The previous chapter dealt with the physical and technological principles of computed tomography (CT). We will now focus on those steps in the CT imaging chain that start from a complete set of projection data in digital form (input) and end with the generation of a digital tomographic image (output). Such steps of the imaging chain are globally referred to as “image reconstruction,” or “tomographic reconstruction.” What precedes this process (e.g., the X-ray generation, interaction with matter, and detection scanner technology) as well as what follows it (e.g., conversion of images to standard image formats, post-processing, transferring to a database, printing, and referring) are outside the scope of this chapter. Even though the next sections are, in some cases, presented in the context of medical imaging, the results obtained are fully applicable to non-medical CT scanners whose working principle is very similar to their clinical or preclinical counterparts (see, e.g., Section III, Chapters 45 and 46, for industrial CT).

The mathematical prerequisites for reading this chapter are the Fourier analysis and a basic knowledge of the theory of linear systems (see also Section I, Chapters 14 and 15, for a general introduction to image quality in X-ray imaging).

### 33.1 Objects and Images

In the context of CT imaging, we define the object function $f$ as follows

$$f(x,t,E) = \mu(x,t,E)$$ \hspace{1cm} (33.1)

that is, the function representing the spatial distribution of linear attenuation coefficients of the object under study at each spatial position $x = (x, y, z)$ of the laboratory and at each time $t$. The dependency of $f$ on the time implies that the attenuation properties of the object at any given point $x$ can change over time: in a medical context, for instance, this can happen when contrast agents (whose kinetics of distribution depend on several variables that are outside the scope of this chapter) are injected or, simply, when the entire patient or just single organs move and then change their position with respect to the reference coordinate system. In most formulæ, the dependency on $t$ will be implicit, and we will assume that the patient’s body is still in the time interval spent by the scanner for acquiring a complete dataset; moreover, we will assume that any exogenous contrast agent has already reached a stationary distribution in the body at the time of CT acquisition.

The reader must bear in mind that the object function $f$ represents the “ground truth,” that is, the actual spatio-temporal distribution of X-ray attenuation properties of the patient, which is independent from any physical measurement process. The data acquisition process in a CT scanner can be modeled as
\[ m_{ij} = P_{ij} f + \text{noise} \quad \text{(33.2)} \]

where \( m \) is the measured attenuation of the X-ray beam on the detector pixel with index \( i, j \) at the \( k \)-th projection angle, and \( P \) is a projection operator acting on \( f \). The above notation emphasizes the fact that the projection operator gives rise to a discrete set of measurements. For our purposes, let us remember three important facts about the real acquisition process in CT:

- The number of available samples \( m_{ij} \) is always finite.
- Noise is always present in the samples \( m_{ij} \).
- The projection operator \( P \) is not linear (this is either due to X-ray beam polychromacity—see Chapter 2—and/or to moderate non-linear behavior of some digital X-ray detectors—Chapter 24).

No matter how many independent measurements we perform in practice, the inverse problem of reconstructing \( f \) from the set of measurements \( m \) can never be solved exactly because a multitude of objects \( f' \) are all compatible with the available data. This kind of inverse problem is referred to as ill-posed, as the existence of a solution is not guaranteed or, even if a solution exists, it may be not unique. For an in-depth overview of inverse problems in imaging the reader should refer to other textbooks: see, for instance, Bertero and Boccacci (1998) or Barrett and Myers (2013). The best that we can do in tomography is to find an estimate of \( f \) that we will denote throughout this chapter by the symbol \( \hat{f} \). Such an estimate is referred to as image in the context of tomographic imaging (unlike projection imaging where the “image” is the projection itself).

The basic idea underlying analytical reconstruction methods is that the real, discrete, and nonlinear operator \( P_{ij} \) in Equation 33.2 can be approximated by linear and continuous operators or integral transforms of \( f \). These operators mimic the real acquisition process in an idealized fashion, and analytical inversion formulae can be found for them. Eventually, practical reconstruction is performed by adapting the inversion formulae in order to work with discrete data. All the discrepancies between real and ideal data are neglected, which is the main source of artifacts in the reconstructed images. By an abuse of notation, we could relate the image \( \hat{f} \) to the object in the following way

\[ \hat{f} = f * \cdots \ast \text{PSF} + \text{noise + artifacts} \quad \text{(33.3)} \]

where \( \ast \cdots \ast \) denotes \( n \)-dimensional convolution, with \( n \) being the dimensionality of the object function’s domain (i.e., \( n = 2 \) or \( 3 \) in this chapter) and PSF stands for point spread function. In the above formula, the PSF is not only related to the blurring that is always present because of the physical limitations of the acquisition system, but it is also dependent on the specific implementation of the reconstruction formula used. It follows from Equation 33.3 that, in absence of noise and artifacts, one could obtain \( f \) from its image just by a deconvolution. Unfortunately, the presence of noise and the discrete nature of \( \hat{f} \) in practical cases prevent us from performing straightforward deconvolution without applying some kind of regularization (see, for instance, Barrett and Myers 2013).

In this chapter, we will take into consideration both exact and approximated algorithms. Rigorously speaking, exact algorithms are such that \( \| f - \hat{f} \| < \varepsilon \) for any arbitrary \( \varepsilon > 0 \) if we use ideal data, that is, if we exclude noise, patient-related and physical-related image artifacts, and if we can arbitrarily refine the sampling step in the radial, axial, and angular directions. All algorithms not satisfying the above condition are referred to as approximated algorithms. Numerical simulations allow testing image reconstruction formulae in ideal conditions by forcing the noise and detector blurring to zero and by increasing the sampling step arbitrarily.

### 33.2 Definitions and Theorems

#### 33.2.1 Line Integrals and 2D Radon Transform

Let us first consider the case of a 2D object function \( f(x, y) \); that is, a single 2D slice of the patient in a given plane \( z = \text{cost} \) perpendicular to the axis of rotation AOR = \( z \). In the real world, such a 2D slice always has a finite thickness depending on the beam collimation width and on the detector pitch in the axial direction, as well as on the detector binning scheme used during the acquisition process. In the context of this chapter, we will neglect the physical thickness of the object function and we will just consider it as a continuous and integrable function with compact support on a plane perpendicular to the \( z \) axis. In that plane, we consider a Cartesian coordinate system \( Oxz \) and a second system \( Ox'y' \) with the same origin and rotated by an angle \( \phi \) with respect to the first one

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{(33.4)} \]

where \( R \) is the rotation matrix

\[ R = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \quad \text{(33.5)} \]

We will refer to \( Oxz \) as the laboratory coordinate system, and \( Ox'y' \) as the gantry coordinate system (see Figure 33.1). A straight line \( l \) forming an angle \( \phi \) with the \( y \) axis and placed at a distance \( x' \) from the origin can be described by the equation

\[ xc \cos \phi + ys \sin \phi - x' = 0 \quad \text{(33.6)} \]

Hence, each line in the object plane can be univocally represented by the pair of parameters \( x', \phi \), that can be regarded as signed polar coordinates because \( x' \in [-\infty, \infty] \) and \( \phi \in [0, \pi] \).

We define the line integral of the function \( f \) over \( l \) as follows

\[ p(x', \phi) = \int_{-\infty}^{\infty} f(x', y') dy' \quad \text{(33.7)} \]

In Equation 33.7, \( f(x', y') \) is the object function in the gantry coordinate system \( Ox'y' \). In what follows, it may be convenient...
where the symbol \( \mu \) is used. We have then mapped the object function \( f(x, y) \) to the set of its line integrals \( p \), which is called the Radon transform (RT) of \( f \). We will denote the RT in \( \mathbb{R}^2 \) by the symbol \( \mathcal{R}_f \), that is, \( p = \mathcal{R}_f f \).

Interestingly, the 2D Radon transform of the object function \( f(x, y) \) is strictly related to the set of physical measurements performed in a first generation CT scanner, that is, scanners working in parallel-beam acquisition geometry. It does make sense to start our discussion with the parallel-beam case because of its simplicity, and also for historical reasons: in fact, the EMI Mark I head scanner (Hounsfield 1973) employed that geometry. For each gantry angle \( \phi \) and radial position \( x' \), the X-ray intensity recorded by the detector is (by assuming a monoenergetic radiation)

\[
I(x', \phi) = I_0 e^{-\int_{-\gamma_{\text{max}}}^{\gamma_{\text{max}}} \mu(x', y') dy'}
\]

(33.9)

Dividing the above equation by the unattenuated intensity \( I_0 \) and taking its natural cologarithm we get

\[
-\ln \frac{I(x', \phi)}{I_0} = \int_{-\gamma_{\text{max}}}^{\gamma_{\text{max}}} \mu(x', y') dy' = p(x', \phi)
\]

(33.10)

which means that the set of all the X-ray projections of an object along all possible lines intersecting it is equivalent to the RT of that object. Rigorously speaking, as already explained in Section 33.1, there are evident differences between the real set of physical measurements and the 2D RT because (i) X-ray radiation produced by most sources (e.g., X-ray tubes) is polychromatic and (ii) the line integrals are only available on a finite set of \( x', \phi \) positions, where \( i = 1, \ldots, N \) and \( j = 1, \ldots, M \).

