The known risks associated with exposure to radiation and the need for high-quality radiographic images to ensure accurate diagnoses make it essential that X-ray imaging systems be designed and maintained to provide high-quality images for low acceptable exposure levels. The ability of an X-ray detector to produce high signal-to-noise ratio images is described by the detective quantum efficiency (DQE). This chapter discusses the fundamental principles of the DQE and provides a summary of important linear systems concepts required to understand and interpret its meaning, including the modulation transfer function (MTF), line-spread function (LSF), sampling, signal aliasing, and noise aliasing. An introduction to cascaded-systems analysis is provided as a means of understanding signal and noise properties of X-ray detectors and generating theoretical models of the DQE. These models are important to establish performance benchmarks and to help guide the development of new detectors. It is shown there remains a large difference in DQE values between detectors in use today, and highlights the importance for informed purchase decisions and ongoing maintenance to ensure continuing high standards of care.

14.1 Introduction

X-ray based imaging systems have been under development for over 100 years, since the discovery of X rays by Röntgen (1896),
and the complete dependency of modern medicine on X-ray technologies continues to motivate research to improve costs, efficacy, and safety. While the health benefits of diagnostic radiography to patients are enormous, potential health risks from incorrect diagnoses and exposure to ionizing radiation makes it critical that X-ray imaging systems be designed, manufactured, and maintained to provide the best possible image quality, while exposing patients to low acceptable levels of radiation.

The past two decades have seen a dramatic shift from film-screen based X-ray imaging systems to digital technologies. These digital systems generally consist of a converter layer in which incident X-ray quanta interact and deposit energy, and a sensor layer in which charges liberated either directly (e.g., from a photoconductor such as selenium) or indirectly (e.g., from the absorption of light generated in a phosphor or scintillator such as cesium iodide) contribute to a signal measured by an array of sensor elements. The output image generally consists of a two-dimensional matrix of numerical values.

Exposure to radiation is associated with a carcinogenic risk that is assumed to increase linearly with absorbed dose and is estimated as approximately $5 \times 10^{-3}$ per mSv of whole-body exposure. While the risk is generally very low for individual examinations (typically between $10^{-6}$ and $10^{-4}$), it has been estimated that approximately 1 in 200 of all cancer deaths at present are attributable to radiation exposure from diagnostic X-rays (Berrington de González and Darby 2004). These risks are accepted because the immediate benefits of medical imaging are compelling and assumed to outweigh the risks (see Section IV, Chapter 66 of this book for radioprotection issues).

Achieving high-quality images is particularly important for difficult diagnostic tasks. For example, screening mammography may require the detection of subtle lesions and fine image structures. A recent study of cancer detection rates in a breast cancer screening program showed that digital radiography (DR) systems (direct and indirect types) performed better than computed radiography (CR) systems (for CR, see Section I, Chapter 12 of this book), resulting in a 30% higher cancer detection rate (4.7 per 1000 versus 3.4) (Chiarelli et al. 2013, 2015; Yaffe et al. 2013), with the difference attributed to differences in image quality and, in particular, image, signal-to-noise ratio (SNR).

Image noise refers to random variations in image brightness that may obscure low-contrast structures in the image. These variations may be caused by technical limitations of the X-ray detector or from the random nature of X-ray production and detection. Figure 14.1a illustrates a typical high-quality mammography image. In comparison, Figure 14.1b shows the same image after blurring to simulate a loss of spatial resolution, and image noise has been added to Figure 14.1c to simulate a loss of SNR. It is clear from this illustration that both a loss of spatial resolution and loss of SNR lead to reduced image quality, making it more difficult to visualize low-contrast structures and fine details.

Due to the statistical nature of X-ray production, transmission, and detection, image quality is related to the number of X-ray photons detected. For these reasons, the number of photons, $N$, detected in a specified region is Poisson-distributed, so that the variance (in repeated similar measurements) is equal to the mean, $\sigma_N^2 = N$, corresponding to a measurement SNR of $\sqrt{N}$. As a direct consequence, image quality SNR can generally be improved by increasing patient radiation exposures. Unfortunately, higher patient exposures do not ensure higher image quality, as some system deficiencies such as low quantum efficiency (fraction of incident photons detected) can be masked by increasing patient exposures. In the breast cancer screening study, the average radiation dose with CR was greater than that with DR, even though image quality with DR was better.

