Micromagnetism as a Prototype for Complexity

3.1 Introduction

Magnetostatic dipole–dipole interactions play an important role in determining properties of nanomagnetic systems along with the exchange coupling and magnetocrystalline anisotropy, which were discussed in Chapter 2. The integral–differential equations of micromagnetics lead to complexity. These equations are nonlinear because the magnetization is a vector of constant magnitude. The nature of the dipole–dipole interactions favors patterns that are divergence free in volume and induce magnetic charges on the surfaces that create demagnetizing fields opposed to applied fields. The emerging field of spintronics with spin-polarized currents in semiconductors proceeds with the magnetic moment of the electron as just the tail of the dog. Much of the work on giant magnetoresistance (see Chapters 4 and 5), tunneling magnetoresistance (see Chapters 10–12), and magnetic random access memories (see Chapter 35) is performed by using intelligent ways to avoid the complexities of dipole–dipole interactions. The movement to ultrathin films was motivated by reducing the role of dipole–dipole interactions in directing the process of magnetization, while at the same time eliminating the variation of magnetization in the z-direction, perpendicular to the plane of the ultrathin film. Ewing understood the importance of the dipole–dipole interaction in ferromagnetism as early as in 1890. Ewing gave the name to “hysteresis” [1]. In 1935, Landau gave the first 3D model of a structure that minimized the effects of the dipole–dipole interaction (see Figure 3.1). In the 1970s, a beautiful new technology was created using magnetic bubbles* for which understanding of the dipole–dipole interactions was critical. This chapter is intended as an ode to complexity in anticipation that emerging technologies may yet benefit if the current view of dipole–dipole interactions is not far from sight. The concept of splay saving is a recurrent theme in this ode. That concept underlies the recent work of Riccardo Hertel’s group on the propagation of domain walls along a nanotube of permalloy [2].

* The past introduced J. A. Ewing to those of the new millennium. The present was dedicated to Alex Hubert and contains a description of the critical role of magnetoelasticity in a current carrying iron whisker, a subject beyond the scope of this chapter.

† The most convenient reference finder today is Google, which includes Wikipedia, that points to additional references.
This chapter is about the 3D patterns of magnetization in a nanobrick of iron with dimensions at the current limit of lithography. This nanobrick is a parallelepiped with typical dimensions $X = 130$ nm, $Y = 80$ nm, and $Z = 50$ nm. The iron nanobrick is compared with an ellipsoid of comparable dimensions to show that the main actor in this drama is a vortex, capable of playing many roles. It is anticipated that some of these roles will aid nanoscience and nanotechnology. This chapter presents calculations of the behavior of the nanobrick and provides insight into the consequences of magnetic dipole–dipole interactions and their role in computational micromagnetics. The vortex states discussed here have minimal problems from surfaces and can be manipulated by sufficiently small fields and currents to be attractive for nanoscale devices. Hubert has given the name “swirl” for the circulating pattern of magnetization in the region where a vortex meets a surface [3]. The external magnetic fields from the vortices intersecting the opposing surfaces in ultrathin films are small because the magnetic charge on one end of the vortex cancels the effects of the charge on the other end. In a nanobrick, the swirls on the two ends of the vortex can be moved far apart so that the external fields can be greater than one quarter of the saturation induction of iron (see Figure 3.2). The swirls can be moved easily and rapidly over large distances. They become highly nonlinear oscillators that serve as prototypes for the complexity of bifurcations and chaos. The calculations for the nanobrick also serve as a primer on the effective use of modern codes for micromagnetism.

3.2 Symmetry Breaking in a Cycle of Magnetization

For an ellipsoid in a mathematical model with continuous variation of magnetization, the application of a large applied magnetic field $\mathbf{H}$ along the $k$-direction, parallel to the $z$-axis, can result in a uniform magnetization. For an iron nanobrick in a high field, this is not the case. The magnetization is along $k$ at the $z$-axis, but deviates from $k$ for finite $x$ and $y$, for coordinates with the origin at the geometric center of the brick. At high fields, that deviation is outward at the top surface and inward at the bottom surface. This is called the flower state. The flower state is discussed in the caption of Figure 3.3. Discussion of the visual patterns and details of the calculations are to be found in the figure captions throughout this chapter.

As the field is lowered, the magnetization pattern changes continuously, starting at a critical field, from the flower state to the curling state in which the magnetization has components that circulate about the central axis (see Figure 3.4). This is explained below as a result of the dipole–dipole interaction and the concept of splay saving. The sense of the circulation, clockwise (cw) or counterclockwise (ccw), is determined while breaking the reflection symmetry of the flower state. The energy contours present a three-tined fork [4]. The paths to both the right and the left of the fork become lower in energy than the path along the central line. Yogi Berra, catcher for the New York Yankees, said famously, “When you come to a fork in the road, take it.” To help the computer “take it,” a current $i_z$ is passed up the $z$-axis.

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* The book as a whole reflects Alex Hubert’s hobby of collecting seashells. The complexities of magnetic domain patterns as so beautifully shown in the work of Rudi Schäfer are organized as one categorizes a biological kingdom.

† References given to the mathematics of bifurcations.
3.2.1 Breaking Inversion Symmetry and Choosing a Handedness

The inversion symmetry of the flower state may or may not be maintained in the curling state. Note that the inversion symmetry for a magnetic dipole is opposite to that of an electric dipole. Consider the moments at two points equidistant from the center of an inversion symmetry on opposite ends of a line through that center. For electric dipoles, the moments point in opposite directions. For magnetic dipoles they point in the same direction. If the inversion symmetry were maintained on the transition to the curling state, the circulation at the bottom surface would be opposite to that on the top surface (see Figure 3.5, where the “top” and “bottom” are along the x-axis). This leads to complex magnetization patterns, as the field is lowered [1] with the formation of two vortices, one of which exits the nanobrick leaving the other to arrive at the same final state, which is achieved by lowering the $H_z$ field.

Here the discussion is focused on reaching that state by lowering $H_z$. To have the circulation in the same sense on both the surfaces, it is necessary to break the inversion symmetry. A controlled way of achieving this is to pass a small current along the z-axis while lowering $H_z$.

![Figure 3.5](image-url) Curling with inversion symmetry for an iron nanobrick with dimensions $112 \times 70 \times 42 \text{nm}^3$. The moments near the front surface at $x = X/2$ show a $cw$ circulation. The moments near the back surface at $x = -X/2$ show a $ccw$ circulation. The line of moments joining the front and back surfaces shows a Néel wall in the form of an end-over-end helix. Inversion symmetry is maintained. Note that the two cones labeled A are connected by a line through the center of the brick and have the same orientation. The rendering of the magnetization as cones shows an object that does not have inversion symmetry (a line joining the tips of the cones does not go through the center of the brick). It is the magnetization at the grid points that has the inversion symmetry. The field along the x-axis, where the two surfaces are separated by 112 nm, was used for this illustration of inversion symmetry. When the field is along the z-axis, the two surfaces are separated by 42 nm. Then the exchange energy presents a sufficiently large barrier that the $cw/ccw$ structure is suppressed (see Figure 3.6). The distinction in the meaning of inversion symmetry between a polar vector and an axial vector is illustrated in the lower left corner using the polar vector $i$ for the current in a ring to produce an axial vector $m$. The two current vectors, equidistant from the center of inversion symmetry, point in opposite directions while the magnetizations they produce point in the same direction.
3.2.2 Response of Isosurfaces to Changing Fields

In the presence of the field, from a small current \( i_p \), the flower pattern develops a slight swirl even at the highest fields. This is analogous to the magnetization of a ferromagnetic material in a small field above its Curie temperature. When the temperature is lowered below \( T_C \), the response to the small field increases rapidly as the spontaneous magnetization develops. In the nanobrick, below a critical field \( H_z \), there is a spontaneous contribution to the angle of rotation of the moments away from their direction in the flower state. The swirls are centered on the \( z \)-axis (see Figure 3.6). As the swirls develop with decreasing \( H_z \), the action is first concentrated at the core of the swirl. As the field is decreased to zero, the moments in the corners are the last to fully participate in the curling pattern, turning from pointing out along a line at 45° to pointing perpendicular to that line.

The direction cosines of the magnetization \( \mathbf{M} \) are denoted by \( m_x, m_y, \) and \( m_z \). Starting from 1, \( m_z \) decreases with distance from the \( z \)-axis. Tubes of constant \( m_z \), called \( m_z \)-isosurfaces, connect the top and bottom surfaces (see Figure 3.7). The cross-sections of the tubes in planes of constant \( z \) are elliptical and centered on the \( z \)-axis, reflecting the geometry of the brick. The ellipses are largest in the midplane \( z = 0 \). The magnetization does not lie in the \( m_z \)-isosurfaces except in the midplane. Elsewhere it has a radial component as well as a circulating component. The sense of the circulating component has been forced to be the same at all \( z \) by the small current along \( z \) while decreasing \( H_z \). The radial component is inward for \( z < 0 \) and outward for \( z > 0 \) as a result of the dipole–dipole interaction. The \( m_z \)-isosurfaces define a vortex structure. The role of vortices in magnetism has often been overlooked.