The graphical representation of \( p(x', \phi) \) in the 2D Radon domain \( Ox'\phi \) is called a sinogram. The choice of such a name relies on the fact that the Radon transform maps points of the space domain to sinusoids of the Radon domain. For example, let us find the analytical expression of the RT of the function \( f = \delta_1(x-1, y) \), that is, a point-like 2D object placed at a point \( P_1 = (x_P, y_P) = (1, 0) \). By invoking the properties of the delta function and using Equation 33.8 above, we can show that, for this case, we have \( p(x', \phi) = \delta_1(\cos \phi - x') \); the corresponding sinogram is shown in Figure 33.2.
The RT is a linear operator (see, for instance, Deans 2007). Due to its linearity, the sinogram of an arbitrary function can be seen as the linear superposition of all the sinusoids corresponding to the (infinite number of) points of the function \( f \). This observation should help the reader to better understand Equation 33.8 as it gives a graphical interpretation of the integral on the right-hand side. Furthermore, linearity also implies that if \( f = f_1 + f_2 \), then \( R_2 f = R_2 f_1 + R_2 f_2 \) (of course this example can be extended to an arbitrary number of factors).

For instance, Figure 33.3 shows a simple compound object made up of two circles with different densities, along with its sinogram. In the context of 2D image reconstruction, the sinogram will be regarded as the input of any reconstruction algorithm.

### 33.2.2 Central Section Theorem

Let us consider the two frequency spaces \( Ouw \) and \( Oud'v' \), associated with the laboratory and gantry coordinate systems, respectively. The rotation matrix \( R \) in Equation 33.5 allows one to transform the laboratory coordinates into the gantry coordinates

\[
\begin{pmatrix} u' \\ v' \end{pmatrix} = R \begin{pmatrix} u \\ v \end{pmatrix}
\]

Let us now consider an arbitrary projection \( p \) acquired at angle \( \phi \). We denote by \( P = \mathcal{F}_p \) the 1D Fourier transform (FT) of \( p \) with respect to the radial coordinate \( x' \)

\[
P(v, \phi) = \int_{-\infty}^{\infty} p(x', \phi) e^{-j 2 \pi v x'} \, dx' = \int_{-\infty}^{\infty} f(x', y') e^{-j 2 \pi v x'} \, dx',
\]

where \( v \) is the frequency variable associated with the spatial coordinate \( x' \). In this context, it is useful to express the polar coordinates of the frequency domain in a signed form, that is, \( v \in ]-\infty, \infty[ \) and \( \phi \in [0, \pi[ \).

We can now look for some relationship between the above 1D FT of the projection and the 2D FT of the object function. To this purpose we rewrite the function \( F = \mathcal{F}_f \) in the gantry coordinate system as

\[
F(u', v') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') e^{-j 2 \pi (u' x' + v' y')} \, dx' \, dy'.
\]

By comparing Equations 33.13 and 33.12 and arguing that in signed polar coordinates \( u' = v \), it is easy to show that

\[
F(u, 0) = P(v, \phi)
\]

or equivalently, by using the laboratory coordinates,

\[
F(u, v) \big|_{v \in \cos^{-1} \sin \phi = 0} = P(v, \phi).
\]
The restriction of the function \( F \) to the line of equation \(-u \sin \varphi + v \cos \varphi = 0\) (or, simply, \( \nu' = 0 \)) is referred to as the central section of \( F \), taken at angle \( \varphi \). Equations 33.14 and 33.15 are two equivalent statements of the central section theorem (CST). In the literature, this theorem is also referred to as the Fourier slice theorem.

The theorem just stated is a cornerstone of analytical image reconstruction as it provides a direct link between the object and its projections in the frequency domain. Reconstruction algorithms based on the CST fall in the category of the so-called Fourier-based methods. As a practical interpretation of CST, we could say that, due to Equation 33.15, each projection of \( f \) at an angle \( \varphi \) gives us access to a “sample” (i.e., a central section) of the 2D FT of \( f \).

Intuitively, by taking as many independent projections as possible, we can progressively refine our knowledge of \( F \) and eventually reconstruct \( f \) by taking the inverse 2D FT of such an approximated (due to the finite sampling) estimate of \( F \). We will see in the following section that the approach just mentioned leads to one possible solution to the problem of image reconstruction, called Direct Fourier Reconstruction (DFR). Figures 33.4 and 33.5 show the link between \( p \) and \( f \), in both the space and frequency domains.

### 33.2.3 X-ray Transform and 3D Radon Transform

When dealing with three-dimensional (3D) objects \( f = f(x) \), it is necessary to extend the definition of the Radon transform given above appropriately. In the laboratory coordinate system, let us define the plane integral of \( f \) as follows

\[
p(r, \theta, \varphi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) \delta_t(x \cos \varphi \sin \theta + y \sin \varphi \sin \theta + z \cos \theta - r) dx dy dz
\]

where \( r, \theta, \varphi \) are the 3D spherical coordinates (see Figure 33.6). In a more compact form, Equation 33.16 can be rewritten by

\[
p(x', \varphi) \quad \rightarrow \quad P(u, \varphi)
\]

where \( P(u, \varphi) \) is the 1D FT of the parallel-beam sinogram of \( f \) taken at angle \( \varphi \).

---

**FIGURE 33.4** Central section theorem (also known as Fourier slice theorem). The 1D Fourier transform (FT) of a row of the parallel-beam sinogram of \( f \) taken at angle \( \varphi \) is equal to a central section of the 2D FT of \( F \) taken at the same angle.

**FIGURE 33.5** Central section theorem. The sequence of operations \((\mathcal{R}_2 + \mathcal{F}_1)\) is equivalent to the sequence \((\mathcal{F}_2 + \delta_1(\nu'))\), where the operator \( \delta_1(\nu') \) works as a sampling operator of the function \( F \) by extracting its central section along the line \( \nu' = 0 \).

**FIGURE 33.6** Definition of spherical coordinates in \( \mathbb{R}^3 \). Each plane is univocally defined by a unit vector \( \alpha \) and its distance from the origin \( r \).
using the two vectors \( \mathbf{x} = (x, y, z) \) and \( \mathbf{\alpha} = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta) \),

\[
p(r, \mathbf{\alpha}) = \iiint_{\mathbb{R}^3} f(\mathbf{x}) \delta(\mathbf{x} \cdot \mathbf{\alpha} - r) \, d^3 \mathbf{x}
\]

(33.17)

which is a convenient definition for the 3D Radon transform \( \mathcal{R}_3 f \), defined at any point \((r, \mathbf{\theta}, \varphi)\) of the 3D Radon space. Despite the apparent complexity of the definition above, its geometrical interpretation is very simple: \( p(r, \mathbf{\alpha}) \) is the plane integral of the function \( f \) over the plane \( \mathbf{x} \cdot \mathbf{\alpha} - r = 0 \), that is, the plane perpendicular to the unit vector \( \mathbf{\alpha} \) and placed at a distance \( r \) from the origin of the laboratory coordinate system. In other words, the 3D RT maps a 3D object function into the set of its plane integrals. It is easy to see that the definition in Equation 33.17 reduces to the one in Equation 33.8 when moving from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \). As plane integrals haven’t an immediate link to the available data acquired with CT scanners, another integral transform is often used to describe projection data of 3D objects, referred to as the X-ray transform \( g = \mathcal{X}_3 f \)

\[
g(\mathbf{x}', \mathbf{\alpha}) = \int_{\mathbb{R}} f(\mathbf{x}' + t\mathbf{\alpha}) \, dt
\]

(33.18)

The 3D X-ray transform maps 3D object functions in their sets of line integrals. We used a different letter \( g \) in place of \( p \) to distinguish between plane integrals and line integrals.

To better understand the previous definition, let us remember that the parametric equation of a line in \( \mathbb{R}^3 \) passing through the point \( \mathbf{x}' \) and parallel to the vector \( \mathbf{\alpha} \) is \( \mathbf{x} = \mathbf{x}' + t\mathbf{\alpha} \), with \( t \in \mathbb{R} \). Again, this new transform also reduces to the 2D RT in \( \mathbb{R}^2 \), that is, \( \mathcal{X}_2 = \mathcal{R}_2 \). The previous identity does not hold in the 3D space. The X-ray transform is particularly useful, especially when dealing with divergent projection data, for example, with data acquired with third generation scanners operating in fan-beam or cone beam geometry. For such geometry, one can set the reference point \( \mathbf{x}' \) in Equation 33.18 to the position of the X-ray focal spot \( \mathbf{x}_{\text{foc}} \) at each gantry position; in this case, the function \( g(\mathbf{x}_{\text{foc}}, \mathbf{\alpha}) \) represents the set of line integrals acquired at a given gantry angle, with \( \mathbf{\alpha} \) defined in a compact angular interval depending on the physical extension of the X-ray detector and on the scanner geometry. In the next section, we will employ the X-ray transform to describe the projection data of CT scanners operating in fan-beam and cone beam geometry. To ease the notation, in that context we will use a slightly different parameterization of the function \( g \) in order to emphasize its dependence on the radial, axial, and angular variables that are more relevant for the geometry under consideration. Anyway, we will also keep the same letter \( g \) in the new parametrization in order to stress the fact that its values maintain the same physical meaning of the \( \mathcal{X}_2 f \) function, that is, a set of line integrals of \( f \).

33.3 2D Reconstruction in Parallel-Beam Geometry

This section deals with inversion formulas for the Radon transform in the simplest case of 2D parallel-beam geometry. We will see that, in 2D, exact solutions to the inversion problem are available. Even though modern CT scanners work in fan-beam or cone beam scanning geometries, the derivation of inversion formulae in parallel-beam geometry is very useful since

1. Rebinning techniques can be used to transform fan-beam data to parallel-beam data.
2. Parallel-beam reconstruction algorithms can be directly used in several positron emission tomography (PET) or single photon emission computed tomography (SPECT) scanners (for SPECT, this is true only if parallel-hole collimators are used).