To achieve the best possible images, it is, therefore, critical that facilities establish acceptance testing of new equipment and ongoing quality assurance programs to ensure manufacturers’ specifications are maintained. Most quality assurance programs are designed to identify common system artifacts and characterize metrics of performance, such as spatial resolution and the ability to see low-contrast objects of various sizes.

The purpose of this chapter is to introduce metrics that are used to characterize image quality and system performance, and identify key physical factors that determine performance characteristics. In particular, the modulation transfer function (MTF)

![FIGURE 14.1](image-url) (a) Typical high-quality X-ray mammogram. (b) Degraded mammogram illustrating reduced spatial resolution (reduced MTF). (c) Degraded mammogram illustrating reduced image signal-to-noise ratio (reduced DQE). (Original image courtesy of T.M. Deserno, Department of Medical Informatics, RWTH Aachen, Germany.)
and detective quantum efficiency (DQE) are key metrics, and described in detail here.

14.2 Background

Starting in the 1940s, there was much scientific interest in classifying the signal and noise performance of various optical detectors such as television cameras and photoconductive devices. Rose (1946, 1948a,b) and contemporaries (Fellgett 1949; Jones 1949; Zweig 1964) showed image SNR is ultimately limited by the statistical properties of the number of quanta used to produce the image, with image quality increasing with the number of quanta. The quantum efficiency of a detector describes the fraction of incident quanta that interact, and is an important indicator of detector performance, imposing an upper limit on image quality. However, other physical processes may further degrade image SNR, and Rose proposed the concept of a useful or equivalent quantum efficiency. This is what we now call the detective quantum efficiency (DQE) (Jones 1949; Zweig 1965) given by

\[
DQE = \frac{\text{NEQ}}{\overline{N}_o}, \quad (14.1)
\]

where \( \text{NEQ} \) is the noise equivalent number of quanta, given by \( \text{NEQ} = \text{SNR}^2 \), and \( \overline{N}_o \) is the average number of quanta incident on the detector.

Imaging detectors consist of an array of sensor elements that are rarely completely independent of each other. This results in a sharing of imaging through elements that results in a blurring of images. For this reason, performance metrics must be expressed as a function of spatial frequency (described below). The DQE was described as a function of spatial frequency to the medical imaging community by Shaw (1963, 1978) and Dainty and Shaw (1974) for the description of X-ray film-screen systems. They showed that image quality (in terms of the SNR) could be expressed in terms of an equivalent number of quanta per unit area, independent of the imaging technology used. We now call this the noise equivalent quanta (NEQ):

\[
\text{NEQ}(u) = \text{SNR}^2(u), \quad (14.2)
\]

where \( u \) is the spatial frequency variable in cycles/mm. The NEQ describes the number of quanta per unit area required to produce a specified SNR with an ideal imaging detector, and is a system-independent measure of image quality describing how many quanta an image is “worth.” Knowing that the squared SNR obtained with an ideal detector is equal to \( \overline{q}_o \), the average number of quanta per unit area, incident on the detector, gives our primary definition of the DQE as

\[
DQE(u) = \frac{\text{NEQ}(u)}{\overline{q}_o}. \quad (14.3)
\]

The DQE can be viewed as the ratio of what an image is worth (in NEQ) to what it cost (number of X-ray quanta incident on the detector).

14.3 Linear Systems Theory

In this section, linear systems theory is introduced and a description is given of important principles and relationships required to characterize system performance in the spatial frequency domain. While most results are expressed in one-dimensional geometry in terms of position \( x \) and spatial frequency \( u \), similar relationships hold true in two dimensions in terms of position vector \( \mathbf{r} \) and spatial frequency vector \( \mathbf{k} \). In this section, it is assumed the detector acts as an ideal noise-free system, and only X-ray quantum noise will affect image quality.

A linear system implies the output signal scales with the input signal. If a system has a transfer characteristic, \( S[\cdot] \), such that an input \( h(x) \) produces an output \( S(h(x)) \), then for any two inputs \( h_1(x) \) and \( h_2(x) \) the system is linear if and only if

\[
S[h_1(x) + h_2(x)] = S[h_1(x)] + S[h_2(x)] \quad (14.4)
\]

and

\[
S[ah(x)] = aS[h(x)] \quad (14.5)
\]

for any real constant \( a \).

14.3.1 Impulse–Response Function, IRF

An impulse input \( \delta(x - x_o) \) has unity area and negligible width (Figure 14.2). The corresponding output, \( S[\delta(x - x_o)] \), is called the impulse–response function (IRF) of the system. That is,

\[
\text{IRF}(x, x_o) \equiv S[\delta(x - x_o)]. \quad (14.6)
\]

The shape of the IRF is arbitrary, and defined to have unity area.