![Figure 3.6](image)

**Figure 3.6** The curling pattern that develops on reducing \( \mu_0 H_0 \) from 0.9 T for the nanobrick in Figure 3.5 is called the I-vortex state. The circulation is \( \text{cw} \) (or \( \text{ccw} \)) in all planes of constant \( z \). The I-vortex state persists to zero field for the dimensions 102 x 70 x 42 nm³, but, if \( Z > 42 \) nm, the I-vortex state becomes unstable at low field. When there is a computational cell centered at the origin, the cells along the \( z \)-axis all have \( m_z = 1 \). Here the center of the vortex is midway between two cells on the \( y \)-axis displaced in \( x \) by 1 nm on either side of the vortex center. When the cells are grouped in bundles of 49 cells in a plane to produce the cones shown here, the center of the bundles nearest the center of the swirl are displaced by 7 nm from the axis. This is far enough from the center of the swirl that the magnetization lies close to the surface when \( H \) goes to zero. The development of the I-vortex state is followed here with the 42 nm high brick in the presence of a small current \( i_p \). Note the progress of the corner moments. These change little at high fields. At lower fields, they rotate so that they are parallel to one of the two corner edges. At the lowest fields, they lose their radial component and at the same time turn down in response to the dipole field from the central core of the vortex structure. Note that the hollow cone in the lower left corner of the large panel for 0.00 T appears darker at the bottom of the cone where the light enters. This cone points into the iron nanobrick as is the case for all corners. It is more evident in the lower left.
3.2.3 Displacing the Swirls

The vortex structure can be moved off the center by bias fields perpendicular to the z-axis. The vortex moves in the direction perpendicular to the bias field with the swirls remaining above one another, but the ellipsoidal cross-sections distort and need not be centered with respect to the swirls—this is for equilibrium configurations. In dynamic responses the swirls move with respect to each other. The $m_1$-isosurface takes on a life of its own. The tubes can bulge, twist, and bend. If the bias fields are derived from a current $i_{x}$ along the x-axis in the +x-direction, the upper swirl moves in the +x-direction and the lower swirl moves in the −x-direction as a result of fields in the +y-direction at the top and in the −y-direction at the bottom. The $H_x$ fields from $i_{x}$

FIGURE 3.7 The S-vortex state develops from the I-vortex state with decreasing $H_x$ for the iron brick with dimensions $156 \times 100 \times 60$ nm$^3$. Each cone represents 9 cubes with 4 nm edges in a plane of constant $y$. The coarse grid is used in this calculation for visual purposes, even though it is too crude for accurate assessment of critical fields. The width of the iron nanobrick was increased by one computational cell compared to Figures 3.3 and 3.4 in order to show the central cross-section, $y=0$, of the I-vortex state. The formation of the S-vortex state is followed using $m_1$-isosurfaces that connect the top and bottom swirls of the I-vortex state. The intersections with the midplane at $y=0$ for $\mu_0 H_x > 0.4$ T or at $y=4$ nm for $\mu_0 H_x < 0.4$ T are shown for the contour lines in that plane for $m_1 = 0.95$ and $m_1 = 0.80$ with the central white regions corresponding to $m_1 > 0.95$ and the gray regions to $0.8 < m_1 < 0.95$. A small bias current in the x-direction creates a field $H_x$ that is positive at the top and negative at the bottom, tilting the magnetization in the +x-direction at the top and in the −x-direction at the bottom. To make this clearer, a contour for $m_1 = 0$ is shown in the middle of the central white region. The tilt of the $m_1 = 0$ contour in the second panel at 0.75 T is the result of the bias current. The larger displacement of the swirls in the panel at 0.50 T is almost all the result of a spontaneous displacement of the swirls in opposite directions to form the S-vortex state. In zero field, the swirls reach their maximum displacement and appear near $x = (X - Y)/2$, $y = y_1$, $z = Z/2$ and $x = -(X - Y)/2$, $y = y_1$, $z = -Z/2$, where $y_1$ is a small displacement (see Figure 3.9). At $Z = 60$ nm, the I-state vortex can persist as a structure in unstable equilibrium all the way to $H_x = 0$. The width of the contours at $z = 0$ is essentially the same for the unstable I-state vortex and the S-state vortex.
are largest at the top and bottom surfaces. When a uniform \( H_y \) field is superimposed on the field from \( i_s \), one can independently manipulate the two swirls in any given \( H_z \).

### 3.2.4 Onset of the Landau Structure

If \( i_s \) is maintained while \( H_z \) is lowered, there is a critical \( H_z \), below which the displacements of the swirls in opposite directions increase rapidly with decreasing \( H_z \). The development of the \( S \)-vortex state from the \( L \)-vortex state is shown in Figure 3.7. The correspondence of the \( S \)-vortex state with the well-known Landau configuration is not at first obvious, if the dimensions are at the threshold for the instability of the \( L \)-vortex state. When viewed in the central cross-section, \( z = 0 \), the pattern more closely reflects the Landau configuration (see Figure 3.8). When \( X \) and \( Y \) are much larger than the values used here, the swirls sit at the ends of the 180° Bloch wall of the Landau configuration at \( x = (X - Y)/2, y = y, z = Z/2 \) and \( x = -(X - Y)/2, y = y, z = -Z/2 \), where \( y \) is a small displacement (see Figure 3.9 for a portrait of the \( S \)-vortex, which is the heart of the Landau configuration). If the current \( i_s \) were in the \( -x \)-direction while \( H_z \) was lowered through the critical \( H_z \), the swirls would be positioned at \( x = -(X - Y)/2, y = -y, z = Z/2 \) and \( x = (X - Y)/2, y = -y, z = -Z/2 \). The statics and dynamics of switching between these two configurations, called the \( S^\prime \)-vortex and the \( S \)-vortex, are the principal focus of this chapter.

### 3.2.5 Landau Structure

Landau structure, as shown in Figure 3.1, was postulated for a large iron brick, where the anisotropy causes a clearer distinction between the domains and the domain walls. The Landau structure does not have inversion symmetry, but the structure with a diamond in the center, often observed in the 1950s by researchers at General Electric and General Motors, does have inversion symmetry. Landau put the walls at 90° to avoid a discontinuity in the component of \( m \) perpendicular to the wall. The magnetization was assumed to be in the \( z \)-plane everywhere except in the 180° Bloch wall. The Bloch wall creates surface charges, which Néel eliminated by having the magnetization lie in the surface as it turns through 180°. These are the Néel caps on the Bloch wall that were first calculated by LaBonte in the 1960s in his treatment of a never-ending Bloch wall (see Figure 3.10). Later, it was pointed out that the Néel cap was an extension of one or the other of the end-closure domains with that extension, terminating in a swirl at the opposite end [5]. The swirls satisfy the topological necessity of the magnetization pointing out of the surface in at least two places. One of the first scanning electron microscopy with polarization analysis (SEMPA) experiments was to show that this is the case with iron whiskers [6]. Structures where both the swirls are on the same surface are possible [7], but not considered here. Which end of the Bloch wall claims the swirl depends on the sense of rotation.
in the Néel cap, which itself depends on the sense of rotation of the magnetization about the $y$-axis, on transversing the Bloch wall in the $y$-direction. The diamond structure, also shown in Figure 3.11, requires additional vortices as there are now two swirls and two half antiswirls on the top and bottom surfaces. But this discussion will be taken up another time.

The Bloch wall with the Néel caps is seen in Figure 3.10 for an S-vortex in the iron nanobrick for a cross-section in the plane $x=0$. This structure is essentially the same as that calculated by LaBonte in the 1960s for a never-ending $180^\circ$ Bloch wall [8]. The vortex structure about a line in the $x$-direction has long been noted, without recognizing that the vertical section of such circulation is actually the core of an $S$-vortex connecting the upper and lower surfaces. The swirls do not appear in LaBonte’s calculation, for they are displaced to infinity in the never-ending Bloch wall. They do not appear in Figure 3.10, because the cross-section is midway between the two swirls. But the displacement of the core of the $S$-vortex does appear in Figures 3.8 and 3.9 as well as in LaBonte’s calculation. The reason for the displacement has long been understood. It is to make room for the Néel caps that not only remove magnetic charge from the surfaces, but also minimize volume charge by curling about the line that is parallel to the $x$-axis.

The $S$-vortex state for a nanoellipsoid is shown in Figure 3.11. The nanoellipsoid avoids the discussion of what happens in the corners for the Landau structure in the nanobrick. The central core of the vortex in the nanoellipsoid is the Bloch wall terminating at the displaced swirls illustrated by an $m_z$-isosurface for $m_z=0.8$, which corresponds to the 3–4–5 triangle with acos $m_z \approx 37^\circ$. (The $m_z$-isosurfaces in Figure 3.7 are also for $m_z=0.8$. In Figure 3.9, the portrait of the $S$-vortex, the $m_z$-isosurfaces are for $m_z=0.95$.) The Néel caps and the closure domains are represented by the $m_z$-isosurfaces for $m_z=\pm 0.95$ in Figure 3.11. The contour lines on the circular slices at various values of $x$, show the $m_z$ components. The principal domains of the Landau structure in the ellipsoid are suggested by contours with $m_z > 0.8$ and $m_z < 0.8$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.10.png}
\caption{Cross-section in the plane $x=0$ for the $156 \times 100 \times 60$ nm$^3$ iron nanobrick. Each cone represents a 4 nm cubic computational cell. The almost vertical bowed line in the center is the contour for $m_z=0$. It is centered along the top and bottom surfaces, but bows to the right by 4 nm at the center of the nanobrick. In the white region about $y=4$ nm and $z=0$, $m_z > 0.95$. This is the core of the $S$-vortex and the center of the $180^\circ$ Bloch wall of the Landau structure. The light line at $z=0$ is also the contour for $m_z=0$. Along that line the cones rotate through an angle greater than $180^\circ$ from $y=-Y/2$ to $y=Y/2$. This is the Bloch wall separating the two principal domains of the Landau structure, where, in the white region on the far left, $m_z < -0.95$, and in the white region on the far right, $m_z > 0.95$. In the gray region on the left, $-0.95 < m_z < -0.8$. In the gray region on the right, $0.95 > m_z > 0.8$. In the gray region in the center, $0.95 > m_z > 0.8$. This figure is in one-to-one correspondence with the LaBonte’s calculation of the cross-section of a never-ending Bloch wall from the 1960s, which showed clearly the existence of the Néel caps to the left of center at both surfaces. The magnetization in the Néel caps and the core of the $S$-vortex circulate about the white bulge on the left, which forms the core of a partial vortex in the $-x$-direction. This partial vortex accounts for the component ($m_y$) that accompanies the transformation of the $I$-vortex to the $S$-vortex, as shown in Figure 3.14.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.11.png}
\caption{The Landau structure in an ellipsoid ($260 \times 161 \times 100$ nm$^3$) is calculated using Hertel’s finite element program TetraMag. The isosurfaces at $m_z=+0.95$ and $m_z=-0.95$ delineate the combined closure domains and Néel walls at each end of the core of the Bloch wall, outlined by the isosurface $m_z=0.8$. The core of the Bloch wall terminates at the surfaces in swirls. The Bloch and Néel walls separate the regions of high magnetization in the $+x$ and $-x$ directions, indicated by the darker regions of the five planes perpendicular to the $x$-axis. This pattern is topologically equivalent to the structure of Figures 3.7 through 3.9. It is generated by the same sequence of fields used in Figure 3.7 to produce the Landau structure starting from the high-field flower state. But, the flower state exists only to the extent that the finite elements are not sufficiently effective in producing an exact ellipsoid. The ellipsoid is simpler than the brick because there are no corners or edges. For the ellipsoid, this configuration is one of the eight ground states, which differ in the choice of polarization with respect to the $z$-axis, the handedness of the circulation about the $z$-axis and whether the upper swirl is on the left in an $S$-state as shown here or on the right in an $S'$-state. The two swirls and the core of the Bloch wall are displaced from the $y=Y/2$ plane as shown for the brick in Figure 3.9. The swirls can be displaced in opposite directions by the field from a current in the $x$-direction as shown in Figure 3.7 for the brick. A uniform $H_x$ will move both the swirls toward one end or the other of the ellipsoid. A large $H_x$ will drive an $S'$-pattern into an $I'$-pattern (see Figure 3.16) near the end of the ellipsoid. Then, on decreasing $H_x$, the $I'$-pattern becomes an $S$-pattern again, if biased by a field in $H_x$; if the bias field is in $-H_x$, the $I'$-pattern becomes an $S'$-pattern. In a high enough $H_x$ field, the swirls move together out from one end of the ellipsoid.}
\end{figure}
3.2.6 Manipulating the Landau Structure