In the previous section, the object function has been regarded as the “input” of several operators, or integral transforms, giving some kind of projection data as “output” (line integrals in the case of \( \mathcal{X}_1 \) and \( \mathcal{R}_2 \), or plane integrals for \( \mathcal{R}_3 \)). In this section, we will deal with the more complex problem of switching the output with the input, that is, we will now focus on the inverse problem of estimating \( f \) by starting from all the available projection data. This is what is being done everyday with CT scanners.

Even though analytical reconstruction methods don’t allow accurate physical modeling (as done in iterative methods), they still remain indispensable tools in CT imaging as they are generally easy to implement and less computationally intense than iterative methods. As mentioned above, the peculiarity of analytical reconstruction methods is that Radon or X-ray transform inversion is first obtained assuming that all the relevant functions (object and projections) are continuous functions defined in a compact support in \( \mathbb{R}^n \), and by neglecting the discrepancy between real data and ideal data. Practical algorithms are then obtained by discretization of the continuous-space analytical inversion formulae. In most cases, data preprocessing steps are introduced in order to compensate for either image noise or data nonlinearity.

33.3.1 Direct Fourier Reconstruction (DFR)

Let us suppose to know exactly the continuous 2D FT of the object, \( F = \mathcal{F}_2 f \). In this case, we could reconstruct \( f \) by just taking the inverse 2D FT of \( F \), that is, \( f = \mathcal{F}_2^{-1} F \). Indeed, we have already shown that the CST Equation 33.14 provides a direct link between the projections of \( f \) and \( F \) in signed polar coordinates, as

\[
P(v, \phi) = \int_{-\infty}^{\infty} F(u', 0) \, du'
\]

(33.13)

The previous identity can be intuitively exploited to formulate a reconstruction algorithm, called Direct Fourier Reconstruction (DFR):

Algorithm 33.1: DFR

1. Take the 1D FT of \( p \), \( P = \mathcal{F}_1 p \), for all the available projection angles \( \phi \).
2. Exploit the CST by taking samples of the function \( F \) in the frequency domain from the results of point one, thus obtaining an approximated version \( \tilde{F} \) of the 2D FT of \( f \).
3. Reconstruct the image by taking the inverse 2D FT of \( \tilde{F} \), that is, \( \tilde{f}_{\text{DFR}} = \mathcal{F}_2^{-1} \tilde{F} \).
As a consequence of the central section theorem, the available samples of $F$ are arranged in a radial grind in the frequency space.

This reconstruction method is attractive for its conceptual simplicity and also because it can take advantage of the very efficient techniques available for the computation of the discrete FT, or DFT (and of its inverse), known as fast Fourier transform or FFT. From a practical point of view, the data in a real CT acquisition are only available as an array of discrete samples, with sampling step $\Delta x'$ in the spatial domain and $\Delta \nu = 1/\Delta x'$ in the frequency domain. That is, the available samples of $F$ are arranged in a polar grid as shown in Figure 33.7.

Taking the (discrete) inverse 2D FT would require $F$ to be sampled on a rectangular grid, which implies an interpolation procedure, also referred to as “gridding” in this context. Unfortunately, interpolation in the frequency domain is generally a hard task as it leads to image artifacts when the inverse FT is performed to convert data back to the spatial domain. Hence, more advanced polar-to-Cartesian interpolation strategies than simple linear or bi-linear interpolation are employed.

### 33.3.2 Filtered Back Projection (FBP)

A different reconstruction formula arises if we rewrite the inverse 2D FT of $F$ as follows

\[
 f(x,y) = \int_0^\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi xu + yv} \, du \, dv \\
= \int_0^\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(v,\phi) p(v,\phi) e^{j2\pi x (\cos \phi + y \sin \phi)} \\
= \int_0^\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(v,\phi) q(v,\phi) e^{j2\pi x (\cos \phi + y \sin \phi)}
\]

where $dv \, du = du \, dv / |\phi'|$, and $|\phi'|$ is the Jacobian of the Cartesian to signed cylindrical coordinate transformation. Readers with a basic background in the theory of signal processing should figure out immediately that the inner integral in Equation 33.19 is equivalent to a 1D filtering of the sinogram $p(x', \phi)$ along the radial variable $x'$, with a filter with frequency response equal to $|\phi'|$. By applying the convolution theorem, this equation can be rewritten as

\[
f(x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} dx' p(x',\phi) h(x \cos \phi + y \sin \phi - x')
\]

where we have defined the auxiliary function $q = p * h = F_{-1}^{-1}(P|\phi|)$. Here, and in what follows, we will refer to the function $q$ as the filtered sinogram.

The inversion formula just written leads to the most used reconstruction algorithm in CT, which is called filtered back projection (FBP). This name is better suited when the inner integral is performed in the frequency space (see Equation 33.19); if, instead, the inner integral is written in the form of a convolution in the space domain, as in Equations 33.20 and 33.22, then the same algorithm is conventionally called convolution back projection (CBP). From a practical point of view, filtering of 1D signals in the frequency domain is more efficient than that in the space domain, because computer implementations of this task can take advantage of very efficient strategies for the calculation of forward and backward FFT (both using general purpose central processing units or CPUs, hardware-based tools such as Field Programmable Gate Arrays or FPGAs or, more recently, also graphic processing units or GPUs). Practical issues of implementation of the FBP/CBP methods will be discussed later in this chapter. From now on, we will only use the FBP acronym to denote both methods, assuming that it will appear clear from the context which method is discussed.

As the name suggest, the FBP algorithm is made up of two basic “blocks”: a 1D filtering (the inner integral of Equations 33.19 through 33.22) and a 2D back projection (the outer integral in the same equations). It must be noted that the integral definition of the ramp filter kernel $h$ in Equation 33.21 is
meaningful only in the sense of distributions, as the function \( |v| \) is not \( L^1 \) integrable and thus it does not admit an inverse FT. Nevertheless, an explicit expression for the ramp filter impulse response \( h \) can be found in practice by assuming that the object function \( f \) is band-limited, that is, \( F(\nu, \phi) = 0 \) for \( |\nu| > \nu_{\text{max}} \). If \( f \) is band-limited, it follows from the CST that \( P(\nu, \phi) = 0 \) for \( |\nu| > \nu_{\text{max}} \) (in other words, \( p \) is also band-limited) and hence the inner integral in Equation 33.22 can be restricted to the compact interval \([-\nu_{\text{max}}, \nu_{\text{max}}]\) without loss of generality. A natural choice for the limiting frequency can be made by remembering that, in practice, all sinograms are sampled with step \( \Delta x' \) and hence we can set \( \nu_{\text{max}} = \nu_{\text{Nyq}} = 1/(2\Delta x') \). Due to the low-pass behavior of all real X-ray detectors, the spectrum amplitude of \( p \) for spatial frequencies greater than \( \nu_{\text{max}} \) should be negligible in most cases. In practical applications, a further modification of the ramp filter is done by mean of apodization windows, that we will generically denote as \( A(\nu) \) in this context

\[
\tilde{h}(x') = \int_{-\nu_{\text{max}}}^{\nu_{\text{max}}} dv A(\nu) v e^{j2\pi \nu x'}
\]

\[
= \int_{-\nu_{\text{max}}}^{\nu_{\text{max}}} dv \Pi(\frac{v}{2\nu_{\text{max}}}) A(\nu) v e^{j2\pi \nu x'}.
\]  

(33.23)

where \( \Pi \) is the boxcar (or rect) function. Kak and Slaney reported an analytical expression of \( \tilde{h} \) in the simplest case of a flat apodization window (i.e., \( A(\nu) = 1 \)) with cutoff frequency equal to \( 1/\Delta x' \) (this is also known as a Ramachandran–Lakshminarayan filter or, for short, Ram–Lak filter) (Figure 33.8)

\[
\tilde{h}_{\text{Ram–Lak}}(x') = \frac{1}{(2\Delta x')^2} 2\text{sinc} \left( \frac{x'}{\Delta x'} \right) - \text{sinc}^2 \left( \frac{x'}{2\Delta x'} \right),
\]

(33.24)

with \( \text{sinc}(\chi) = \sin(\pi \chi)/\pi \chi \) is the cardinal sine function and \( \Delta x' \) is the sampling step, linked to the limiting frequency by the relation

\[
\Delta x' = \frac{1}{2\nu_{\text{max}}}.
\]

(33.25)

The notation just used emphasizes the conceptual differences between the real filter kernel \( \tilde{h} \) and the distribution \( h \). In most CT workstations, the type of window \( A \) is one of the (indeed, very few) parameters of image reconstruction that the user is allowed to change when using FBP. Briefly, we will just say that such windows allow the user to modify the shape of the ramp filter by attenuating the high spatial frequency and then reducing the noise in the final image. We can now write an inversion formula for \( p \), which is the basis for all practical implementations of the FBP algorithm

FIGURE 33.8 Effect of the application of the ramp filter on the sinogram of the object shown in Figure 33.3. It is worth noting that the ramp filter introduced negative values in the sinogram space, as visible in the linear plot on the right.
with obvious meaning of the modified filtered sinogram \( \tilde{q} = p * h \). In the above equation, we have assumed that \( f \) (and hence \( p \)) are both band-limited. It is worth noting that this must be meant as just a practical approximation, because band-limitedness in the frequency domain does imply an infinite extension of the object in the space domain. On the other hand, as already explained above, the low-pass behaviour of most physical systems for X-ray detection partly justifies having neglected the information content at spatial frequencies \( |l| > \nu_{\text{Nyquist}} \). The final outcome of such an approximation is that aliasing artifacts always appear in images reconstructed by FBP; these artifacts become particularly evident when sharp edges and high-contrast details (rich in high spatial frequencies at the interfaces with the softer background materials) are present in the scanned object. Hip prostheses or dental implants are typical examples of objects where we could not neglect the spectrum amplitude above the Nyquist frequency; for such objects, aliasing artifacts appear (along with photon starvation and beam hardening artifacts) in the FBP-reconstructed images unless specific correction strategies are applied.