The IRF can be used to determine the expected output from a linear system. If an input, \( h(x) \), is approximated with a very large number of impulses on spacings, \( \Delta x \) (Figure 14.3), small relative to the width of the IRF and scaled by \( h(x)\Delta x \) to preserve area, the output \( S[h(x)] \) is the superposition of an IRF for each impulse:

\[
\begin{align*}
(a) & \quad \text{Spatial domain} \\
& \quad \delta(x - x_o) \\
& \quad \downarrow \quad \text{IRF}(x - x_o) \\
& \quad \downarrow \quad x_o \\
& \quad x \\
(b) & \quad \text{Spatial domain} \\
& \quad \delta(x - x_1) + \delta(x - x_2) \\
& \quad \downarrow \quad \text{IRF}(x - x_1) + \text{IRF}(x - x_2) \\
& \quad \downarrow \quad x \\
& \quad x \\
& \quad \text{FIGURE 14.2} \quad (a) \text{ An impulse input } \delta(x - x_o) \text{ results in an impulse–response function output IRF}(x - x_o). \quad (b) \text{ For linear systems, the output is the superposition of multiple impulse responses.}
\end{align*}
\]
FIGURE 14.3 The signal $h(x)$ is approximated with a large number of scaled delta functions, each with area $h(j\Delta x)\Delta x$.

$$S[h(x)] \approx \sum_{j=-\infty}^{\infty} h_j \text{IRF}(x,j\Delta x)\Delta x,$$

(14.7)

In the limit of $\Delta x \to 0$, the summation becomes the integral

$$S[h(x)] = \int_{-\infty}^{\infty} h(x')\text{IRF}(x,x')dx',$$

(14.8)

which is called a superposition integral. Equation 14.8 is a full description of the expected output signal from a linear system for any input, $h(x)$.

### 14.3.2 Linear and Shift-Invariant (LSI) Systems

An important special case of linear systems exists for systems that also have a shift-invariant impulse–response function. The IRF can, therefore, be expressed in terms of this one shifted argument, given by:

$$\text{IRF}(x,x_0) \Rightarrow \text{IRF}(x-x_0).$$

(14.9)

and the superposition integral simplifies to:

$$S[h(x)] = \int_{-\infty}^{\infty} h(x')\text{IRF}(x-x')dx',$$

(14.10)

which is called the convolution integral. It is a complete description of the expected output of a linear and shift-invariant system for any input, $h(x)$.

### 14.3.3 Convolution Integral

The convolution integral is of fundamental importance in the imaging sciences. It gives the expected output signal when any input, $h(x)$, is passed through any LSI system. A general statement of the convolution of $h(x)$ with $f(x)$ is

$$d(x) = \int_{-\infty}^{\infty} h(x')f(x-x')dx' = h(x) * f(x).$$

(14.11)

Selected properties of the convolution integral are listed in Table 14.1. The left column of Figure 14.4 illustrates how

<table>
<thead>
<tr>
<th>TABLE 14.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Properties of the Convolution Integral, where $(dd/dx)f(x)$ and $\Pi(x/a)$ Is a Rectangular Function of Width, $a$, Centered at $x = 0$</td>
</tr>
<tr>
<td>Comutative: $f(x) * h(x) = h(x) * f(x)$</td>
</tr>
<tr>
<td>Distributive over addition: $f(x) * [h_1(x) + h_2(x)] = f(x) * h_1(x) + f(x) * h_2(x)$</td>
</tr>
<tr>
<td>Associative: $f(x) * h_1(x) * h_2(x) = f(x) * [h_1(x) * h_2(x)]$</td>
</tr>
<tr>
<td>Multiplication with a constant: $a[f(x) * h(x)] = a[f(x)] * h(x) = f(x) * ah(x)$</td>
</tr>
<tr>
<td>Convolution with an impulse: $f(x) * h(x-x_0) = f(x-x_0)$</td>
</tr>
<tr>
<td>Differentiation of a convolution: $\frac{d}{dx}[f(x) * h(x)] = f'(x) * h(x) = f(x) * h'(x)$</td>
</tr>
<tr>
<td>Integration of a convolution: $\int_{-\infty}^{\infty} f(x) * h(x)dx = \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} h(x)dx$</td>
</tr>
<tr>
<td>Differentiation as a convolution: $f'(x) = f(x) * \delta'(x)$</td>
</tr>
<tr>
<td>Integration as a convolution: $\int_{-\infty}^{\infty} f(x)dx = f(x) * \frac{\Pi(x/a)}{a}$</td>
</tr>
</tbody>
</table>

FIGURE 14.4 Signal-transfer characteristics can be represented equivalently as either a convolution with IRF($x$) in the spatial domain (left column), or as multiplication with $T(u)$ in the spatial frequency domain (right column).
a sinusoidally varying signal might be passed through the
convolution integral.

