+(-lambda)^(c-(q-1)*m-z)/(factorial(q-1).*(mu*(c-q*m-z)/q+1)); end; G = 1-alpha*exp(mu*(c-z)).*sm1; g = alpha*exp(mu*(c-z)).*sm2; function [G g] = getd2(z, c, alpha, mu)
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\[ G = 1 - \alpha \exp(\mu (c - z)) \]
\[ g = \mu \alpha \exp(\mu (c - z)) \]

Program P2

This program calculates the probabilities of filling and emptying of a reservoir of standard size \( c \), drift \( m \), and exponential inflow with parameter \( \mu \) using a number of reservoir stages \( k \) using the integral equations approach.

function \([p1, p2] = \text{PegramIntExp}(c, m, k, \mu)\)
\[ dk = c/k; \]
\[ y = 0:dk:c; \]
\[ C = \text{gencdf}(y' - m - c, \mu); \]
\[ rp1 = \text{repmat}(y, k+1, 1); \]
\[ rp2 = \text{repmat}(y', 1, k+1); \]
\[ A = -\text{genpdf}(rp2 - m - rp1, \mu); \]
\[ A(:, 1) = A(:, 1)/2; \]
\[ A(:, end) = A(:, end)/2; \]
\[ A = A * dk; \]
\[ A = A + \text{eye}(res + 1); \]
\[ ff1 = \text{inv}(A) * C; \]
\[ ff = ff1(2:end-1); \]
\[ pp = ff'; \]
\[ pl = ff1(1); \]
\[ p2 = 1 - ff1(end); \]
\[ \text{fprintf(‘\nempty = %10.9f, full = %10.9f ‘, ff1(1), 1 - ff1(end));} \]

function \(y = \text{gencdf}(x, \mu)\)
\[ x = x + 1/\mu; \]
\[ y = 1 - \exp(-x*\mu); \]
\[ y(x < 0) = 0; \]

function \(y = \text{genpdf}(x, \mu)\)
\[ x = x + 1/\mu; \]
\[ y = \mu * \exp(-x*\mu); \]
\[ y(x < 0) = 0; \]

Program P3

This program calculates the probabilities of filling and emptying of a reservoir of standard size \( c \), drift \( m \), and exponential inflow with parameter \( \mu \) using a number of reservoir stages \( k \) using the difference equations approach.

function \([p1, p2] = \text{PegramDiff}(c, m, k, \mu)\)
\[ h = c/k; \]
\[ Q(1, 1) = \text{gencdf}(-m, \mu); \]
\[ Q(k+2, 1) = 1 - \text{gencdf}(c - m, \mu); \]
\[ Q(1, k+2) = \text{gencdf}(-c - m, \mu); \]
\[ Q(k+2, k+2) = 1 - \text{gencdf}(-m, \mu); \]
\[ i = 1:k; \]
\[ Q(2:k+1, 1) = \text{gencdf}(i*h - m, \mu) - \text{gencdf}((i-1)*h - m, \mu); \]
\[ Q(2:k+1, k+2) = \text{gencdf}(i*h - m - c, \mu) - \text{gencdf}((i-1)*h - m - c, \mu); \]
\[ j = 1:k; \]
\[ Q(1, j+1) = \text{gencdf}((-j + 0.5)*h - m, \mu); \]
\[ Q(k+2, j+1) = 1 - \text{gencdf}(c - (j - 0.5)*h - m, \mu); \]
\[ i = \text{repmat}((1:k)', 1, k); \]
j=i;
Q(2:k+1,2:k+1) = gencdf((i-j+.5)*h-m, mu) - gencdf((i-j-.5)*h-m, mu);
P=Q^1000;
p1=P(1);
p2=P(end);
fprintf('Pempty = %10.9f, Pfull = %10.9f 
', p1, p2);

function y = gencdf(x, mu)
x = x/mu;
y = 1-exp(-x*mu);
y(x<0) = 0;

Program P4

function [fe ff] = criticalmvnar(c, m, r)
%criticalmvnar calculates the probabilities of being empty and full of a
% reservoir with AR(1) Gaussian inflow through the
% distribution of the critical drawdown and filling periods
% For a reservoir with standard size c, drift m, and Gaussian inflow with
% first-order autocorrelation coefficient r, this function uses the
% multivariate normal integral to solve for the probabilities fe and ff
% of being empty and full, respectively, via the calculation of the
% distribution of the critical drawdown and filling periods.
% nmx=500;
mp=50000;
cr=r.^(0:nmx);% This is the ACF for AR(1). Change for other models
% 1)Full to empty and 2)Empty to full
f11=zeros(1,nmx);
f22=zeros(1,nmx);
e11=zeros(1,nmx);
e22=zeros(1,nmx);
f11(1) = normcdf(-c,m,1);
f22(1) = 1-normcdf(c,m,1);
hi = 0;
for i=2:nmx
    mu1 = (1:i)*m;
    ll1 = [-c*ones(1,i-1); -inf] - mu1;
    ul1 = [zeros(1,i-1); -c] - mu1;
    A = tril(ones(i,i));
    [f11(i) ee1(i)] = qscmvnv(mp, toeplitz(cr(1:i)), ll1, A, ul1);
    if f11(i) > hi; hi=f11(i);end;
    if f11(i)<0.001*hi; break;end;
end;

hi = 0;
for i=2:nmx
    mu2 = (1:i)*m;
    ll2 = [zeros(1,i-1); c] - mu2;
    ul2 = [c*ones(1,i-1); inf] - mu2;
    A = tril(ones(i,i));
    [f22(i) ee2(i)] = qscmvnv(mp, toeplitz(cr(1:i)), ll2, A, ul2);
    if f22(i) > hi; hi=f22(i);end;
    if f22(i)<0.001*hi; break;end;
end;

fe = sum(f11);% Probability of being empty
ff = sum(f22);% Probability of being full
References