The integration with respect to the angular variable in all the equations above in this section is called “back projection,” (BP) representing the second step of the FBP algorithm, which follows the application of the ramp filter to each radial section of the sinogram. We will denote the back projection operator by the symbol \( R^a \), defined in two dimensions by means of the identity

\[
\tilde{f}_{\text{FBP}}(x,y) = \int_0^\pi \int_{-\infty}^{\infty} dx' \int_0^\pi dx \, p(x',\phi) \tilde{h}(x \cos \phi + y \sin \phi - x') = \int_0^\pi d\phi \tilde{q}(x \cos \phi + y \sin \phi, \phi)
\]

(33.26)

\[
= \int_0^\pi d\phi (p * \tilde{h})(x \cos \phi + y \sin \phi, \phi)
\]

(33.27)

By comparing Equations 33.27 and 33.20, it appears clear that the back projection is not the inverse of the projection operator, or Radon transform. In fact, in Equation 33.27 we got the object function \( f \) by applying \( R^a_2 \) to \( q \) (i.e., the filtered sinogram) instead of \( p \). We could then write

\[
f = R^a_2 (R_2 f * h).
\]

(33.28)

As a further, empirical demonstration that \( R^a = R^{-1} \), let us consider a rectangular object, as shown in the left side of Figure 33.9, and let us try to reconstruct \( f \) supposing only two projections at zero and 90 degrees. Each projection produces a one-dimensional profile \( p \), corresponding to a row of the sinogram. Let’s now suppose we start from an empty image in the \( Oxy \) plane, by trying to “reverse” the forward projection process that produced those two profiles. Practically, this is similar to smearing back (i.e., back projecting) each profile on the image plane along the original direction of projection, as shown in the right side of Figure 33.9. Each back projected profile will give rise to an intermediate image. By superimposing all the intermediate images of the acquired profiles we get a BP-reconstructed version of the object function, \( \tilde{f}_{\text{FBP}} \). It is intuitive that the quality of the reconstruction increases as the number of back projected profiles is increased, even though we are not focusing on sampling considerations in this example.

One problem with this approach is that, if we skip the ramp-filtering step, the back projected profiles will never superimpose in a destructive way because the projection \( p \) is positive definite (as it comes from the line integrals of a positive definite physical quantity, \( \mu \)). Hence, by BP alone, we cannot avoid the “tails” of the reconstructed image away from the object’s support. The ultimate reason why BP alone isn’t able to reconstruct the original object, apart from the insufficient number of views in this example, is more understandable by looking at the radial grid in the frequency

FIGURE 33.9 Example of back projection for a simple rectangular object. The linear super-position of the back projected view is constructive inside the support of the object, but is incoherent outside. As a consequence, non-zero values are reconstructed even outside the physical extension of the object.
The sparsity of the samples of the object’s 2D FT at high spatial frequency with respect to those at lower spatial frequency has to be compensated in some way. High frequency samples of the spectrum must be weighted more than low frequency ones in order to compensate for such differences in sampling density. In other words, a filter must be applied to the data before back projection. Because the sample density decreases linearly with \( |l| \), the ramp filter appearing in Equation 33.19 is needed to counterbalance the undersampling at high frequencies.

Let us now write a possible workflow for a practical implementation of the FBP algorithm.

This implies a discretization strategy that will be discussed later:

**Algorithm 33.2: FBP**

1. Select the apodization window \( A(\phi) \) and compute the discrete version of the modified filter kernel \( \hat{h} \);
2. Apply the 1D ramp-filter to the sinogram: for each available projection angle \( \phi \), take the 1D DFTs \( P \) and \( \hat{H} \) of the sinogram and of the modified filter, respectively, and multiply them in the frequency domain; afterwards, compute the filtered sinogram \( \hat{q} \) as the inverse 1D DFT of the product \( P \cdot \hat{H} \), that is, \( \hat{q} = F^{-1}_{1D}(P \cdot \hat{H}) \);
3. Reconstruct the image by back projecting each row of the filtered sinogram \( \hat{q} \) on the image plane along its own direction of projection \( \phi \), that is, \( \hat{f}_{BP} = R^2_{x,y} \hat{q} \).

### 33.3.3 Practical Considerations in FBP Reconstruction

Apart from a constant normalization factor, the discretized version of the FBP inversion formula can be written as

\[
\hat{f}_{BP}(x,y) = \sum_{k=0}^{N_{ax}-1} \hat{q}(x_k \cos \phi_k + y_k \sin \phi_k, \phi_k),
\]

where the back projection integral has been replaced by a back projection summation. A practical problem with the implementation of this formula is that the available radial positions \( x'_l = l \Delta x', \ l = 0, \ldots, N_t - 1 \), in the pixelized detectors almost never coincide with \( \hat{q} \) in Equation 33.29. Hence, interpolation of the actual values of the radial argument is required to calculate the value of the filtered sinogram in the desired position. Figure 33.10 shows the effect of two different types of interpolation in back projection for a simple dataset. In most practical cases, linear interpolation gives acceptable results. Nevertheless, nearest neighbor interpolation is more computationally efficient.

A common shape for the apodization window widely used in practical FBP reconstruction is due to Hamming and is depicted in Figure 33.11, in comparison with the product \( \text{lr} A(\phi) \):

\[
A_{\text{Hamming}}(\phi) = \begin{cases} 
\frac{\xi + (1 - \xi) \cos \left( \frac{\pi v}{v_{\text{max}}} \right)}{} & \text{if } |v| < v_{\text{max}}, \\
0 & \text{if } |v| \geq v_{\text{max}}. 
\end{cases}
\]

The parameter \( \xi \) in the previous equation acts as a smoothing factor and can be varied in the interval \( \xi \in [0.5; 1] \) (\( \xi = 0.5 \) gives the maximum smoothing, while \( \xi = 1 \) reduces to the flat window).

The effect of angular sampling should also be taken into account in practical reconstruction. As a rule of thumb, the number of angular projections over a \( \pi \) interval should be of the same order of magnitude as the number of radial bins in the sinogram (see Kak and Slaney 1988). In Figure 33.12, the effect of angular undersampling is evaluated for the same object of the previous examples reconstructed by FBP. The streaks visible for large sampling steps are aliasing artifacts coming from angular undersampling. This type of artifact is very well known in FBP reconstruction.

The most time consuming step in FBP reconstruction is the back projection. For a \( N \times N \) image matrix with a number of projections \( M \) over 180 degrees of the same order of \( N \), this operation implies that approximately \( N^3 \) pixel projections onto the detector array must be performed. Some investigators have proposed fast back projection strategies with complexity of order \( O(N^2 \log N) \) (see, for instance, Turbell 2001). In some circumstances, the complexity of the back projection can be reduced by exploiting the symmetries of the imaging system. For instance, let us suppose to reconstruct an \( N \times N \) image, with \( N \) even, and suppose that the number \( M \) of projections over 180 degrees is a multiple of four. In this case, the calculation of the detector position \( x'_l = x_l \cos \phi_k + y_l \sin \phi_k \) for a pixel at point \( x_l, y_l \) will give exactly the same result for the other three pixels located on the other three quadrants of the \( xy \) plane (see Figures 33.8 and 33.11).

### 33.3.4 Back Projection-Filtration (BPF)

We have shown in Equation 33.28 that the back projection operator \( R^\ast \), which is the dual operator of the Radon transform \( R^\ast \), is different from the inverse Radon transform \( R^{-1} \). When we omit the pedix in the RT symbol, it is understood that we are referring to the transform in two dimensions. We have seen in the previous section that we can still obtain an approximated (blurred) version of \( f \) by performing a simple, unfiltered back projection of the sinogram, that is, \( \hat{f}_{BP} = R^\ast P = (R^\ast R)f \). The function \( \hat{f}_{BP} \) is called the laminogram of the object function \( f \). We can state that the laminogram and the object function are linked by the operator \( R^\ast R \), having an impulse response \( h' \) such that

\[
\hat{f}_{BP} = R^\ast Rf = h' \ast f
\]

In order to find an analytical expression for the 2D filter \( h' \), it is easier to study the problem in the frequency domain by writing \( \hat{F}_{BP} = H' \cdot \hat{F} \), where \( H' = H'(u, v) \) is the frequency response of the blurring filter. Hence, we rewrite Equation 33.31 as follows

\[
\hat{f}_{BP}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{H'}(v, \phi) F(v, \phi) e^{2\pi i (x \cos \phi + y \sin \phi)} dv \, d\phi
\]

If we set

\[
H'(v, \phi) = \frac{1}{|v|}
\]
in Equation 33.32 then we get

\[
\tilde{f}_{BP}(x, y) = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dv \frac{1}{|v|} P(v, \phi) e^{j2\pi vx \cos \phi + vy \sin \phi} \\
= \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dv P(v, \phi) e^{j2\pi vx \cos \phi + vy \sin \phi} \\
= \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dv P(v, \phi) e^{j2\pi vx \cos \phi + vy \sin \phi} \\
= \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dv P(v, \phi) e^{j2\pi vx \cos \phi + vy \sin \phi}
\]

which demonstrates that the choice of the 2D blurring filter \(H^\prime\) defined in Equation 33.33 is coherent with the definition of the laminogram in Equation 33.31. In other words, we have found out that by backprojecting the unfiltered sinogram we get an image given by the convolution of \(f\) with a 2D blurring filter with frequency response \(1/\sqrt{u^2 + v^2} = 1/|v|\).