14.3.4 System Characteristic Function, \( T(u) \)

The need to evaluate the convolution integral in Equation 14.11
limits the physical insight it provides towards understanding
system performance. A more useful approach comes from
examining the special case of an input, \( h(x) = e^{2\pi u x} \), that varies
sinusoidally with position, expressed as a complex exponential.
The output is given by

\[
S\left[ e^{2\pi u x} \right] = \int_{-\infty}^{\infty} \text{IRF}(x') e^{2\pi u x (x-x')} dx',
\]

(14.12)

\[
e^{2\pi u x} \int_{-\infty}^{\infty} \text{IRF}(x') e^{-2\pi u x'} dx',
\]

(14.13)

where the final integral is recognized as being the Fourier trans-
form of \( \text{IRF}(x) \), which we call \( T(u) \). Therefore,

\[
S\left[ e^{2\pi u x} \right] = T(u) e^{2\pi u x},
\]

(14.14)

showing that the output is identical to the sinusoidal input scaled
by the (possibly complex) characteristic function, \( T(u) \), as illus-
trated in Figure 14.4. The impulse–response function and the
system characteristic function are Fourier pairs:

\[
T(u) = F[\text{IRF}(x)].
\]

(14.15)

The Fourier transform decomposes a function into sinusoidal
basis functions. If a specified input, \( h(x) \), has the Fourier trans-
form, \( H(u) \), then \( h(x) \) can be expressed as the inverse Fourier
transform of \( H(u) \), and the corresponding output is

\[
S[h(x)] = S\left\{ \int_{-\infty}^{\infty} H(u) e^{2\pi u x} du \right\}.
\]

(14.16)

Representing the output as \( d(x) \) and noting that \( d(x) = h(x) \ast
\text{IRF}(x) \) gives

\[
d(x) = \int_{-\infty}^{\infty} h(x') \text{IRF}(x-x') dx',
\]

(14.17)

\[
= \int_{-\infty}^{\infty} H(u) e^{2\pi u x} \int_{-\infty}^{\infty} \text{IRF}(x') e^{-2\pi u x'} dx' du,
\]

(14.18)

\[
= \int_{-\infty}^{\infty} H(u) T(u) e^{2\pi u x} du.
\]

(14.19)

Also, since the inverse Fourier transform gives

\[
d(x) = \int_{-\infty}^{\infty} D(u) e^{2\pi u x} du,
\]

\[
D(u) = H(u) T(u).
\]

(14.20)

This is a key result. It shows that signal-transfer characteristics of an LSI system can be expressed either as convolution with \( \text{IRF}(x) \) in the spatial domain, or equivalently as multiplication with \( T(u) \) in the spatial frequency domain. The Fourier components, \( H(u) \), of the input are passed unchanged through the system, other than a scaling by \( T(u) \). This relationship is illustrated graphically in Figure 14.4, and shows that image contrast is, in fact, a function of frequency and fully described by \( T(u) \). In many situations, expressing an imaging problem in the spatial frequency domain offers a complimentary perspective to what is obtained from a spatial domain analysis. The ability to move fluently between the two domains is essential to being able to solve many imaging problems.

14.3.5 Modulation Transfer Function

Closely related to the characteristic function, \( T(u) \), is the modula-
tion transfer function (MTF), defined as

\[
\text{MTF}(u) \equiv \frac{|T(u)|}{T(0)}.
\]

(14.21)

The MTF has tremendous practical value. For instance, if the
IRF is a real-only even function (often true with quantum-based
imaging systems), both the MTF and characteristic function are
also real and even functions, and the MTF is a complete descrip-
tion of the expected system response within a scaling factor.
However, if the IRF is asymmetric, the characteristic function
is partly odd and complex. The MTF is a useful but incomplete
description of such a system, because it does not include the
system “phase” information.