Near the critical field, it is easy to drive the swirls back and forth between their small offset positions. At $H_s = 0$ it is harder. But it can still be done. It is easier to do this dynamically with the right time-sequence of fields. The moving swirls form a highly nonlinear oscillator. They can move over distances of 50 nm in 200 ps. The swirls carry with them localized external magnetic fields with $\mu_0 H_s \approx 0.5$ T from the surface magnetic charges. At resonance, they will oscillate as long as the energy is supplied to compensate for the damping losses. As the swirls oscillate back and forth in the $x$-direction, they also make excursions in the $y$-direction, as they follow paths of almost constant energy. This is in contrast with the motion of the swirls in varying fields, where the paths are perpendicular to the paths of constant energy. It is easier to go around a barrier than to climb over it. The ability to easily and quickly move well-localized sources of large external magnetic fields is a phenomenon waiting to be exploited in the world of nanoscience and nanotechnology.

3.2.7 Core Reversal by Pair Creation

When $H_s$ is increased in the $-k$ direction, the swirls move back toward the z-axis, reaching the axis at a critical field that is lower in magnitude than the critical field for forming the displaced swirls on the decreasing field in the $k$-direction. During all these changes, the $m_z = 1$ line continues to go from the bottom surface to the top surface. But at high enough field in the $-k$ direction, a pair of singularities are created that propagate from near the mid-plane outward toward the surfaces, reversing the core of the vortex as they propagate (see Figure 3.12).

The two singularities are singular in a continuum model. On a finite grid, the core of the vortex centers itself on a position between the grid points, so that the singularity itself is a mathematical point between the grid points. This should be the case for a real lattice, where the center of the vortex would lie between the atoms. But that is a classical description for which there is no quantum mechanical calculation to support the concept of the atoms maintaining a rigid magnetic moment during the reversal. The magnetic moment density can vary in direction as well as magnitude across an atom [9].

### Figure 3.12
Pair creation in a reversed applied magnetic field $\mu_0 H_s = -0.8$ T. The centers of the propagating singularities are in the middle of the four central cones in planes of constant $z$ in each of the panels A, B, C, D on the right. The panel on the left is a cut in a plane of constant $y$ through the two cones just above that center. By symmetry, the two central columns of the cones on the left are two views of each cone of the four central cones. In panels A, B, C, D, the four central cones all rotate in unison as one views this as a sequence of layers one above the other. It is also the time sequence for any given layer as the singularity passes up the axis. For the upper singularity, $\partial m_z/\partial z$ is positive and in the lower singularity, it is negative. For splay canceling, a negative $m_z$ is required for the first case and a positive $m_z$ for the lower case, as $\nabla \cdot m = m_z/r + \partial m_z/\partial z$. As $m_z$ is positive in both regions, there is splay saving in the case of the lower singularity. The lower one propagates more quickly as the splay saving lowers the energy barrier.

3.3 Landau–Lifshitz–Gilbert Equations and Their Numerical Solutions

The above description is for a vortex that intersects the top and bottom surfaces. Starting with fields in the $x$- or $y$-directions, vortices can form with swirls on the end and/or side surfaces. In either case, the removal or reversal of these fields returns the system to the state where the swirls are on the top and bottom surfaces. This process can be quite complex with the swirls moving from one surface to another or by the vortices leaving the brick and then reentering. Starting with the Landau structure in zero field, there is a rich landscape of the responses, steady and dynamic, to fields and field gradients applied in the $x$–$y$ plane.

All of the above can be seen by solving the Landau–Lifshitz–Gilbert (LLG) equations of motion. A modern review of micromagnetics is given by 33 authors in 500 pages of Vol. 2 of the *Handbook of Magnetism*. The chapters on numerical methods are quite detailed with hundreds of equations [10].

There are few cases where these equations can be treated by analytical techniques. The LLG equations are simultaneous integral–differential equations for the direction cosines of the components of the magnetization. The differential part comes from the exchange energy responsible in the first place for
ferromagnetism. Any variation of the moments from parallel alignment locally increases the negative exchange energy. The integral part comes from the dipole–dipole interactions in which, if there are \( N \) moments, there are \( N(N-1)/2 \) pairs to consider.

### 3.3.1 Zeeman Energy

The independent variable is the applied magnetic field \( H_{\text{ext}} \). The Zeeman energy is \( E_{\text{ze}} = -\mathbf{u} \cdot \mathbf{B} \) of a magnetic moment \( \mathbf{u} \) in an externally applied magnetic induction, \( \mathbf{B} \), defined by the force on a moving charge. In free space, \( \mathbf{B} = \mathbf{u} \cdot H_{\text{ext}} \), where \( \mu_0 = 4\pi \times 10^{-7} \text{H/m} \), is an arbitrary constant called the permeability of free space in SI units, which were enacted by a one-vote margin at a conference in 1931 dominated by electrical engineers with little appreciation of magnetism. Magneticians have resisted the adoption of these units in favor of Gaussian units for reasons of good physical insight. In Gaussian units \( \mu_0 = 1 \). The magnetic moment of a nanobrick is \( \mathbf{u} = \langle \mathbf{M} \rangle V \) where \( \langle \mathbf{M} \rangle \) is the average magnetization in the volume \( V \). In SI units, the magnetization \( \mathbf{M} \) is replaced by the magnetic polarization \( \mathbf{J} \), where \( \mathbf{u} = \langle \mathbf{J} \rangle / \mu_0 \). In terms of the unit vector \( \mathbf{m} \) representing the direction of magnetization or magnetic polarization, the Zeeman energy in SI units is \( E_{\text{ze}} = -J_m \mathbf{H}_{\text{ext}} \cdot \mathbf{V} = -M_m \mathbf{m} \cdot \mathbf{H}_{\text{ext}} V \), where \( J_m \) and \( M_m \) are the spontaneous polarization and the spontaneous magnetization, respectively, of the nanobrick. For iron, \( J_m = 4\pi \times 1714 \times 10^4 \text{T} = 2.154 \text{T} \), where the 1714 comes from \( M_m \), for iron in Gaussian units and the 104 comes from the conversion of Gauss to Tesla. In Gaussian units the polarization has the same units as the magnetization but differs by the factor \( 4\pi \); that is, \( J_m = 4\pi M_m \). In SI units, \( J_m = \mu_0 M_m \). This means that one cannot convert from SI to Gaussian just by setting \( \mu_0 = 1 \). The magnetic susceptibility, which is dimensionless in both systems, differs by that factor of \( 4\pi \). The engineers removed the \( 4\pi \)’s in Maxwell’s equations, from where they properly belong, in front of the source terms. In Maxwell’s equations \( M \) appears as a current source given by \( j_m = \nabla \times \mathbf{M} \). The \( 4\pi \) will reappear when the dipole–dipole interaction is considered. The anisotropy energy and the exchange energies depend only on the directions of magnetization, so the question of units does not affect their energy expressions.

### 3.3.2 Anisotropy

When a magnetic moment is rotated with respect to the atomic lattice, there is a change in energy because of spin–orbit coupling to the lattice. The derivatives of the anisotropy energy with respect to the magnetization components produce effective fields acting on the magnetization. These are not real fields, and do not necessarily obey the Rabi–Schwinger theorem [11], that a magnetic moment in a real field can be treated classically, but they are treated as real fields in the LLG equations. As the anisotropy fields play very minor roles in the behavior of an iron nanobrick, quantum mechanical subtleties will be ignored here. The magnetization patterns for an iron nanobrick and those for a permalloy nanobrick are indistinguishable in zero field at the level of visually comparing graphs of lines of constant components of the reduced magnetization. To see where anisotropy is important in spintronics, visit Chapter 2 of this book.

### 3.3.3 Exchange Field

The effective field from the exchange interaction can be written using the Laplacian of the direction cosines of the magnetization \( \nabla \cdot \mathbf{m} \), leading to nine terms in the torque equations representing curvatures in each of the components in each of the three directions. The coefficient \( A \) of the Laplacian \( \nabla^2 \) is called the exchange stiffness constant. Due to \( |\mathbf{m}| = 1 \) and hence \( \mathbf{m} \cdot \partial \mathbf{m} / \partial \mathbf{x} = 0 \), the exchange energy density \( e_{\text{ex}} \) can be written as

\[
e_{\text{ex}} = A(\nabla \cdot \mathbf{m})^2 = -A \mathbf{m} \cdot \nabla^2 \mathbf{m},
\]

(3.1)

where the Laplacian can also be written as

\[
\nabla^2 \mathbf{m} = \nabla(\nabla \cdot \mathbf{m}) - \nabla \times (\nabla \times \mathbf{m}).
\]

(3.2)

In a divergence-free pattern of magnetization, the only contributions to the exchange field come from \( \nabla \times \mathbf{m} \). In polar coordinates, there are 15 terms in the Laplacian. Even so, it is easier to think about divergence-free patterns using the Laplacian rather than \( \nabla \mathbf{m} \). The simplest magnetization pattern for a vortex is \( m_\theta = 0 \), \( m_\phi = \tanh(\alpha r) \), and \( m_z = \text{sech}(\alpha r) \), where \( \alpha \) is a constant.