The identity in Equation 33.31 suggests another reconstruction method consisting of the 2D deblurring of the laminogram \(\tilde{f}_{BP}\). Deblurring in the frequency space is intuitively done by taking the reciprocal of the blurring filter \(H^\prime\) with frequency response \(\tilde{H}^\prime = 1/H^\prime = \sqrt{u^2 + v^2}\), and then writing

**FIGURE 33.10** Effect of the choice of interpolation in pixel-driven back projection. (a) For \(\phi_k = 0\), no interpolation is necessary. Nearest neighbor (b) or linear (c) interpolation are used in general.
Basically, an implementation of the BPF algorithm consists of switching the filtering and back projection steps of the FBP algorithm, that is, by first back projecting the unfiltered sinogram in the image space and then deblurring the intermediate image by applying the 2D filter \( \hat{H}' \). Also in this case, 2D apodization windows \( A'(\nu, \varphi) \) can be used in practice to control the image noise in the reconstructed image.

Algorithm 33.3: BPF

1. Reconstruct the intermediate image \( \tilde{f}_{BP} \) (i.e., the laminogram of \( f \)), by back projecting each row of the unfiltered sinogram \( p \) on the image plane along its own direction of projection \( \phi \) that is; \( \tilde{f}_{BP} = R_f^p \).
2. Select the 2D apodization window \( A'(\nu, \varphi) \) and compute the discrete version of the deblurring filter \( A'\hat{H}' \);
3. Multiply the 2D deblurring filter \( A'\hat{H}' \) to the function \( \tilde{F}_{BP} = \mathcal{F}_2 \tilde{f}_{BP} \) in the frequency space;
4. Reconstruct the image by taking the inverse 2D DFT of the result of the previous point, that is, \( \tilde{f}_{BPF} = \mathcal{F}_2^{-1}(A'\hat{H}' \tilde{F}_{BP}) \).

Figure 33.14 shows the conceptual difference between FBP and BPF reconstruction. Due to both historical reasons and computational efficiency considerations, the FBP approach to image reconstruction is most widely employed in practice in real CT scanners.
We will now extend the results obtained in the previous section in order to reconstruct CT slices from projections acquired with divergent beams. This is very useful in practice because almost all today’s CT scanners are based on acquisition of divergent projections. The 2D fan-beam geometry is shown in Figure 33.15.

The practical implementation of this geometry can rely on both flat or curved detectors: in this chapter, we will just consider the most common cases of centered flat detectors (especially useful when flat X-ray detectors are used) and circular arc detectors (which is the standard for clinical CT scanners).

33.4 2D Reconstruction in Circular Fan-Beam Geometry

We will now extend the results obtained in the previous section in order to reconstruct CT slices from projections acquired with divergent beams. This is very useful in practice because almost all today’s CT scanners are based on acquisition of divergent projections. The 2D fan-beam geometry is shown in Figure 33.15.

The practical implementation of this geometry can rely on both flat or curved detectors: in this chapter, we will just consider the most common cases of centered flat detectors (especially useful when flat X-ray detectors are used) and circular arc detectors (which is the standard for clinical CT scanners).
where the fan-beam coordinates can be calculated for each desired bin of the Radon space with the following formulae:

\[
\begin{align*}
\sigma(x') &= \frac{D_1 x'}{\sqrt{D_1^2 - x'^2}}, \\
\gamma(x') &= -\sin^{-1} \frac{x'}{D_1}, \\
\beta(x', \phi) &= \phi + \gamma(x').
\end{align*}
\]

The reconstruction then consists of filling up the parallel-beam sinogram \( p \) by taking, for each position of the \( x', \phi \) space, the corresponding line integral in the fan-beam sinogram.

From a practical point of view, mapping the \((\sigma, \beta)\) coordinates or the \((\gamma, \beta)\) coordinates to the Radon coordinates requires some type of interpolation.

### 33.4.2 Full-Scan (2\(\sigma\)) FBP Reconstruction in Native Fan-Beam Geometry

Another approach to the reconstruction of fan-beam data is to adapt the FBP formula to the specific geometry under consideration. The derivation reported here is based on those of Kak and Slaney (1988) and Turbell (2001), based on the results of (Herman and Naparstek 1977). Let us first rewrite the FBP reconstruction formula by expressing the image point location in (unsigned) cylindrical coordinates, \( r, \varphi \), where \( r \in [0, \infty[ \) and \( \varphi \in [0, 2\pi[ \)

\[
f(r, \varphi) = \frac{1}{2} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dx' p(x', \phi) h[r \cos(\phi - \varphi) - x']
\]

and let’s replace the Radon coordinates \( x', \varphi \) with the coordinates of the fan-beam sinogram by using the inverse relations of Equation 33.37

\[
\begin{align*}
x'(\sigma) &= \frac{D_1 \sigma}{\sqrt{D_1^2 + \sigma^2}}, \\
\phi(\sigma, \beta) &= \beta - \tan^{-1} \frac{\sigma}{D_1},
\end{align*}
\]

for the case of a flat detector and

\[
\begin{align*}
x'(\gamma) &= -D_1 \sin \gamma, \\
\phi(\gamma, \beta) &= \beta + \gamma,
\end{align*}
\]

for the case of a curved detector. By changing the variables defined in Equation 33.39 for the flat detector and Equation 33.40 for the curved detector into Equation 33.38 and by using the following auxiliary functions

\[
\begin{align*}
U(r, \varphi, \beta) &= \frac{D_1 + r \sin(\beta - \varphi)}{D_1}, \\
L(r, \varphi, \beta) &= \sqrt{[D_1 + r \sin(\beta - \varphi)]^2 + [r \cos(\beta - \varphi)]^2}
\end{align*}
\]
we obtain the following inversion formulae for the fan-beam data (see Kak and Slaney 1988 for the complete derivation):

\[
f(r, \varphi) = \frac{1}{2} \int_0^{2\pi} \frac{1}{U^2} d\beta \int_{-\infty}^{\infty} d\sigma \left( D_i \left( \frac{1}{D_i^2 + \sigma^2} \right) \right)
\]

\[
g^{(f)}(\sigma, \beta) h(\sigma' - \sigma) = \frac{1}{2} \int_0^{2\pi} d\beta \int_{-\infty}^{\infty} d\gamma (D_i \cos \gamma) g^{(c)}(\gamma, \beta) h^{(c)}(\gamma' - \gamma).
\]  

(33.42)  

(33.43)

where \( h^{(c)} \) is a weighted version of the ramp filter given by

\[
h^{(c)}(\gamma) = \left( \frac{\gamma}{\sin \gamma} \right)^2 h(\gamma).
\]  

(33.44)

The above inversion formulae can be used for reconstruction of fan-beam data from flat (Equation 33.42) and curved detectors (Equation 33.43). As one can see, they are in the form of filtered back projection, with some difference with respect to the parallel-beam formula. First, the fan-beam data must be pre-weighted with a factor that only depends on the radial coordinate \((D_i/\sqrt{D_i^2 + \sigma^2})\) for the flat detector and \( D_i \cos \gamma \) for the curved one. The filter kernel is the standard ramp filter, \( h \), for the case of the flat detector. For the curved detector, the change of variables introduced a bit different, weighted form (see Equation 33.44) to distinguish it from the standard one. Finally, the back projection is done projection-wise by taking into account a space-dependent weighting factor \((U \) and \( L \) \) for the flat and curved detectors, respectively) as defined in Equation 33.41.

The geometrical interpretation of these factors can be understood by looking at Figure 33.16. For the purpose of keeping the next equations as compact as possible, we also define the four weighting factors

\[
W_{fB}^{(f)}(r, \varphi, \beta) = U^2(r, \varphi, \beta); \quad W^{(f)}(\sigma) = \frac{D_i}{\sqrt{D_i^2 + \sigma^2}}.
\]  

(33.45)

where the \( W^{(f)} \) factors are applied to the fan-beam data before filtering and the \( W_{fB}^{(f)} \) factors are applied on back projection.

The reader should note that the integration limit in the back projection integral has been changed from \( \pi \) to \( 2\pi \) and a factor \( 1/2 \) has been added to compensate for the effect of this change. The reconstruction formulae in Equations 33.42 through 33.43 must be used, assuming that the fan-beam data were acquired on a rotation interval of the gantry starting from 0 to \( 2\pi \) (i.e., 360 degrees). This acquisition modality is referred to as full scan acquisition. Similarly to the parallel-beam geometry, in this case the ramp filter is also commonly modified in practice by mean of apodization windows and then truncated in the frequency space according to the sampling theory (see Equation 33.23). Then, we can write the reconstruction formulae for the implementation of the FBP algorithm in fan-beam geometry as follows

\[
\tilde{f}_{FBP}(r, \varphi) = \frac{1}{2} \int_0^{2\pi} d\beta \tilde{g}^{(f)}
\]  

(33.46)

where \( \tilde{f}^{(f)}(\sigma, \beta) = \left[ (w^{(f)} g^{(f)}) * \tilde{h} \right](\sigma, \beta) = (g^{(f)} * \tilde{h})(\sigma, \beta) \)

(33.47)

where \( \tilde{g}^{(c)}(\gamma, \beta) = \left[ (w^{(c)} g^{(c)}) * \tilde{h}^{(c)} \right](\gamma, \beta) = (g^{(c)} * \tilde{h}^{(c)})(\gamma, \beta) \)

(33.48)

Algorithm 33.4: Fan Beam FBP, Full Scan

1. Select the apodization window \( A(\sigma) \) and compute the discrete version of the modified filter kernel \( \tilde{h} \)
2. If curved detector geometry is employed, multiply the modified filter kernel by the factor \((\gamma/\sin \gamma)^2) \)
3. Multiply the fan-beam sinogram \( g^{(f)} \) by the radial weighting factor \( W^{(f)} \), obtaining the weighted fan-beam sinogram \( g^{(f)} \).
4. Apply the 1D ramp filter to the weighted fan-beam sinogram; for each available projection angle $\beta$, take the 1D DFTs $G^{(w)}_u = F_u G^{(w)}_u$ and $H = F_u h$ of the weighted fan-beam sinogram and of the modified filter, respectively, and multiply them in the frequency domain; afterwards, compute the filtered sinogram $\tilde{q}^{(w)}$ as the inverse 1D DFT of the product $G^{(w)}_u \cdot H$; 
5. Reconstruct the image by back projecting each row of the weighted filtered fan-beam sinogram $\tilde{q}^{(w)}/w_{\beta}^{(w)}$ on the image plane; the back projection is performed by following the original direction of each acquired line integral at each gantry angle; 
6. Multiply the entire reconstructed image by 1/2.