The function

\[
\text{OTF}(u) \equiv \frac{T(u)}{T(0)}
\]

(14.22)

is sometimes called the optical transfer function (OTF), and
retains this phase-transfer information.

14.3.6 Cascading Processes

Imaging systems are complex, and often better described as a
cascade of multiple processes. Each process has an input, out-
put, and well-defined characteristic function. For instance, when
one LSI process described by the IRF, \( \text{IRF}_1(x) \), is cascaded into a
second described by \( \text{IRF}_2(x) \), the associative property of the
convolution integral gives

\[
d(x) = [h(x) \ast \text{IRF}_1(x)] \ast \text{IRF}_2(x) = h(x) \ast [\text{IRF}_1(x) \ast \text{IRF}_2(x)] = h(x) \ast \text{IRF}(x),
\]

(14.23)

where \( \text{IRF}(x) = \text{IRF}_1(x) \ast \text{IRF}_2(x) \) and

\[
D(u) = H(u) T_1(u) T_2(u) = H(u) T(u),
\]

(14.24)
where $T(u) = T_1(u)T_2(u)$. The MTF of the cascaded process, therefore, is

$$\text{MTF}(u) = \frac{|T_1(u)T_2(u)|}{|T_1(0)T_2(0)|}.$$  \hspace{1cm} (14.25)

The MTF is not multiplicative if the characteristic function has an imaginary term (asymmetric IRF).

### 14.3.7 LSI Systems in Two Dimensions

The concepts discussed so far have all been expressed in terms of the one-dimensional variables $x$ and $u$, but are more generally described in two- (or more) dimensional variables, $(x, y)$ or $(u, v)$ and $(r, k)$.

#### 14.3.7.1 Point-Spread Function

The point-spread function, $p(x, y)$, describes the system response to a point impulse response. This is equivalent to the impulse–response function, although not all authors normalize to unity area. When normalized, the two-dimensional point-spread function (PSF) and two-dimensional characteristic function are Fourier pairs:

$$F_p(x, y) = T(u, v).$$  \hspace{1cm} (14.26)

#### 14.3.7.2 Line-Spread Function

The line-spread function (LSF) describes the response of a system to a line delta function, normalized to unity area. This is seen if we consider a line impulse extending forever in the $y$ direction as $\delta_y(x)$. The response is given by the two-dimensional convolution integral:

$$\text{LSF}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta_y(x') \text{PSF}(x-x', y-y') dx' dy'$$  \hspace{1cm} (14.27)

$$= \int_{-\infty}^{\infty} \text{PSF}(x, y) dy,$$  \hspace{1cm} (14.28)

for a normalized PSF. The LSF describes the response of a system in one direction, when details of the response in the orthogonal direction have been “integrated out,” and is normalized to unity area by definition.

Taking the Fourier transform of both sides and evaluating for $v = 0$ gives

$$F_{\text{LSF}}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{PSF}(x, y)e^{-2\pi i (u x + v y)} dx dy |_{v=0}$$  \hspace{1cm} (14.29)

$$= F^2(\text{PSF}(x, y))|_{v=0} = T(u, v)|_{v=0}$$  \hspace{1cm} (14.30)

and

$$\text{MTF}(u) = \text{MTF}(u, v)|_{v=0} = |F_{\text{LSF}}(x)|.$$  \hspace{1cm} (14.31)

### 14.3.8 Image Wiener Noise Power Spectrum

Fourier methods are also key for describing noise in imaging systems. This section describes how Fourier methods are used, and how to develop an understanding of image noise in both the spatial and spatial frequency domains.

#### 14.3.8.1 Parseval’s Theorem

An important relationship given by Parseval’s theorem (Papoulis 1991) states

$$\int_{-\infty}^{\infty} f^*(x)g(x) dx = \int_{-\infty}^{\infty} F^*(u)G(u) du$$  \hspace{1cm} (14.32)

for wide-sense stationary (WSS) noise processes (mean and correlation are the same in all parts of the image). From the Fourier relationships, $f(x) \ast g(x) \leftrightarrow F(u)G(u)$ and $f(x) \ast g(x) \Leftrightarrow F(u)G(-u)$, when $f(x)$ and $g(x)$ are real-only, $G(-u) = G^*(u)$ and, therefore,

$$f(x) \ast g(x) \leftrightarrow |F(u)|^2.$$  \hspace{1cm} (14.33)

and

$$f(x) \ast g(x) \leftrightarrow |F(u)|^2.$$  \hspace{1cm} (14.34)

The intensity of a quantity is often proportional to the square of the amplitude, as the “energy” or “power” of an electrical signal may be proportional to the square of the voltage or current, respectively, and kinetic energy is proportional to velocity squared. It is, therefore, common to talk about the “energy” of an image signal as being the integral of the square of the image signal:

$$E = \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(u)|^2 du,$$  \hspace{1cm} (14.35)

which is a statement that “energy” is conserved by the Fourier transform.