There are no derivatives with respect to \( \phi \) or \( z \), there is no \( m \), component, and \( \nabla \cdot \mathbf{m} = 0 \). In this case there remain five terms in the Laplacian. When the dot product of the Laplacian and the magnetization is considered, two of the terms cancel and two of the terms combine, leaving the local exchange energy density \( e_{\text{ex}}(r) \) as

\[
e_{\text{ex}}(r) = A \left( \frac{\text{sech}^2(\alpha r)}{\alpha^2} + \frac{\tanh^2(\alpha r)}{r^2} \right).
\]

(3.3)

Even after these simplifications, the integration of Equation 3.3 requires approximations [12]. It would be helpful to have the total exchange energy in terms of \( \langle m_z \rangle \), but that requires the evaluation of \( \int r \text{sech}(\alpha r) \, dr \), which also does not have an analytic expression. Progress in micromagnetics is difficult without numerical methods. The exchange fields calculated from the simple case using \( m_z = \text{sech}(\alpha r) \) and \( m_z = \tanh(\alpha r) \) do not quite point in the same direction as the magnetization, indicating that these are not self-consistent solutions of the torque equations. They are, however, quite good approximations for the magnetization around a vortex in an ultrathin circular disk (one for which there is no significant dependence on \( z \), when the full micromagnetic

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* See p. 131 for the approximation to the integral of \( \tanh^2(\rho \lambda / \rho) \). This reference follows micromagnetics from a single iron atom imbedded in a lattice to a 4 nm thick circular disk of 96 nm diameter. The present chapter continues that development to thicknesses where the patterns of magnetization are fully 3D.
calculation includes all the dipole–dipole interactions [12]. In such situation, the parameter α is close to 1/λ, where λ is the exchange length discussed below.

3.3.4 Dipole–Dipole Interactions

The treatment of the dipole–dipole interaction is different in the finite element program TetraMag by Riccardo Hertel,* and the finite grid program LLG Micromagnetic Simulator by Scheinfein.† In TetraMag, the moments in each element are used to solve Poisson’s equation for the magnetic potential, from which the fields are derived. In the LLG Micromagnetic Simulator the dipoles on a uniform grid are summed using fast Fourier transforms, which require that every grid point be treated similarly so that the interactions depend only on the vector, connecting any two grid points. Using fast Fourier transforms reduces the calculation from the order of $N^2$ to the order of $N \ln(N)$. Fortunately, the equations of micromagnetics have attracted mathematicians who have brought sophisticated methods to bear on the problem of treating the dipole–dipole problem, sufficiently sophisticated to be beyond the intent of this article.

There is a problem in the finite grid approach in treating boundaries that are not aligned with the grid. This is avoided here by choosing the parallelepiped as the object of interest. One solution to the jaggy-edge problem is to treat the local regions on the order of $(n(n-1)/2$ and farther regions at the level of $N \ln(N)$ for all $N$ points, adding $Nn(n-1)/2$ calculations, where $n$ is the number of points in the local region. It takes but a small $n$ for $n(n-1)/2$ to be bigger than $N \ln(N)$.

The dipole–dipole energy is written using the demagnetizing field. This is taken from the fact that the sources of $\nabla \times B$ are currents and there are no sources for $\nabla \cdot B$. The vector $H$ is a mixed vector combining the source vector $M$ with the field vector $B$. In Gaussian units the combination defines $H$ as $H=\mu_B B−4\pi M$, where the $4\pi$ belongs in front of the source vector. In SI units, $H$ is defined as $\mu H=B−J$. The sources of $\nabla \cdot H$ are real currents. The sources of $\nabla \times H$ are magnetic charges given by $\nabla \cdot M$. The part of $H$ that derives from these charges is called the demagnetizing field $H_d$. In SI units, $\nabla \cdot H_d=−(J/\mu_0) \nabla \cdot M$. In Gaussian units $\nabla \cdot H_d=−4\pi M, \nabla \cdot M$. The demagnetizing field energy is the integral over the nanobrick to get $E_{dem}=−(1/2) \int H_d \cdot J \, dV$ in SI units and $E_{dem}=-4\pi M \cdot J \, dV$ in Gaussian units. The factor $(1/2)$ comes from these being self-energies. For a sphere uniformly magnetized in the $z$-direction, in Gaussian units $H_{dz}=4\pi(1/3) M \hat{k}$ and $M=M \hat{k}$ for which $E_{dem}=−(1/2) \int H_d \cdot M \, dV$. In SI units, $J=\mu B$ and $H_d=(1/3)\mu_0 \mu B$ for which $E_{dem}=−(1/2) J \cdot J \, dV$ or $E_{dem}=−(1/2) J \cdot J \, dV$. In all cases, the $(1/2)$ is the demagnetizing coefficient $N$ for a sphere. In the general ellipse, $N_1+N_2+N_3=1$. In Gaussian units, $4\pi N$ is called the demagnetizing factor.

3.3.5 Torque Equation

The basic equation of micromagnetics is the torque equation for the precession of an electron in a magnetic induction $B$. The electron has a magnetic moment $\mu=−g\mu_s S$ and an angular momentum $\hbar \gamma_s$ where $\hbar$ is the reduced Planck’s constant, $g$ is very close to 2 and $S=1/2$. The minus sign appears because the spin and the moment are in opposite direction for the negatively charged electron. The ratio of the angular momentum to the magnetic moment of the electron is $\gamma_s=−g\mu_s/h$. These are used in a classical equation of motion, where the angular momentum of the magnetic moment is $L_\mu=\mu/\gamma_s$ and the torque acting is $\mu \times B$; that is,

$$\left( \frac{1}{\gamma_s} \right) \frac{d\mu}{dt} = \mu \times B. \quad (3.4)$$

$B$ can be replaced by $\mu_0 H$ in the torque. Dividing both sides by a small volume and letting $\mu$ stand for the moment in that volume, $\mu$ can be replaced by $M$ to give

$$\left( \frac{1}{\gamma_0} \right) \frac{dM}{dt} = \mu_0 M \times H, \quad (3.5)$$

which is in SI units, but produces the torque equation in Gaussian units by replacing $\mu_0=4\pi \times 10^{-7} \text{H/m}$ by a dimensionless 1. It is common practice to absorb the $\mu_0$ into the gyromagnetic ratio to write

$$\left( \frac{1}{\gamma_0} \right) \frac{dM}{dt} = M \times H, \quad (3.6)$$

where $\gamma_0=\mu_0 \gamma_s$. As $M$, the magnetization, appears on both sides of the equation, it can be replaced by $\mathbf{J}$, the magnetic polarization ($\mathbf{J}=\mu_0 M$), to obtain

$$\left( \frac{1}{\gamma_0} \right) \frac{d\mathbf{J}}{dt} = \mathbf{J} \times \mathbf{H} = \mathbf{J} \times (−\nabla J_\text{ext}). \quad (3.7)$$

The reason this is done is to treat the $H$ in Equation 3.7 as an effective field in an expression for the free-energy density in which the Zeeman term is $J_\text{ext}=−\mathbf{M} \cdot \mathbf{B} = −\mathbf{J} \cdot \mathbf{H}$. The price for using the magnetic polarization as the variable in the torque equations is to put some $\mu_0$’s into expressions for the anisotropy and exchange energies where there is no physical reason for them to exist there. The other terms are derived from expressions

*A concise summary of TetraMag is found in TetraMag—A general-purpose finite-element micromagnetic simulation package and high resolution large-scale micromagnetic simulations with hierarchical matrices by Riccardo Hertel and Attila Kakay. These are available by inserting TetraMag Hertel into the Google search box.

†A full description of Michael R. Scheinfein’s LLG Micromagnetic Simulator is available by inserting Scheinfein micromagnetic simulator into the Google search box.
where the variable \( \mathbf{m} \) is the direction of the polarization (and the magnetization).

\[
\nabla f(e_{\text{int}}) = \frac{\nabla_m(e_{\text{int}})}{I_z} = \frac{\nabla_m(e_{\text{int}})}{\alpha M_s}.
\]

(3.8)

The \( \mathbf{H} \) in Equation 3.7 becomes an effective field. It includes the applied field, the demagnetizing field from all the other magnetic moments, the exchange field from the variation of the moment direction with position, the anisotropy field, and the damping field. The damping field was formulated by Gilbert to be proportional to the rate of change of the components of the magnetization [13]. The damping in iron comes from the repopulation of the Fermi surfaces of spin-up and spin-down electrons as the magnetization direction is rotated locally. These spin currents dissipate energy. Spin currents can be created externally and used to cause the magnetization to rotate by forced repopulation of the Fermi surfaces. The two processes differ in the sign of the coefficient of the contribution of \( \mathbf{m} \) to the effective field. There is much more about the subject of damping in Bretislav Heinrich’s chapter in this book (Chapter 2). Spin currents are not discussed in this chapter because the author has not applied them to the nanobrick, yet.

3.4 Applying the Micromagnetic Equations of Motion

This chapter describes in some detail the results of calculations for the iron nanobrick. The parameters are those of iron except for the calculation of the equilibrium configurations, where the damping coefficient \( \alpha \) is greatly increased from the low value of iron to the value that gives critical damping. The dynamic calculations use \( \alpha = 0.02 \) and the equilibrium calculations use \( \alpha = 1.0 \). For \( \alpha = 0.02 \), the time constant for the approach to equilibrium is typically 10s of ns. For \( \alpha = 1.0 \) the time constants can be <1 ns, unless the system is near a critical point for which the torques vanish. Then, the time can be too long to compute and the usual approach to equilibrium, as an exponential, changes to a 1/t approach. As critical points are of interest in describing magnetic configurations, it is necessary to have techniques to obtain the answers more quickly.

The time steps used in micromagnetic calculations are small compared to the time resolution adequate to describe the fastest of dynamic responses. The torque equations have mathematical instabilities that often require that the time steps be as small as a fraction of a femtosecond. The time step must be smaller when the grid size is smaller. The grid size itself should be smaller by at least a factor of two than the exchange length \( \lambda \) given by

\[
\lambda^2 = A/K,
\]

where \( A \) is exchange-stiffness constant of Equation 3.1 and \( K \) is the magnetostatic-energy density; \( K = \mu_0 M_s^2 / 2 \). In classical micromagnetics [14], the language is that of Gaussian units, where \( M, H, \) and \( B \) all have the same dimensions; there, \( \mu_0 \) is replaced by \( 4\pi \) in the magnetostatic energy used in the definition of the exchange length.