33.4.3 Data Redundancy and Short-Scan Reconstruction

The last step in the previous algorithm for full-scan fan-beam FBP consisted of dividing the reconstructed image by two (see Equation 33.46). This normalization step was necessary because the fan-beam data acquired on a $2\pi$ interval are redundant, that is, every line integral $x' \phi$ crossing the object is acquired exactly twice. More specifically, the periodicity of the fan-beam sinogram

$$g^{(c)}(\sigma, \beta) = g^{(c)}(-\sigma, \beta + \pi - 2\tan^{-1}\frac{\sigma}{D_1})$$

$$g^{(e)}(\gamma, \beta) = g^{(e)}(-\gamma, \beta + \pi - 2\gamma).$$

(33.49)

Hence, when a full scan is performed, each line integral acquired at a given gantry angle $\beta$ has a redundant one (acquired in the opposite direction) in the projection taken at $\beta' = \beta + \pi - 2\gamma$ (this notation also applies to a flat detector, where $\gamma = \tan^{-1}(\sigma/D_1)$). We will now derive the minimum rotation interval of the gantry that allows us to obtain a minimally redundant dataset, that is, a fan-beam dataset where every line integral of the object is acquired at least once. First of all, let us notice from Equation 33.37 that

$$\beta = \phi + \gamma = \phi - \sin^{-1}\frac{x'}{D_1},$$

(33.50)

and then, for a given angle $\phi$ of the Radon space, the radial sampling inside a circle of radius $x'_{\text{FOV}}$ is completed by varying the gantry angle $\beta$ in the range

$$\phi - \sin^{-1}\frac{x'_{\text{FOV}}}{D_1} \leq \beta < \phi + \sin^{-1}\frac{x'_{\text{FOV}}}{D_1},$$

(33.51)

or, equivalently,

$$\phi - x'_{\text{FOV}} \leq \beta < \phi + x'_{\text{FOV}},$$

(33.52)

where $x'_{\text{FOV}}$ is the semi-aperture of the detected X-ray fan-beam. A complete dataset, in the sense of Radon inversion, is such that all the angles $\phi \in [0, \pi]$ must be acquired; thus, Equation 33.52 can be rewritten as

$$-\gamma_{\text{FOV}} \leq \beta < \pi + \gamma_{\text{FOV}}$$

(33.53)

which leads to

$$0 \leq \beta < \pi + 2\gamma_{\text{FOV}}.$$  

(33.54)

The minimum rotation interval leading to a complete fan-beam dataset is then $\beta \in [0, \pi + 2\gamma_{\text{FOV}}]$, that is, a half-rotation plus the angular aperture of the detected X-ray fan-beam.

A minimally redundant acquisition in CT is also referred to as a short-scan acquisition. The advantages of short-scan acquisitions in CT over full-scan acquisitions are mainly related to the improvement of temporal resolution, as the time required for gathering a complete dataset is almost halved with respect to the gantry revolution time. From the reconstruction point of view, a short-scan dataset can’t be reconstructed by simple application of the formulae in Equation 33.46 because it contains partially redundant data. In other words, some line integrals are acquired once while some others are acquired twice. In this case, the application of a global normalization factor (as we have done by the 1/2 factor in Equation 33.46) just won’t work. Figure 33.17 shows the redundant data of a short-scan acquisition (with a flat detector) as shaded areas in the $\sigma, \beta$ plane. One possible solution to the problem of partial redundancy is to apply a weighting window such that $w_{\text{short}} = 1/2$ on the redundant area and $w_{\text{short}} = 1$ elsewhere. This solution would cause a discontinuity along the radial direction in the sinogram, leading to severe artifacts when the ramp filter is applied. Parker (1982) proposed a smooth weighting window with continuous derivatives:

$$w_{\text{short}}(\gamma, \beta) = 
\begin{cases}
\sin^2\left(\frac{\pi}{4}\frac{\beta}{\gamma_{\text{FOV}} + \gamma}\right) & \text{if } 0 \leq \beta < 2(\gamma + \gamma_{\text{FOV}}) \\
1 & \text{if } 2(\gamma + \gamma_{\text{FOV}}) \leq \beta < \gamma + \pi \\
\sin^2\left(\frac{\pi}{4}\frac{\beta}{\gamma_{\text{FOV}} - \gamma}\right) & \text{if } \gamma + \pi \leq \beta < \pi + 2\gamma_{\text{FOV}} \\
0 & \text{if } \pi + 2\gamma_{\text{FOV}} \leq \beta < 2\pi
\end{cases}$$

(33.55)

![FIGURE 33.17 Data redundancy in a short scan fan-beam sinogram (the figure refers to the case of a flat detector). Primed letters denote data points that are redundant copies of the non-primed ones. Shaded areas contain line integrals that are acquired twice during a short scan acquisition.](image-url)
The short-scan weighting window above has been written using curved detector coordinates, but it can be easily adapted for the flat detector by using $\gamma = \tan^{-1}(\sigma/D)$. Figure shows an example of a Parker-type weighting window for a flat detector geometry. When the partial redundancy is not corrected (i.e., full-scan reconstruction with global weighting is done), artifacts appear as shown in the figure. Reconstruction of short-scan fan-beam data requires then that the fan-beam sinogram $g$ in Equations 33.47 and 33.48 must be further pre-weighted by a short-scan weighting window (e.g., like the Parker window in Equation 33.55) in order to pre-correct for partial data redundancy. For short-scan reconstruction, we can now write

$$\tilde{g}_{FBP, Short}^{(s)} = \int_0^{\pi/4} \frac{d\beta}{W_{FBP}^{(s)}} \tilde{g}_{Short}^{(s)}, \quad (33.56)$$

where

$$\tilde{g}_{Short}^{(s)} = [W^{(s)}g^{(s)}W_{Short}] \ast \tilde{h}^{(s)}. \quad (33.57)$$

For completeness, we report here the algorithm for short-scan fan-beam FBP reconstruction even though the only additional step required as compared to the full-scan algorithm is the pre-weighting with the short-scan window $w_{Short}$, which replaces the global weighting by the $1/2$ factor (Figure 33.18).

**Algorithm 33.5: Fan Beam FBP, Short Scan**

1. Multiply the short-scan sinogram by the short-scan weighting window $w_{Short}$.
2. Execute the same steps of the FAN BEAM FBP, FULL SCAN algorithm reported above (Algorithm 33.4), except for the last step of global weighting by the factor $1/2$.

### 33.5 3D Reconstruction

This section deals with the problem of reconstructing 3D objects from projection data. Intuitively, the most straightforward method is by stacking 2D slices reconstructed in selected planes, by properly selecting the $z$-sampling step $\Delta z$ depending...
on the instrumentation and on the specific goal of the imaging procedure. Early generations of single-slice CT (SSCT) scanners allowed this type of quasi-3D (or non isotropic 3D) imaging to be performed by serial scanning and reconstruction of contiguous 2D slices, even though long acquisition times were required for the reconstruction of significantly long body segments. Multi-slice CT (MSCT) scanners improved the efficiency of sequential scanning, even though the real breakthrough in 3D scanning was the introduction of helical (or spiral) scanning trajectories in the early nineties, that is, almost a decade before the widespread use of the modern MSCT technology (see Kalender et al. 1990, Kalender 2011). This section starts with a short discussion of helical CT reconstruction. Afterwards, an introduction to intrinsically volumetric imaging based on the inversion of the 3D Radon transform is given. As we will see later, exact reconstruction methods based on inversion formulae for the 3D Radon transform rely on stringent conditions that must be met by the acquired data; these methods are generally difficult to implement. Approximate reconstruction methods that attempt to invert the divergent 3D X-ray transform are commonly used in practice: the most used approximated algorithm for cone beam CT reconstruction, presented in 1984 by Feldkamp, Devis, and Kress (also known as the FDK method Feldkamp et al. 1984) is described at the end of this section.