When we talk about “noise power” in an image, we are talking about the mean-square value of the fluctuation of a signal about its mean value due to random noise. In the Fourier domain, $W_\delta(u)$ is the Wiener “noise power spectrum” (NPS) of the random function, $d(x)$:

$$W_\delta(u) = \lim_{X \to \infty} \frac{1}{X} \langle \tilde{D}_x(u)\tilde{D}_x(u) \rangle = \lim_{X \to \infty} \frac{1}{X} \langle \tilde{D}_x|^2 \rangle,$$  \hspace{1cm} (14.36)

where $\sim$ indicates a random variable and $\tilde{D}_x(u)$ is the Fourier integral of $d(x)$ evaluated over distance $X$.

#### 14.3.8.2 Wiener–Khinchin Theorem and NPS

The Wiener–Khinchin theorem (Papoulis 1991; Barrett and Myers 2004) provides another key Fourier relationship for the
description of WSS image noise. If $\hat{d}(x)$ is a zero-mean random function describing fluctuations in image brightness, then:

\[
\langle \hat{D}_x(u_1)\hat{D}_x(u_2) \rangle = \int_{x_1}^{x_2} \int_{x_1}^{x_2} \langle \hat{d}(x_1)\hat{d}(x_2) \rangle e^{2\pi i(x_1u_1-x_2u_2)} \, dx_1dx_2 \quad (14.37)
\]

Thus, the Wiener–Khinchin theorem gives the noise power spectrum of $\hat{d}(x)$ as:

\[
W_D(u) = \lim_{X \to \infty} \frac{1}{X} \langle \hat{D}_x(u)^* \rangle = \int_{-\infty}^{\infty} K_x(x) e^{2\pi mu} \, dx \quad (14.40)
\]

in units of $|\hat{d}(x)|^2$ and, more generally, the Wiener NPS and autocovariance are Fourier pairs:

\[
K(x) \Leftrightarrow W(u). \quad (14.41)
\]

### 14.3.9 Digital Systems

Digital images consist of a matrix of numerical values that represent image brightness on a uniformly-spaced grid of picture elements (pixels). One consequence of this is that both signal and noise may exhibit spectral aliasing artifacts, as described in this section. As before, use is made of a graphical approach to represent image signals in both spatial and spatial frequency domains. Discrete samples are represented by Dirac $\delta$ functions scaled by discrete numerical sample values. One-dimensional geometry is assumed for simplicity, but results are easily extended into two- (or more) dimensions.

#### 14.3.9.1 Sampling and Aliasing

A continuous function, $d(x)$, can be approximated using a set of discrete values, $d_n$, using the sifting property of $\delta$ functions (Bracewell 1978), which says the value of $d(x)$ at $x = x_n$ is given by $d(x)|_{x=x_n} = \int_{-\infty}^{\infty} d(x) \delta(x - x_n) \, dx$. It follows, therefore, that $d(x)$ can be evaluated at uniformly-spaced positions with a two-step process:

1. Multiplication of $d(x)$ by a set of $\delta$ functions at the desired sample points.
2. Integration over each resulting scaled $\delta$ function.

The first step is illustrated in Figure 14.5. Each horizontal pair forms a Fourier pair, related through the Fourier transform (not the discrete Fourier transform). The function, $d(x)$, may be bounded (have finite spatial extent) or not, but it is assumed that its Fourier transform, $D(u)$, exists. As shown in the left column, $d(x)$ is multiplied by $\sum \delta(x - nx)$, an infinite set of $\delta$ functions at uniform spacing, $x_s$, resulting in the sequence of scaled $\delta$ functions, $d'(x)$:

\[
d'(x) = d(x) \sum_{n=-\infty}^{\infty} \delta(x - nx_s) = \sum_{n=-\infty}^{\infty} d_n \delta(x - nx_s) \quad (14.42)
\]

where $d_n = d(x)|_{x=x_n}$. The $\dagger$ notation is used to indicate a sampled function represented as a sequence of scaled $\delta$ functions. While $d(x)$ is typically a dimensionless quantity, $d'(x)$ has units of $[d(x)x^{-1}]$, and is a generalized function ($\delta$ functions have units inverse to the argument). The area of each scaled $\delta$ function is equal to the corresponding sample value.