The use of a grid spacing that is too large can lead to results that are completely misleading. In a treatment of the magnetization processes in a nanobrick with square sides, \( X = Y > Z \), it was shown that the moments in the \( I_z \)-vortex have \( m_z = 0 \), when the grid is greater than \( 3\lambda \) [15]. Critical fields are sensitive to the grid size even for grid size \(<\lambda/10\).

3.4.1 Bias Fields

The fields from currents used to break the symmetry at critical points also provide a means of avoiding the prohibitively long computation times. Such bias fields are called anticipatory fields [16], as they select among three prongs of a three-tined fork. Without a proper bias field, one can stay on the central line, even though the other lines lead to lower energies. If an anticipatory field is not used, the middle line will be abandoned after numerical round-off errors propagate exponentially with time for more than 10-time constants. Away from critical points, the total energy comes to equilibrium exponentially, while the components of the magnetization and individual terms in the energy come to equilibrium as damped oscillators. The convergence of a calculation is achieved when the extrapolation of the exponential or the damped oscillators to infinite time no longer changes significantly with time. To find a critical field, it is convenient to use a bias field that takes the magnetization from one configuration to another quickly and then analyze the results of that calculation to obtain what would have happened in the absence of that bias field. This is called the path method [17].

3.4.2 Internal Energy

The path method relies on the insensitivity the internal energy \( E_{\text{int}} \) to the values of the external fields needed to reach configurations that have the same values of the average components of the magnetization \( \langle m_x \rangle, \langle m_y \rangle, \langle m_z \rangle \). \( E_{\text{int}} \) is the sum of all the energy terms excluding the Zeeman term. The energy, divided by the volume \( V \), along an equilibrium path can be written as

\[
\frac{E}{V} = -\langle \mathbf{m} \rangle \cdot (I_z\mathbf{H} - \nabla_m(E_{\text{int}}))|_{\langle m \rangle = \langle m^* \rangle},
\]

(3.9)

where \( \langle m^* \rangle \) is \( \langle m \rangle \) at some point along the path. Knowing the gradient of the internal energy \( E_{\text{int}} \) with respect to the components of the average magnetization as a function of \( \langle m \rangle \) along an equilibrium path, one can calculate the field necessary to reach equilibrium (not necessarily stable) for a given \( \langle m^* \rangle \) along that path. The path method assumes that knowledge of \( E_{\text{int}} \) along a path that is

* The theoretical foundation is of current interest as described in the chapter by Bret Heinrich and in the interesting exchange in Physical Review Letters that can be accessed by entering Hickey Replies in the Google search box.

† These are still worth reading today, not only for the physics but also for the style. Brown was also an English teacher.
close to a given equilibrium path will give the same result; that is, \( V_{\text{int}}(E_{\text{int}}) \) is assumed to be insensitive to small bias fields. The calculations of the magnetic response to applied fields are carried out for sufficient number of fields to determine the functional form of \( E_{\text{int}} \) but no more than necessary. Once one has an analytic expression for \( E_{\text{int}}(\langle m \rangle) \) of a particular type of pattern, one can produce the entire dependence of the magnetization on field. When this works, it can save many orders of magnitude in computation time. To determine whether it works can take time, but that need only be done once for a given type of system.

### 3.4.3 Using Anticipatory Fields

An example of the use of the path method and anticipatory fields is the calculation of the field for a vortex parallel to the z-axis to reenter an iron nanobox [15] after being driven out through the \( y=Y/2 \) surface by a field \( H_y \). After the vortex leaves, the magnetization is in a C-state, which can be viewed as a virtual \( I_y \)-vortex just outside the brick. When \( H_y \) is reduced sufficiently, the vortex should reenter the surface through which it exited. The threefold fork in this case is the fact that the virtual vortex must choose a direction of the magnetization for the core in order to reenter. The virtual vortex does not have a polarization before it enters unless the C-state itself has a bias in the direction of the vortex that exited, which can happen for particular geometries. If a bias field along the z-axis is not applied, the system stays on the central line and the vortex does not reenter until long after round-off error provides an initial bias. If a bias field is applied along the z-axis, it changes slightly the \( H_y \) at which the vortex enters, but it changes greatly, by many orders of magnitude, how long one would have to wait for that to happen.

### 3.4.4 Using the Path Method

A hypothetical example of the path method is given in Figure 3.13 where it is supposed that the dependence of the internal energy \( E_{\text{int}} \) upon \( \langle M \rangle \) is shown in the left panel as the solid curve for an arbitrarily constructed \( E_{\text{int}} = E_x(\langle m_x \rangle)^2 + E_y(\langle m_y \rangle)^2 + E_z(\langle m_z \rangle)^2(1 - \langle M_z \rangle)^2 \) to produce four inflection points and also mimic the approach to saturation. The system is in unstable equilibrium for the region between the inflection points at \( a \) and \( b \). The dashed line shows \( \Delta E_{\text{int}}/\Delta \langle M_z \rangle \), which can be calculated using the regions of unstable equilibrium if the time scale of changes is appropriate for obtaining close to equilibrium configurations while the configuration as a whole is moving in time.

The path method works only for large damping. For small damping, the magnetization moves on a path of almost constant energy, while the path method has the magnetization moving in the direction of maximum gradient of the energy.

### 3.5 Around the \( \langle M_z \rangle-H_z \) Hysteresis Loops

The computations are designed to obtain \( E_{\text{int}} \) as a function of \( \langle M \rangle \) and \( \langle M_z \rangle \) as a function of \( H_z \) for each configuration using appropriate bias fields when necessary. Hysteresis loops are shown for \( \langle m_z \rangle \) and \( \langle m_x \rangle \) in Figure 3.14 with and without a small bias field from a current \( i_e = 0.1 \) mA. The iron brick has dimensions 50×80×130 nm³. The sequence of configurations starting from high \( H_z \) includes the following.

![Figure 3.13](image-url)  
**FIGURE 3.13** Illustration of the path method. The dependence of the internal energy \( E_{\text{int}} \) upon \( \langle M_z \rangle \) is shown in the left panel as the solid curve for an arbitrarily constructed \( E_{\text{int}} = E_x(\langle m_x \rangle)^2 + E_y(\langle m_y \rangle)^2 + E_z(\langle m_z \rangle)^2(1 - \langle M_z \rangle)^2 \) to produce four inflection points and also mimic the approach to saturation. The system is in unstable equilibrium for the region between the inflection points at \( a \) and \( b \). The dashed line shows \( \Delta E_{\text{int}}/\Delta \langle M_z \rangle \), which is shown again in the right panel as the independent variable \( -H_z \) to produce a magnetization curve with hysteresis.
3.5.3 \( I_z \)-Vortex State

The same handedness throughout the nanobrick was achieved by applying a bias field from a current along the \( z \)-axis, which can be removed once the handedness is chosen.

The gray curves in Figure 3.14 were obtained in the presence of a bias field from a current along the \( x \)-axis, \( i_z = 0.1 \text{ mA} \), which anticipates the transformation from the \( I_z \)-vortex state to the \( S_z \)-vortex state as \( H_z \) is lowered. The degree of displacement of the swirls is tracked by \( \langle m_z \rangle \). The inflexion point on the gray curve at \( \mu_0 H_z = 0.32 \text{ T} \) corresponds to the field at which the transition takes place in the absence of the bias field, labeled \( d \). The bias current \( i_z \) does not produce any \( \langle m_z \rangle \) in the flower state but does in the curling state, so that \( \langle m_z \rangle \) tracks the onset of curling at \( b \) and the approach to saturation of that effect at \( a \) on the gray curves.

3.5.4 Landau Type Curling State

The \( I_z \)-vortex distorts spontaneously into either the \( S_z \)-vortex or the \( S_z \)-vortex, depending on the bias field applied from a current along the \( x \)-axis (see Figures 3.5 through 3.9). The presence of the \( S_z \)-vortex is signaled by the presence of \( \langle m_z \rangle \) in the absence of bias fields. The presence of the \( S_z \)-vortex has a minor effect on \( \langle m_z \rangle \) that can be noticed after subtracting the demagnetizing field, the dominant effect of the magnetostatic energy.

3.5.5 Corner States

The Landau type state has four choices of polarization, plus (\( p \)) or minus (\( m \)) for the virtual vortices along the four vertical corners. Without additional bias fields, the four corners for the \( S_z \)-vortex are either \( pppp, ppmm \), or \( mmpp \), labeled \( c \) from the corner \( \langle -X/2, Y/2 \rangle \). For the \( S_z \)-vortex state the sequence is \( pppp, mmpp, mmpp \). Each of these states has its range of stability with hysteresis in the minor loops for the switching of the corners. The minor hysteresis loops at low fields are shown in Figure 3.14 and in more detail in Figure 3.15, where the mean effect of the demagnetizing field has been subtracted. All of this is avoided in the ellipsoid, which has no corners.

3.5.6 S-Vortex in Reversed Field

The \( S \)-vortex is at its maximum extension for \( H_z = 0 \). For \( H_z < 0 \) the core of the \( S \)-vortex is opposite to the field, but most of the magnetization follows \( H_z \) because the energetics is a competition between the magnetostatic energy and the Zeeman energy, both of which are reduced slightly by the core magnetization remaining in the positive \( z \)-direction. The \( S \)-vortex changes back to the \( I \)-vortex in negative field. This is seen in the curve labeled \( A \) for the \( \langle m_z \rangle \) component, which goes to zero for a smaller magnitude of \( H_z \) when the fields are in the direction opposite to the core magnetization; compare points \( d \) and \( h \) in Figure 3.14. The curves labeled \( A \) are calculated for negative fields and then replotted in the first quadrant for direct comparison with the positive fields.
3.6 Discussion of Magnetization Processes

The programs for micromagnetic calculations keep track of the components of the magnetization for each grid point for every-so-many iterations as well as the net magnetizations and each term in the energy of the nanobrick as a whole. These are analyzed to gain insight into the competition among the energy terms and how they lead to the various magnetization patterns.

3.6.1 Contributing Factors

The leading energy term is the Zeeman energy. Even when the external fields are all zero, the configuration in a real nanobrick is one that reflects the past history of the magnetization. On the computer, one can arbitrarily assign a configuration A and calculate the configuration B that minimizes the energy starting from A. Then, if $H_a = 0$, the dominant term in the energy of the iron nanobrick is the dipole–dipole energy. At all other fields the competition is between the Zeeman energy and the dipole–dipole energy with the exchange playing a supporting role and the anisotropy almost no role at all.