### 33.5.1 Reconstruction of Fan-Beam Data from Helical Scans

The technology of helical CT scanning has been already presented in Chapter 32. In the context of image reconstruction, it is customary to assume that the object function is still in the laboratory coordinate system while the source performs a helical trajectory. Let us first remember that the source trajectory in fan-laboratory coordinate system while the source performs a helical rotation, let’s say \( jN \), with \( j = 0, \ldots, N - 1 \) where \( N \) is the number of gantry revolutions, and where \( j \) must be such that

\[
  z_{\text{start}} + jd < z^* < z_{\text{start}} + (j + 1)d. \tag{33.60}
\]

For full-scan reconstruction, we can see from the first \( \theta(z) \) graph of Figure 33.19 that the required angular range of the gantry to build a complete fan-beam sinogram is \( 4\pi \); moreover, the axial positions \( z_{\text{start}} < z^* < z_{\text{stop}} \) are inaccessible, as well as the positions \( z_{\text{stop}} - d < z^* < z_{\text{start}} \). This means that two segments of length \( d \) at the beginning and at the end of the helical trajectory cannot be reconstructed, even though they have been crossed by the X-ray fan-beam for at least one gantry angle. Following the notation of (Kalender et al. 1990) and (Schaller et al. 2000), we denote the reconstruction method just described as 360°LI, where LI stands for linear interpolation.

\[
x_{\text{foc}}(\beta) = \begin{bmatrix}
  D_1 \sin \beta \\
  -D_1 \cos \beta \\
  z_{\text{start}}
\end{bmatrix} \tag{33.58}
\]

where \( z_{\text{start}} \) is the axial position of the plane defining the circular trajectory. For standard helical scan (i.e., with constant radius and pitch), the source trajectory is instead

\[
x_{\text{foc}}(\beta) = \begin{bmatrix}
  D_1 \sin \beta \\
  -D_1 \cos \beta \\
  z_{\text{start}} + \frac{d \cdot \beta}{2\pi}
\end{bmatrix}, \quad \beta \in [0; \beta_{\text{stop}}]. \tag{33.59}
\]
For MSCT scanners, the reader must figure out that each detector row defines a $\partial g(\mathbf{r})$ graph shifted by a factor $k \Delta z$ with respect to the central row, where $\Delta z$ is the axial extent of a given row and $k$ is the row index, with $k = -M/2, \ldots, M/2 - 1$ and $M$ is the number of rows. Optimal reconstruction is done for each arbitrary slice position $z'$ by searching the two closest data points for each gantry angle $\beta$ among all the available detector rows. The extension of the 360LI used method for helical SSCT is denoted by 360MLI in case of helical MSCT.

In most cases, the periodicity condition of the fan-beam sinogram is exploited to further reduce the $z$-distance of the data points to be interpolated. Using this method leads to a family of rebinning algorithms called 180LI (for SSCT) and 180MLI (for MSCT). The reader can refer to Schaller et al. (2000) for further reading on the rebinning techniques in MSCT.

### 33.5.2 Inversion of the Radon Transform in More Than Two Dimensions

We will now focus on the problem of reconstructing a 3D object from 2D projections. As explained in Section 33.2, the X-ray transform of a 3D object $g = \mathcal{X}f$ (made up of line integrals) is more directly linked to the acquired projection data in a CT scan than the 3D Radon transform $p = \mathcal{R}3f$ (which is, instead, a set of plane integrals). Nevertheless, a general inversion formula for the $n$-dimensional RT is well known (see Natterer 1986)

$$f = \frac{1}{2(2\pi)^{n-1}} \mathcal{R}_x^n \left\{ \begin{array}{ll} \left( -1 \right)^{\frac{n-1}{2}} \mathcal{H}_x \frac{\partial^{n-1}}{\partial r^{n-1}} (\mathcal{R}_n f) , & \text{if even} \\ \left( -1 \right)^{\frac{n-1}{2}} \frac{\partial^{n-1}}{\partial r^{n-1}} (\mathcal{R}_n f) , & \text{if odd} \end{array} \right. \right) \tag{33.61}$$

where the Hilbert transform $\mathcal{H}_x$ with respect to an arbitrary variable $x$ is defined as the Cauchy principal value of the integral

$$(\mathcal{H}_x f)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{x-s} \, ds = \frac{1}{\pi} \cdot f(x) \tag{33.62}$$

and the derivatives are taken with respect to the radial variable.

Let us first show that, in two dimensions, the inversion formula above can be linked to one of the formulae obtained in the previous section for 2D reconstruction. For $n = 2$, remembering that we have previously denoted by $\gamma$ the radial variable in the Radon space, Equation 33.61 is written as

$$f(x,y) = \frac{1}{4\pi} \mathcal{R}_x^2 \mathcal{H}_x \left[ \frac{\partial}{\partial \gamma} p(\gamma,\phi) \right]. \tag{33.63}$$

By comparing Equations 33.63 and 33.28, one can see that, apart from a constant multiplicative factor, the two inversion formulae are equivalent provided that

$$\mathcal{H}_x \left[ \frac{\partial}{\partial \gamma} p(\gamma,\phi) \right] = \frac{1}{\pi} x \cdot \frac{\partial p(x',\phi)}{\partial x} = h \cdot p(x',\phi) \tag{33.64}$$

where $h$ is the ramp filter kernel (see Equation 33.21). In other words, we must now demonstrate that the operator $\mathcal{H}_x \partial x' \partial x' $ is equivalent to the ramp filter in the space domain. Switching to the Fourier transform of both hands, we find that

$$\mathcal{F}_x \left[ \frac{1}{\pi x \cdot \partial x} \cdot h \cdot p(x',\phi) \right] = \mathcal{F}_x \left[ \frac{1}{\pi} \cdot \mathcal{F}_x \left[ \frac{\partial p}{\partial x} \right] \right] (33.65)$$

$$= -j \cdot \text{sgn}(\nu) \cdot (2\pi \nu) [\mathcal{F}_x p] = 2\pi \left| \nu \right| \cdot P$$

$$= 2\pi \mathcal{F}_x [h \cdot p]. \tag{33.65}$$

In the previous equation, the sign function $\text{sgn}()$ equals $+1$ when the argument is positive and $-1$ when the argument is negative, and zero otherwise. Hence, we have written the frequency response of the ramp filter as $b \nu = -(1/2\pi) j \cdot \text{sgn}(\nu) \cdot (2\pi \nu)$ and we have used the identities $F_1 [1/\pi \nu^2] = -j \cdot \text{sgn}(\nu)$ and $F_1 [df(x)/dx] = (2\pi \nu) e^{j\nu \cdot x} [F_1 f(x)]$ to show that Equations 33.63 and 33.28 are equivalent inversion formulae for the 2D Radon transform. Noteworthy, the Hilbert transform is not local, as can be seen by its definition in Equation 33.62. As a consequence, the ramp filter is not local and hence it requires that the projection data must be radially untruncated. From a practical point of view, the nonlocality of the ramp filter poses important limitations when only small portions of the object are scanned. Interior (or local) tomography is outside the scope of this chapter. The reader can refer to Wang and Yu (2013), and references therein, for further reading.

Let us now focus on the inversion of the 3D Radon transform. From Equation 33.61, we get for $n = 3$

$$f(x) = -\frac{1}{8\pi^2} \mathcal{R}_x^3 \left[ \frac{\partial^2}{\partial r^2} p(r,\alpha) \right]. \tag{33.66}$$

The application of the above equation to image reconstruction from real data has been the subject of intensive study by several investigators for decades. From a practical point of view, we are interested in the reconstruction of projection data acquired in cone beam geometry, as this is the geometry employed in volumetric CT scanners using 2D (curved or flat) detectors. As a first practical problem, data completeness in the Radon domain is required in order to apply Equation 33.66. Tuy and Smith (Tuy 1983) have shown that an exact reconstruction is possible only if all the planes crossing the object do intersect the source trajectory in at least one point. This is also called the Tuy–Smith sufficiency condition (Tuy 1983). Based on Tuy’s inversion formula, Grangeat (1991) proposed a reconstruction framework in which the radial derivative of the 3D Radon transform can be extracted from line integrals acquired in cone beam geometry. Nevertheless, this solution presents practical issues and numerical instabilities. Defrise and Clack proposed a modification of Grangeat’s method, obtaining an FBP-type algorithm, with shift-variant filtering (Defrise and Clack 1994). More recently, Katsevich proposed exact reconstruction formulae for cone beam data acquired in a helical trajectory (Katsevich 2002) and subsequently extended this to an arbitrary trajectory (Katsevich 2003). Interestingly, the Katsevich formulae are of the FBP type with shift-invariant filtering. It is worth mentioning that the reconstruction formulae provided by Katsevich for 3D reconstruction
have inspired a brand new family of 2D reconstruction algorithms, allowing exact reconstruction of a region of interest (ROI) of an object in less than a short scan (Noo et al. 2002). This new type of fan-beam FBP is based on radial derivatives of the fan-beam sinogram and Hilbert transform, and requires that the fan-beam projections are radially untruncated.

Exact reconstruction has limited application in real-world cone beam CT imaging. The most used source trajectory for most CT and micro-CT scanners is a circle, which does not fulfill the Tuy–Smith sufficiency condition. Several scanning trajectories have been evaluated by investigators in order to handle the data incompleteness, such as perpendicular circles or circles plus lines (Kudo and Saito 1994), circle plus arc (Wang and Ning 1999), or helix with constant or variable pitch and radius (Katsevich et al. 2004). Among them, only the circle and helix with constant pitch and radius have found significant applications. Even though a helical cone beam scan is performed on modern multi-slice CT (MSCT) scanners, the computational burden of exact algorithms is still an issue. On the other hand, exact reconstruction methods based on Radon inversion must somehow handle the problem of data truncation, which is almost always present in practice along the axial direction due to the small axial extent of real-world detectors with respect to the length of a patient. An exact solution to the so-called long-object problem, that is, the problem of reconstructing limited axial segments of infinitely long objects, was first proposed by Tam et al. (Tam 1998, Tam et al. 1998). A quasi-exact solution to the long-object problem, called the zero-boundary (or ZB) method, was proposed subsequently by Defrise et al. (Defrise et al. 2000), based on the previous method developed by (Kudo et al. 1998) for the short-object problem (i.e., the problem of reconstructing an entire object by axially truncated cone beam projections, provided that the source trajectory is such that the projection data include the top and bottom axial boundaries of the object). Hence, approximate algorithms are by far the most used, rather than exact ones, for practical cone beam reconstruction. The most widespread method for approximate cone beam reconstruction was derived by Feldkamp, Devis, and Kress (Feldkamp et al. 1984) and is described in the next section.