The Fourier transform of $d'(x)$, as shown in the lower right of Figure 14.5:

\[
F[d'(x)] = D(u) * u_s \sum_{n=-\infty}^{\infty} \delta(u - nu_s) \quad (14.43)
\]

with units of $[d(x)]$, where $D(u)$ is the Fourier transform of $d(x)$, and $u_s = 1/x_s$ is the sampling frequency. Thus, sampling in the spatial domain has resulted in scaling of $D(u)$ by $u_s$ and the creation of spectral aliases. The aliases overlap if $D(u)$ extends beyond $|u| < u/2$, where $u/2$ is the sampling cutoff frequency. If aliasing occurs, frequency components above $u/2$ are “folded” into frequencies below $u/2$, resulting in aliasing artifacts.

![Figure 14.5](image-url)
Figure 14.6 shows an example of spectral aliasing. As the spokes of the star pattern approach the center, they consist of frequencies greater than $u_s/2$, resulting in the Moiré pattern near the hub. The second step is to integrate across each δ function in $d(x)$ to determine its area. There are many ways to do this, but a convenient method is to weight the integration with a sinc function centered on the desired δ and having zeros at all other δ functions. Knowing that $xδ(x) = 0$, for a δ function at $x = x_o$ we obtain

$$d(x_o) = \int_{-\infty}^{\infty} d'(x')\text{sinc}((x - x')/x_o) dx'\bigg|_{x = x_o} = d'(x) * \text{sinc}(x/x_o)\bigg|_{x = x_o}. \quad (14.45)$$

This process is, therefore, a convolution, corresponding to a low-pass filter in the spatial frequency domain that suppresses all frequencies above the sampling cutoff frequency, $u = u_s/2$.

### 14.3.9.2 Noise Aliasing

The effect of sampling on the Wiener NPS is determined by recognizing that random sample values scaling uniformly spaced δ functions form a random function with periodic behavior and statistical properties that are invariant to a shift of a multiple of the sample spacing, $x_s$ (Gardner and Franks 1975; Papoulis 1991; Cunningham 2000). Such a process is called wide-sense cyclo-stationary (WSCS) if the mean and correlation are invariant to the shift. For example, if $d(x)$ represents a WSCS random process that is represented by the digital samples $d_n = d(nx_s)$ at uniform spacing, $x_s$, then $d'(x)$ given by

$$\hat{d}'(x) = \sum_{n=-\infty}^{\infty} \hat{d}_n \delta(x - nx_s) \quad (14.46)$$

is an infinite train of scaled δ-functions, and is a WSCS random process with period $x_s$. The covariance of $d'(x)$ is $K_{\hat{d}'}(x)$ given by

$$K_{\hat{d}'}(x) = \frac{1}{x_s} \sum_{n=-\infty}^{\infty} K_{\hat{d}}(nx_s) \delta(n - nx_s) = \frac{1}{x_s} \sum_{n=-\infty}^{\infty} \delta(x - nx_s)$$

(14.47)

with units of $[\hat{d}^2(x) x^{-2}]$, where $K_{\hat{d}}(x)$ is the autocovariance of $\hat{d}(x)$. Similarly, the Wiener NPS, $W_{\hat{d}}(u)$, is given by

$$W_{\hat{d}}(u) = F[K_{\hat{d}'}(x)]$$

(14.48)

$$= u_s^2 \left| W_{\hat{d}}(u) + \sum_{n=1}^{\infty} W_{\hat{d}}(u \pm nu_s) \right|, \quad (14.49)$$

with units of $[\hat{d}^2(x) x^{-1}]$, where $u_s = 1/x_s$ is the sampling frequency. It is important to note that these units are different from those of $W_{\hat{d}}(u)$, which are $[\hat{d}^2(x) x^{-1}]$. Note the similarities and differences to signal aliasing in Equation 14.44.