3.6.1.1 Demagnetizing Factor

To a good approximation the dipole–dipole energy is quadratic in $\langle m_i \rangle$ for all the above states of the nanobrick. The Zeeman energy is, of course, linear in $\langle m_i \rangle$. The magnetization curve is then approximately linear in $H_z$ until the flower state is approached at high fields. The magnetization is given to a good approximation by $H_z - H_a < m_z > = 0$, where $H_a < m_z >$ is called the effective demagnetizing field. To emphasize the role of the supporting actors in this drama and to show the nonquadratic terms in $E_{denv}$ a magnetization $< m_z >_0 = H_a / H_z$ is subtracted from each point in Figure 3.14 to produce Figure 3.15, where the value of $H_a$ has been chosen to make the deviation of $\Delta < m_z > = < m_z > - < m_z >_0$ independent of $H_z$ over much of the range of $H_z$.

3.6.1.2 Vortex Contributions to the Fields

To some extent the iron nanobrick behaves like an ideally soft magnetic material, but the core of the vortex has its own life. It builds up during the transition from the flower state to the curling state. It costs exchange energy to do this. During the build up, the exchange energy is proportional to the deviation of $< m_z >$ from unity. The derivative of the exchange energy with respect to $< m_z >$ is constant during this process, giving rise to a constant effective-exchange field that aids the Zeeman field in maintaining the magnetization. Once the vortex breaks free of confinement to the symmetry axis, it is a moving wall in which changes in exchange energy are slight and compensated by changes in magnetostatic self-energy. In the dynamic response in constant applied field, the vortex can moves back and forth between the $S^+$ and the $S^-$ state on paths of constant internal energy in which there are oscillations in the exchange energy.

### Figures

**FIGURE 3.15** The hysteresis loops for $\langle m_z \rangle$ versus $\mu_0 H_z$ after subtracting $H_a / H_z$ from each $\langle m_z \rangle$ in Figure 3.14. The hysteresis in low fields accompanies the reversal of the magnetization in the four corners. All four corners flipped in a single large step in field on decreasing field, but on increasing field in smaller steps, two flip back first and, then, after a second step in field, the other two flip back. When resolved in time, the four flip independently from one another. The onset of the $S$-vortex is signaled by $\Delta < m_z >$, which begin to decrease below $d$. The lower curve is for the magnetization of the core opposite to the applied field. The $S$-vortex goes back to the $I$-vortex at $h$. The higher slope for the reversed field reflects the smaller demagnetizing field for the trapped vortex. Point $j$ is where the core reverses by pair creation and propagation.

Those curves would be obtained directly in the first quadrant if the magnetization process started with large negative fields and then proceeded to positive fields.

3.5.7 Trapped $I_z$-Vortex State

When the field is large in the reversed direction, the magnetization is negative everywhere except in the core of the vortex. The demagnetizing field becomes smaller for the trapped state leading to a higher slope of $m_z$ vs. $H_z$.

3.5.8 Transient Core Reversal State

Starting with the trapped $+I_z$-vortex, as the field becomes more negative, near $-\mu_0 H_a = 0.8$T, a pair of point singularities is created near the origin that propagate up and down the $z$-axis reversing the magnetization of the core to produce the $-I_z$-vortex with the same circulation as maintained in all these processes (see Figure 3.12 and point $j$ in Figure 3.14).

3.5.9 Saturation

At a high negative $H_a$, the $-I_z$-vortex state goes to the flower state with the moments splaying outward from the center of the bottom surface and inward toward the center of the top surface, with most of the magnetization along the $z$-axis.
and the magnetostatic energy, which are equal and opposite to one another. When the vortex is free to move as it is in the S state, the exchange energy has little effect on the magnetization loop.

3.6.1.3 Vortex Contributions to the Magnetization
The core of the vortex does have an effect on the magnetization loop because the core is magnetized. It is a separate permanent magnet that has its own magnetization loop, reversing only in fields of the order of $\mu_0 H_e = 1$ T. The volume of that permanent magnet changes somewhat with field, shrinking as the field is lowered from saturation and continuing to shrink as the field is increased in the reverse direction.

3.6.1.4 Corners as Partial Antivortices
The four corners of the cube are each one-quarter of a virtual vortex that is just outside the corners. These partial virtual vortices are antivortices with a winding number of −1. Each of the corner partial-virtual antivortices can be described using Preisach diagrams (with sloping sides). When a corner reverses, there is a change in the exchange energy in the region between the corners and the core of the S- or I-vortex in the center. The change in exchange energy with $\langle m_i \rangle$ during the reversal of a corner is an exchange field that adds to or subtracts from the applied field at the same time that the magnetization of the corner changes its contribution to the magnetization. The flipping of a corner shifts the sloping line of the demagnetizing field competition with the Zeeman field both sideways from the exchange field effect and also up or down from the change in magnetic moment.

The composite picture is then of five Preisach diagrams added to the ideal soft magnet, plus some exchange energy to be provided for the buildup of the five Preisach regions.

3.6.2 Transition from the Flower State to the Curling State
The flower state has reflection symmetry in the planes $x = 0$ and $y = 0$. It has inversion symmetry with respect to the origin at the center of the nanobrick or the nanoeellipsoid. There are states such as the diamond structure that maintain inversion symmetry while breaking the reflection symmetries; the states, considered here, all have broken the inversion symmetry as well as the reflection symmetry. A current breaks both symmetries to produce curling states.

3.6.2.1 Magnetic Charge Density
The magnetic surface charge density $\sigma_m = n \cdot M$ is positive $M_i$ at the top surface and negative $M_i$ at the bottom. The fields from these charges are radially outward from the positive charge and radially inward toward the negative charge. This produces the flower state at high fields for the flat top of the brick where the surface charge is in the planar surface. The transition to the curling state results from minimizing the volume magnetic charge density $\rho_m = -\nabla \cdot m$ that arises from the change in $m_i$ in the $z$-direction, as the surface is approached.

3.6.2.2 Magnetic Conductors
Both the applied field and the exchange energy favor parallel alignment of the magnetic dipoles, but the dipole–dipole interactions favor minimizing the surface magnetic charge density and at the same time minimizing the volume magnetic charge density. The surface charge density creates a demagnetizing field that opposes the applied field. If there were no anisotropy and the system were large enough, the magnetic surface charge density would create a demagnetizing field that is equal and opposite to the applied field for any shape, just as an electrical conductor produces surface charges to cancel applied electric fields within the conductor. The fact that the magnetic conductor has its charge limited by $M_i$ produces major differences in the response to external fields.

3.6.2.3 Ideally Soft Magnetic Materials
For a magnetic system to act like a conductor, the magnetization pattern has to adjust itself to produce the required surface charges while remaining divergence free within the volume. It can do this if there is no anisotropy and the system is large enough that the increase in exchange energy required by the divergence free pattern is very small. A large enough ferromagnetic body without anisotropy behaves as a magnetic conductor with no net field inside the body, except that in a singly connected body, the magnetic conductor cannot topologically escape the need for two swirls. For a large enough body, the magnetization pattern is divergence free everywhere except in the vicinity of the swirls. This is called the ideally soft magnetic material. This is realized experimentally in iron whiskers with $X = Y = 0.1$ mm and $Z = 10$ mm just below the Curie temperature, where the magnetic anisotropy goes to zero much faster than the spontaneous magnetization [18]. Although this work was inspired by measurements at high temperature, the calculations are all for low temperature, where thermal agitation is completely neglected, except for its effect on the material constants that are those of ambient temperature.

The electrical charge on the surface of an electrical conductor is a very small fraction of the charge on a surface atom. The magnetic charge on the surface of a magnetic conductor is limited by the finite moment of the surface atom. As the charge necessary to cancel an applied field at an edge or corner of a brick goes to infinity, the corners become saturated (in the direction of the net field), as the external field penetrates the surface.

In high fields in the curling state the iron nanobrick also mimics a magnetically soft material as long as the high field is not so high as to force the flower state.

3.6.2.4 Splay Saving
For an ideally soft ferromagnet in not too high fields, the pattern is set by the surface charge density distribution that produces a field equal and opposite to the applied field. For a cylinder in an axial field, that charge density is, to a crude approximation, linear along the cylindrical surface and constant across the top and bottom surfaces. Once there is a component of the
magnetization in the plane of the top surface and the magnetization is saturated in the midplane, there is a variation in $M_z$ with $z$ away from the surface. This would produce magnetic volume charge density $p_m = -\nabla \cdot m$, if $\partial M_z/\partial z$ were not compensated by an equal and opposite contribution to the divergence from $(1/r) \partial(r M_z)/\partial r$. If near the surface $z=Z/2$, the magnetization is approximately given by

$$M_z = M_z \cos \psi(r,z) = \frac{M_z (Z/2 - z)}{\sqrt{b^2 r^2 + (Z/2 - z)^2}}, \quad (3.11)$$

where $b$ is a constant then on the surface

$$M_z = M_z \sin \psi \left( r, \frac{Z}{2} \right) \cos \chi(r), \quad (3.12)$$

and

$$M_z = M_z \sin \psi \left( r, \frac{Z}{2} \right) \sin \chi(r), \quad (3.13)$$

where $\chi(r)$ is the angle that the magnetic moment in the plane makes with the radial vector from the axis. The pattern is more complicated than this simple expression, but it is a good local approximation to the configuration in any small region. The value of $b$ changes slowly with distance from the center of the swirl. For each $b$ there is a $\chi(r)$ for which the contribution to $\nabla \cdot m$ from $(1/r) \partial(r M_z)/\partial r$ cancels the contribution from $\partial M_z/\partial z$. For $\cos \chi = b$ the divergence vanishes. When $b = 1$, the magnetization is completely radial. For $b > 1$, the maximum amount of splay canceling accompanies the radial pattern. As $H_s$ is increased, the position where $b = 1$ moves in toward the center of the swirl. When it reaches the swirl the entire pattern becomes radial. To a first approximation the exchange energy is independent of $\chi$. It is the gradual change in $\chi$ with distance from the core that contributes to the linear increase in exchange energy proportional to $1 - \langle m_z \rangle$. The transition from the flower state to the curling state with lowering of $H_s$ starts with $\chi$ increasing from zero in the core of the swirl. If one artificially decreases the exchange energy in the iron nanobrick by a factor of 10 from its value in iron, the position, where $\chi = 0$ moves outward from the center as $H_s$ decreases. In the iron, the exchange is strong enough to couple the rotations of $\chi$ in all distances from the center of the swirl. So, once the core of the swirl has an increase in $\chi$, all regions have increases in $\chi$, but smaller, depending on the distance from the core. The rotation of the center of the swirl is continuous starting at $\chi = 0$ at the threshold field. If the exchange energy is artificially decreased, that threshold field increases.