### 33.5.3 The Feldkamp–Devis–Kress (FDK) Method

The circular cone beam geometry is depicted in Figure 33.20. The data point coordinates are very similar to those of fan-beam geometry, where a longitudinal (i.e., axial) coordinate $\rho$ was added, taking into account the axial extension of the 2D detector.

The basic idea behind the Feldkamp method, or FDK method, is that for moderate aperture extents of the detected cone beam, the acquisition geometry should not deviate too much from a multi-slice fan-beam acquisition. The discrepancy between cone beam and multi-fan-beam geometry could then be compensated by means of a correction factor. Furthermore, the resulting algorithm should reduce to the fan-beam algorithm when just the axial midplane is reconstructed.

Let us first consider the case of a flat detector, similar to Feldkamp and colleagues in their original article. The cone beam coordinates are denoted by $\sigma$, $\rho$, $\beta$, where $\rho$ is the axial coordinate of the detector and the other two variables have the same meaning as those used for the fan-beam geometry. In analogy with the notation used for fan-beam sinograms, we will denote the cone beam projections as $g^{(s)}(\sigma, \rho, \beta)$. As one can see from Figure 33.20, each detector row ($\rho = \text{cost}$) defines a tilted fan-beam with tilt angle $\kappa$. Let us first start by rewriting the fan-beam reconstruction formula, which should provide an exact reconstruction of the object at $z = 0$: for the flat detector

$$
\tilde{f}_{\text{FBP}}(x, y, 0) = \tilde{f}_{\text{FDK}}(x) \int_{-\infty}^{\infty} d\sigma \left( \frac{D_1}{D_1^2 + \sigma^2 + \rho^2} g^{(s)}(\sigma, 0, \beta) h(\sigma' - \sigma) \right)
$$

(33.67)

Away from the midplane, we can see that the line integrals in the tilted fan-beam are scaled by a factor $\sqrt{D_1^2 + \rho^2} / \sqrt{D_1^2 + \sigma^2}$ with respect to those passing from the central row of the detector. Hence, an approximate reconstruction for $z \neq 0$ can be obtained by compensating for this extra-length of the tilted line integrals in Equation 33.67

$$
\tilde{f}_{\text{FDK}}(x) = \frac{1}{2} \int_{0}^{2\pi} \frac{1}{U^2} d\beta \int_{-\infty}^{\infty} d\sigma \left( \frac{D_1}{D_1^2 + \sigma^2 + \rho^2} \right)^{1/2} \left( \frac{D_1}{D_1^2 + \sigma^2} \right)^{1/2} g^{(s)}(\sigma, \rho, \beta) h(\sigma' - \sigma)
$$

(33.68)

For the curved detector, the scaling factor in the tilted fans is simply $\sqrt{D_1^2 + \rho^2} / D_1$ so that the FDK reconstruction formula can be written as

$$
\tilde{f}_{\text{FDK}}(x) = \frac{1}{2} \int_{0}^{2\pi} \frac{1}{U^2} d\beta \int_{-\infty}^{\infty} d\gamma \left( \frac{D_1 \cos \gamma}{\sqrt{D_1^2 + \rho^2}} \right) g^{(s)}(\gamma, \rho, \beta) h(\gamma' - \gamma)
$$

(33.69)
It is worth noting that the ramp filtering in the FDK method is done row-by-row, similar to the standard fan-beam reconstruction. As one can see from Equations 33.68 and 33.69, FDK differs from fan-beam FBP just due to the different shape of the pre-weighting factor, \( w^{(i)} \), which is also dependent on the axial position \( \rho \).

\[
\begin{align*}
  w^{(i)}_{\text{FDK}}(\sigma, \rho) &= \frac{D_i}{\sqrt{D_i^2 + \sigma^2 + \rho^2}}, \\
  w^{(i)}_{\text{FDK}}(\gamma, \rho) &= \frac{D_i^2 \cos \gamma}{\sqrt{D_i^2 + \rho^2}}.
\end{align*}
\] (33.70)

while the back projection weighting factors \( W^{(i)} \) and the ramp first \( \hat{h} \) are kept identical to those of the fan-beam formula. In this case, the back projection is done in the 3D space by following the original direction of the acquired line integrals. Even if the basic steps are very similar to those reported above for the full-scan fan-beam FBP, we summarize here all the steps required for FDK reconstruction:

Algorithm 33.6: FDK Cone Beam FBP, Full Scan

1. Select the apodization window \( A(\nu) \) and compute the discrete version of the modified filter kernel \( \hat{h} \);
2. If curved detector geometry is employed, multiply the modified filter kernel by the factor \( (\gamma / \sin \gamma)^2 \);
3. Multiply the cone beam projection \( g^{(i)} \) by the weighting factor \( w^{(i)}_{\text{FDK}} \) (see Equation 33.70), obtaining the weighted cone beam projection \( g^{(i)}_w \);
4. Apply the 1D ramp filter to each row of the weighted cone beam projections: for each available projection angle \( \beta \) and for each axial position \( \rho \) of the detector, take the 1D DFTs \( G^{(i)}_w = \mathcal{F}^{(i)}_{\text{le}} \) and \( \hat{H} = \mathcal{F}^{(i)}_{\hat{h}} \) of the weighted cone beam projection and of the modified filter, respectively, and multiply them in the frequency domain; afterwards, compute each row of the filtered cone beam projections \( \tilde{q}^{(i)} \) as the inverse 1D DFT of the product \( G^{(i)}_w \cdot \hat{H} \), that is, \( q^{(i)} = \mathcal{F}^{-1} (G^{(i)}_w \cdot \hat{H}) \);
5. Reconstruct the image by back projecting each weighted filtered cone beam projection \( q^{(i)} / w^{(i)}_{\text{FDK}} \) on the 3D image space; the back projection is performed by following the original direction of each acquired line integral at each gantry angle;
6. Multiply the entire reconstructed image by 1/2.

Figure 33.21 shows how the back projection in cone beam geometry is done in practice. This figure refers to the case of voxel driven back projection for a flat detector: for each voxel location and for each gantry angle, the corresponding point \( (\sigma', \rho') \) is calculated on the detector surface. For that point, the value of the filtered projection \( \tilde{q}^{(i)}(\sigma', \rho', \beta) \) (with proper weighting as explained above) is calculated by bilinear interpolation with the four nearest available data points. The interpolated value is then accumulated on the voxel, and the process is repeated for all the voxels in the target image volume and for all the gantry angles.

Short-scan reconstruction of cone beam data by the FDK method is also possible, even though the data redundancy away from the midplane is not guaranteed for all object functions and hence direct row-wise application of the short-scan weighting window for fan-beam geometry can lead to image artifacts (see, for instance, Maaß et al. 2010). Interestingly, as reported in the appendix of their original article, Feldkamp et al. have shown that their method is exact for \( z \)-invariant object functions, that is, when \( f \) is such that

\[
\frac{\partial f(x, y, z)}{\partial z} = 0.
\] (33.71)

In this case, it is easy to see that the line integrals away from the midplane are just the scaled version on the integrals in the \( z = 0 \) plane, that is:

\[
\begin{align*}
  g^{(i)}(\sigma, \rho, \beta) &= \frac{g^{(i)}(\sigma, 0, \beta)}{w^{(i)}_{\text{FDK}}(\sigma, \rho)} \cdot \frac{w^{(i)}_{\text{FDK}}(\gamma, \rho)}{w^{(i)}_{\text{FDK}}(\gamma, \rho)}.
\end{align*}
\] (33.72)

For this class of objects, data redundancy and periodicity conditions for the cone beam projections can be found similarly to the fan-beam case. This property can have practical applications: for instance, Panetta et al. have exploited the redundancy of cone beam projections of \( z \)-invariant objects for the determination of a subset of the seven misalignment parameters of cone beam CT systems (Panetta et al. 2008).

In practice, the basic idea behind the FDK approximation is that we are assuming that tilted X-ray paths can be approximated by rays that are parallel to the \( xy \) plane, and the error is compensated by a correction factor. This approximation becomes exact in the case of globally or even locally \( z \)-invariant objects, which is realistic when the cone angle \( \kappa \) is small. Indeed, this approximation is clearly unrealistic for some objects that are strongly varying along \( z \); as an example, Figure 33.22 shows an axial view of the multidisk phantom along with a FDK reconstruction. This phantom is sometimes called the “Feldkamp killer,” and the reason for this name is clear by observing the reconstructed image. For “normal” objects, the approximated nature of the FDK method is visible as underestimations of the reconstructed values for increasing cone angles. The difficulty in keeping
the image quality of peripheral slices similar to central slices can be viewed as one of the reasons of the end of the so-called “slice war.” Several decades after the publication of the article of Feldkamp et al., there is still effort in modifying and extending the FDK reconstruction formula. In 1993, Wang et al. generalized the FDK method to an arbitrary source trajectory (Wang et al. 1993). In 2002, the same author proposed a different formulation that allows data to be reconstructed using a transversally shifted detector array (Wang 2002); the big advantage of such a formulation is that one can virtually enlarge the scanner field of view (FOV), thus coping with the problem of large objects and small flat panel detectors.

REFERENCES