### 3.6.2.5 Absence of the Flower State in Ellipsoids

When the calculations are carried out for an ellipsoid, the flower state does not occur. Splay saving works right up to saturation. The swirl in the ellipsoid is not at a flat surface. It is the curvature that forestalls the breakdown of splay saving when $b = 1$ in the above argument. The ellipsoid goes directly from saturation to the swirl with the handedness chosen by a symmetry-breaking field. As the swirl forms, the exchange energy increases directly with $\langle M_z - \langle M_z \rangle \rangle$. The linear increase in exchange energy with decreasing $\langle M_z \rangle$ is a constant exchange field that adds to the applied field. When $H_s$ increases below saturation, the exchange field brings the ellipsoid to saturation at a lower $H_s$ than one would obtain for a paramagnet with infinite susceptibility to reach $M_s$. The magnetization is linear in the applied field with the slope determined completely by the demagnetizing field. The demagnetizing field line is offset by the constant exchange field. For the ellipsoid, this is true for a field along any of the three principal axes. The slope is different for each axis because of the change in demagnetizing factor. The constant offset is different because the curvatures of the surface change the contribution of the exchange energy to the energy of formation of the swirl.

### 3.6.3 Curling States

The curling states include all the structures that occur once a current $i_z$ is applied while lowering $H_s$ from the flower state.

#### 3.6.3.1 Curling in an Ellipsoid

To predict the line of $\langle M_z \rangle$ versus $H_s$ for an ellipsoid, all one needs is Osborn’s formulae [19] for the demagnetizing factors of the ellipsoid and a single number for $\partial E_z/\partial \langle M_z \rangle$ for the chosen axis. The latter can be obtained from a micromagnetic calculation of $\langle M_z \rangle$ at a single field below saturation. Precise agreement with the analytic formulae has been found using TetraMag to calculate the properties of the mathematical ellipsoid with a triangular mesh on the boundaries shown in Figure 3.11. A full micromagnetic calculation of $\langle M_z \rangle$ versus $H_s$ for the approach to saturation for an ellipsoid would require very long computational times because the torques become very low as the swirl saturates. There is an important message to workers in the field of micromagnetics.

#### 3.6.3.2 Message

Modeling the results of the calculations can lead to insights that greatly shorten the computational time for a given problem. The ellipsoid at high fields is an extreme example in which one calculation in a single field, where the torques are large and the relaxation time short, produces the entire magnetization “curve” in the region where swirl remains along the axis and $\partial E_z/\partial \langle M_z \rangle$ remains constant with change in $H_s$. In this case, the field for saturation is determined precisely. The instability field at which the $I_z$-vortex moves off the axis cannot be determined without calculating the pattern changes when the swirls move off the axis.

#### 3.6.3.3 Reversal of a “Stoner–Wohlfarth Particle”

Below a second critical field, the centered position of the swirls on the ends of the principal axis of the ellipsoid becomes an energy maximum and the swirls move off the center, if the dimensions...
of the ellipsoid are sufficiently large. At small enough dimensions the ellipsoid remains “uniformly magnetized” at all fields. The magnetization process is limited to rotations in the Stoner–Wohlfarth model. It is assumed that the exchange energy does not change in the process. This model has served for 60 years as the starting point for understanding magnetization processes as a competition between the Zeeman energy and the anisotropy energy, where the anisotropy energy includes the dipole–dipole interactions and the crystalline anisotropy. Variations of the exchange energy in “uniform” rotation would also appear as an addition to the anisotropy. Even an ellipsoid in the Stoner–Wohlfarth model requires bias fields for the uniform rotation of the magnetization away from a principal axis.

3.6.3.4 Propagating Singularities
In a micromagnetic calculation, one can eliminate all the geometrical biases and cause the small ellipsoid to reverse its magnetization by a nonuniform distortion in which a singularity propagates along the principal axis starting at the swirl. The field must be applied fast enough such that the round-off errors in the numerical calculation do not have sufficient time to nucleate the uniform rotation by the displacement of the swirls. Even then, one needs a symmetry-breaking field to choose the handedness of the swirls. Here again, the numerical round-off error can provide the handedness. The field must be larger than necessary and must be applied fast enough such that the round-off error favors the reversal by singularity propagation rather than by uniform rotation.

3.6.3.5 Pair Creation and Propagation in the $I_v$-Vortex
A rule for the formation of a pair of singularities in micromagnetics has been given by Sebastian Gliga who worked in high energy physics before specializing in magnetism [20]. If enough energy is provided to create each of the singularities in a given region, the program for solving the micromagnetic equations will find that solution in which the pair is created.

As the field is increased in the negative direction for a $+I_v$-vortex state, the magnetization turns to the $-z$-direction everywhere except in the immediate vicinity of the axis of the vortex. There is a wall in which exchange energy becomes higher as the field in the $-z$-direction increases. In a bcc lattice, the singularity appears at the center of a tetrahedron of iron atoms. The four moments can no longer sustain the wall when their $m_z$ decreases to a critical value.

3.6.3.6 Bias Fields in the Stoner–Wohlfarth Model
The importance of bias fields for reversal of magnetization was first pointed out by Smith at the second MMM conference in Boston, in 1956. The subject of the session was the failure of experiments to show switching with the time constant predicted by the Landau–Lifshitz equations. The experimentalists were asking what was wrong with the Landau–Lifshitz equations. Smith showed that if the experiments and the theory are done using bias fields, they agree.

3.6.3.7 $I_v$-Vortex State
In the nanobrick the curling pattern with the swirls centered on the $z$-axis is called the $I_v$-vortex state. The $m_z$-isosurfaces are elliptical in cross-section. The central bulge along the $x$-axis corresponds to the 180° wall of the Landau structure. As the field $H_x$ is lowered, the bulge extends toward the positions $x = \pm (X/2 - Y/2)$. When the $I_v$-vortex state is maintained to $H_x = 0$, for $|x| > (X/2 - Y/2)$ and $y = 0$, magnetization lies almost in the midplane with $m_y = 1$ corresponding to the closure-domain pattern of the Landau structure. For $Z$ above a critical thickness that depends somewhat on $X$ and $Y$, the $I_v$-vortex is not stable for $H_x = 0$, but it is always an equilibrium state that persists as long as there is no symmetry-breaking field or the inevitable effect of computational round-off error has not yet developed. Once one sees the correspondence between the Landau structure and the $I_v$-vortex state, one can view the $I_v$-vortex state as the Landau structure with its two swirls centered on the $z$-axis. Or one can view the Landau structure as a vortex with its two swirls moving off the $z$-axis. The two swirls can be manipulated to move along the top and bottom surfaces distorting the 180° Bloch wall as they move.

3.6.3.8 S-Vortex States
The swirls of the $I_v$-vortex state can be manipulated. In response to a current $i$, in the $+x$-direction, the $I_v$-vortex state takes the $S^+_v$-vortex configuration. For the dimensions of the iron nanobricks chosen for this chapter, the $S^+_v$-vortex configuration appears spontaneously below a critical magnitude of $H_x$. In the absence of bias fields in the $x$- or $y$-directions, the two swirls of the spontaneous $S^+_v$ and $S^-_v$-vortex states have coordinates $(x, y, Z/2)$ and $(-x, y, -Z/2)$, respectively, where $x_1$ and $y_1$ increase with decreasing $H_x$, reaching a maximum at $H_x = 0$. As $x_1$ increases it is accompanied by an increase in $(m_y)$, as shown in Figure 3.14. This occurs because there is a displacement of the core in the $-y$-direction increasing the volume in which the magnetization in the $+x$-direction is dominant. In the midplane, where $x = 0$, the core of the vortex with $m_z = 1$ and the two Néel caps with $m_z = 1$ at the top and $m_z = -1$ at the bottom form a circulating magnetization pattern on one side of the $z$-vortex. The circulation is about an $x$-axis displaced from the midplane in the $-y$-direction, as originally calculated by LaBonte for a never-ending Bloch wall (see Figure 3.10).

The lowest $Z$ for the spontaneous appearance of an $S^-_v$-vortex structure is $Z_{crit} = 25$ nm for $Y = 35$ nm with $X$ varying from 120 to 126 nm. For $X = 119$ nm the $S^-_v$-vortex configuration goes to the $I_v$-vortex state. For $X = 127$ nm, an $S^-_v$-structure is unstable with respect to the formation of an $I_v$-vortex along the $x$-axis. For $X = 130$ nm and $Y = 80$ nm, the spontaneous $S^-_v$-structure occurs for $Z > 42$ nm.

There are also limits on the sizes of the nanobrick for which the $I_v$-vortex state is stable in the absence of a magnetic field. If the nanobrick is too small, the $I_v$-vortex state moves away from the axis and disappears out of the nearby $Y$ face, as $H_x$ is reduced. The range of dimensions $(X, Y, Z)$ and applied fields
(\(H_x\), \(H_y\), or \(H_z\)) for which the \(S'_j\) and \(S_z\) and \(I'_z\)-vortex states are stable has been studied by Templeton [21], which includes the effects of the configurations in the four corners in his elaborate phase diagrams.

### 3.6.3.9 \(I'_z\)-Vortex State

The spontaneous \(S'_j\) state is distorted by applying a uniform field \(H_x\). The swirl on the left moves toward the swirl on the right, which moves only slightly to the right. There is a critical field \(H_x\) (\(\mu_0 H_x \sim 0.1\) T) at which the left swirl catches up to the right swirl. The \(I'_z\)-vortex state is the Landau structure with both swirls on the same end of the \(180^\circ\) Bloch wall. The centers of the two swirls are at the same \(x\)-position, but the small displacements of the swirls in the \(y\)-direction are in opposite \(y\)-directions. The \(m_y\)-isosurfaces are far from symmetric. The \(180^\circ\) Bloch wall is attached to one side of the \(I'_z\)-vortex. The central cross-section of the \(m_y\)-isosurfaces bulges to include the Bloch wall (see Figure 3.16). On lowering \(H_x\), there is a critical field for re-nucleating the \(S'_j\)-vortex that restores the Landau state with the swirls on opposite ends as, \(H_x\) goes to zero. The transitions from \(S'_j\) to \(I'_z\) in \(H_x\) are not quite continuous and hysteresis occurs. The transitions from \(I'_z\) to \(S'_j\) depends on a bias to select between \(S'_j\) and \(S'_j\).

From the view of the swirls, one of the swirls traveled further to form the \(I'_z\)-vortex state in the large \(H_x\). The choice of which swirl travels back along the nanobrick when \(H_x\) is reduced is, again, a three-tined fork. If no decision is made, the \(I'_z\)-vortex state persists for a long time in a region where an \(S'_j\)-vortex would be more stable. A small bias field in \(H_x\) would make the selection. For a finite step in \(H_x\), the dynamics makes the selection. In experimental studies at Grenoble of faceted nanogoms of iron, it is the facets on the top surface that chooses the upper swirl as the one that propagates further on reducing \(H_x\) [22].

The transition to the \(I'_z\)-vortex state from the spontaneous \(S'_j\)-vortex state can also be made by applying a large current, \(i_s\) (\(-20\) mA). The latter produces a large gradient field in the \(z\)-direction (\(\mu_0 H_x = 0.06\) T at the end surface), which stabilizes the \(I'_z\)-vortex state in a first-order jump, with the complication of turning the magnetization in the corners into the direction of the gradient field, that is \(m_{mmn}\) goes to \(m_{ppn}\) for the four corners ordered \(cw\) from \((-X/2, Y/2)\). Removing the large \(i_s\) does not restore the corners to their original configurations.

### 3.6.3.10 Switching

The most dramatic result of the attack on the micromagnetics of the nanobrick was the discovery by computation that the two swirls of the Landau structure when viewed as the ends of the \(S\)-vortex could be switched back and forth, using modest driving fields, over long distances in short times, carrying with them large external fields [23]. The switching of the \(S\)-vortex had already been observed experimentally in 2004, but not specifically identified with the reversal of the positions of the two swirls [24]. The switching between two stable states is discussed here in terms of the energy landscape correlated with the positions of the two swirls. This is a gross simplification, but in equilibrium and for heavily damped dynamics, there is some usefulness in thinking about the internal energy alone and near the equilibrium path.

The combination of the field from a current \(i_s\), producing a field that is \(+H_x\) at the top surface and \(-H_x\) at the bottom, and a uniform \(H_x\), permits the independent manipulation of the positions of the two swirls in any given \(H_x\). The internal energy at equilibrium in the combined fields changes with the positions of the two swirls. At constant \(H_x\), one has an energy landscape with minima at the symmetric positions that the two swirls for \(S'_j\) or \(S'_j\) take in the absence of bias fields. If the motion of the two swirls between \(S'_j\) and \(S'_j\) is determined by a slowly varying current oscillating between \(+i_s\) and \(-i_s\), the internal energy along the path goes through the minima when the current goes through zero. The displacement in \(x\) from that equilibrium has the separation between the two swirls first increasing, reaching a maximum for the highest current, returning to equilibrium as the current goes through zero, and then having the displacement in \(x\) go toward zero as the swirls approach each other. But before they reach each other, the swirls have reached an energy position where it is all down hill toward a stable position that lies beyond the far equilibrium position in zero current. That position will be reached if the current is maintained at the critical current for switching \((i_s = 1.2\ mA\) producing a maximum field \(\mu_0 H_x = 0.06\) T at the surfaces for the nanobrick \(156 \times 96 \times 60\) nm\(^2\)).

### 3.6.3.11 Double-Well Potential

Along the path to the critical current for switching from the \(S'_j\) to \(S'_j\) configuration, the internal energy contour is one-half of a double-well potential. That potential is completely determined at each current up to the critical current. The inflection point on that curve is reached at the critical current. If the motion is calculated using large damping, a path beyond the inflection point can be determined from the damped dynamic response and the full double-well potential can be determined. This path leads through the position where the separation in \(x\) of the swirls goes through zero; but when the separation in \(x\) is zero, the separation
in y is not zero. The paths of the two swirls on the opposite faces of the nanobrick are narrow ellipses tilted in the x-y planes. The double-well potential is defined along the ellipses. The swirls do not pass over the maximum in the potential where the displacements in x and y are both zero.

### 3.6.3.12 Forced Oscillations

An example of the nonlinear oscillations of the S-vortex with large amplitude is found using a 1.28 GHz driving current $i_d = 0.6$ mA with a period of 780 ps. This current is one-half of that necessary to reverse the S-vortex with slowly varying currents. The swirls move in elliptical paths on the top and bottom surfaces that avoid the region of the local maximum in the internal energy at the center of the faces as they follow contours of almost constant energy over the saddle point on the $+y$ and $-y$ sides of the origin. The path on the top surface is shown in Figure 3.17 for one period of oscillation. The path on the bottom surface is the mirror image in either the $x = 0$ or $y = 0$ plane with the two ends of the $m_z$-isosurface moving ccw along the paths. When the swirls pass one another at $x = 0$, the two ends are displaced in $y = \pm 8$ nm. At their greatest separation $x = \pm 36$ nm at $y = 16$ nm.

![Figure 3.17](image)

**FIGURE 3.17** The dynamic response of a nanobrick with dimensions $156 \times 96 \times 60$ nm$^3$ to an ac driving current $i_d = 0.6$ mA with a period $T = 780$ ps. The path of the swirl on the bottom surface is followed in the upper panel by tracing the contours $m_z = 0.95$ in steps of 12.8 ps. The calculation was carried out with a grid of 4 nm cubes, which exaggerates the interaction of the swirl with the grid resulting in the steps of 4 nm in both the x- and y-directions. A much smoother ellipse is obtained for a grid of 1.25 nm. At no time does the central core of the S-vortex lie in a plane, let alone in the $y = 0$ plane, but the contours with $m_y = 0$ in the plane $y = 0$ are used in the bottom panel to reflect the distortions of the $m_z$-isosurfaces, which, to be fully appreciated, require 3D movies.

### 3.6.3.13 Mired at the Central Maximum

In the dynamic response with low damping, the path of either swirl can go through the origin, but generally not at the same time. But in one dynamical calculation, the two swirls came through the origin at the same time during 1 cycle of a damped oscillation. The two swirls then stayed there for a time equal to the period of the nonlinear oscillator. During this time, a higher harmonic of the dynamic response corresponding to a wave propagating up and down the z-axis provided the driving force to allow the two vortices to move away from the central energy maximum.

### 3.6.4 Motion Pictures

Most of the switching in Figure 3.17 takes place in 200 ps in each direction. The time between contours 16 and 24 is 100 ps. There is a crack-the-whip effect as the swirls rush past each other, followed by a dwell time before they make an assault on the barrier between the two stable S-states. For a lower driving current, swirls make it over the barrier in some attempts and then not in others, only to regroup and try again on the next cycle. Whether a swirl makes it over the barrier depends on the phase of oscillations along the vortex core. Such motions are not easily envisioned even with movies of the magnetization patterns. One can view the tubes of constant $M_z$ with 3D glasses or one can have a changing camera angle to add rotation of the observer to follow the 3D shape of the tubes and the oscillations that propagate along the tube, as the tube itself wanders through space. No DVD is included with this chapter, but movies are sometimes submitted as supplemental material that is available online; an example is the visual display of the small amplitude oscillations in a nanobrick [25].

### 3.6.5 What Causes the S-Vortex to Form Spontaneously?

If the exchange energy in the iron nanobrick is reduced by a factor of 10, the $S_z$-vortex is stable to a field that is about twice that of the iron nanobrick with the correct exchange energy. The lowering of the magnetostatic dipole–dipole interaction energy is the driving force for the spontaneous formation of the $S_z$-vortex state. One gets the sign correctly if one compares the demagnetizing energy for the magnetization along the z-axis, where the demagnetizing factor is largest, to along the x-axis, where it is smallest. But a vortex is far from a state where the magnetization is uniform. There is magnetic charge in the center of the swirls at the surfaces. The $m_z$-isosurfaces move inward toward the center of the swirl as the swirls move off the center. This decreases the amount of charge within any given radius from the center of the swirl. This decreases the self-energy of those charges. What causes the isosurfaces to move inward is the distance to the surfaces at $x = \pm X/2$. For the $I$-vortex, the isosurfaces move outward, if X is increased. If the swirl moves toward the X/2 end, it slightly increases the outward displacement of the isosurface toward the
receding \(-X/2\) end, but increases more the inward displacement of the isosurface toward the approaching \(X/2\) surface. Thus, there now is a partial understanding of the full Landau structure some 65 years after it was first drawn without the details of what happens at the point where the walls meet the surfaces and each other.

Acknowledgments

Bretislav Heinrich and the author started their studies of the magnetic response of iron whiskers over 40 years ago. Dan S. Bloomberg, Murray J. Press, Scott D. Hanham, Amikam Aharoni, T.L. Templeton, and J.-G. Lee all contributed to the partial understanding of a long list of phenomena observed in whiskers. The micromagnetics of a nanobrick is a continuation of that work. The nanobrick and the nanoellipsoid provide simple examples of 3D magnetic configurations in a singly connected body. Only the configurations with a single vortex have been described above. The larger the nanobrick, the more room there is for more vortices and the more the need for more computing power to fully appreciate the complexity they bring with them. In over 1000 pictures in their treatise *Magnetic Domains*, Rudi Schäfer and the late Alex Hubert have shown how complex and beautiful the patterns of magnetization can be. Riccardo Hertel and Mike Scheinftein have provided tools for and participated in the attack on this complexity. Bill Gates has removed his 2 Gbyte memory barrier for the personal computer. Attila Kakay has now implemented micromagnetism for parallel processing using the hundreds of central processing units on the graphic processor unit [26]. My work has just begun.

References

18. For the years 1971–1975 the *Proceedings of the MMM Conference* were published in books by the American Institute of Physics. These are known in magnetism as the “lost years” because the work was so infrequently cited and still are not referenced in the Web of Science. (Now AIP now has its conference proceedings online at $28 per paper.) In those years Heinrich and Arrott studied iron whiskers at the Curie temperature. The conjectured curling pattern just below \(T_c\) appeared in the seldom-cited and hard-to-find paper by B. Heinrich and A. S. Arrott, in *Proceedings of the International Conference of Magnetism ICM-73*, vol. IV, Publishing House NAUKA, Moscow, pp. 556–561, 1974.
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