### 14.4 Cascaded-Systems Theory

Quantum-based imaging systems generate images through a cascade of random physical processes. For example, X-ray interactions and energy absorption are random processes. Light generation in a phosphor is another, as is collection and detection of light. However, within the assumptions of LSI systems with WSS or WSCS noise processes, system performance can be described using a cascade of simple physical processes. In this section, it is shown that a few elementary processes can be cascaded to produce accurate models of performance that make a connection between system design parameters and DQE. This work is based on early studies of noise transfer by Shockley and Pierce (1938), Mandel (1959), and Zweig (1964, 1965), extended to the Fourier domain by Rabbani et al. (1987) and Rabbani and VanMetter (1989), and more recent contributions (Cunningham et al. 1994; Siewerdsen et al. 1997; Cunningham and Shaw 1999; Yao and Cunningham 2001; Zhao et al. 2001; Cunningham and Yao 2002; Ganguly et al. 2003; Sattarivand and Cunningham 2005; Hajdok et al. 2008; Friedman and Cunningham 2010; Yun et al. 2013a,b).

#### 14.4.1 Elementary Processes

The input to a cascaded model is a distribution of X-ray quanta incident on a detector. Representing each photon as a Dirac δ function, a distribution of incident quanta can be described as the random point process, $\tilde{q}(x)$, where

$$\tilde{q}(x) = \sum_{i=1}^{N} \delta(x - x_i). \quad (14.50)$$

In the point process notation, the points may represent locations of image quanta or events, as described in each case.

#### 14.4.1.1 Quantum Gain

Quantum gain (Figure 14.7) represents a conversion from one type of quantum to another, such as conversion of X-ray quanta
to optical quanta in a radiographic screen, where each interacting quantum is converted to $\tilde{g}$ secondary quanta and $\tilde{g}$ is a random variable characterized by a mean $\overline{\tilde{g}}$ and variance $\sigma_\tilde{g}^2$. It is assumed $\tilde{g}$ has the same mean and variance for each interacting X-ray photon. For the radiographic screen, this also requires that all incident X-ray quanta have the same energy. A spectrum of energies can be accommodated by integrating results over the spectrum.

The process of quantum gain can be represented as

$$\tilde{q}_{\text{out}}(x) = \tilde{g} \tilde{q}_{\text{in}}(x),$$

(14.51)

where mean and Wiener NPS are transferred as

$$\overline{\tilde{q}_{\text{out}}} = \overline{\tilde{g}} \overline{\tilde{q}_{\text{in}}},$$

(14.52)

$$W_{\text{out}}(k) = \tilde{g}^2 W_{\text{in}}(k) + \sigma_\tilde{g}^2 \overline{\tilde{q}_{\text{in}}},$$

(14.53)

respectively.

### 14.4.1.2 Quantum Selection

Quantum selection (Figure 14.8) is a special case of quantum gain in which each quantum is selected or not according to the random variable $\tilde{g}$, where $\tilde{g}$ has a value of 0 or 1 only and mean $\alpha$.

**Figure 14.8** Quantum selection is a special case of quantum gain in which each quantum is selected or not according to the random variable $\tilde{g}$, where $\tilde{g}$ has a value of 0 or 1 only and mean $\alpha$.

![Quantum Selection Diagram](image)

Quantum selection is a special case of quantum gain in which each quantum is selected or not according to the random variable $\tilde{g}$, where $\tilde{g}$ has a value of 0 or 1 only and mean $\alpha$.

The quanta in $q_{\text{in}}(r)$ are transferred to the output quantum image, $q_{\text{out}}(r)$, and the output NPS is given by $W_{\text{out}}(k) = \alpha \overline{\tilde{q}_{\text{in}}}$, and the output NPS is given by

$$W_{\text{out}}(u) = \alpha^2 [W_{\text{in}}(u) - \overline{\tilde{q}_{\text{in}}}] + \alpha \overline{\tilde{q}_{\text{in}}}. $$

(14.56)

**Figure 14.9** illustrates a sparse quantum image, $q_{\text{in}}(r)$ (left), being passed through a quantum selection process with a probability, $\alpha = 0.5$, resulting in the output quantum image, $q_{\text{out}}(r)$ (right). The quanta in $q_{\text{in}}(r)$ are uncorrelated and, hence, the input NPS is given by $W_{\text{in}}(k) = \overline{\tilde{q}_{\text{in}}}$, and the output NPS is given by

$$W_{\text{out}}(k) = \alpha \overline{\tilde{q}_{\text{in}}}. $$

(14.57)

### 14.4.1.3 Quantum Scatter

Image-blurring mechanisms including scattering of optical quanta in a radiographic screen are fundamentally quantum-scattering processes. That is, each quantum is randomly relocated to a new position, with a probability described by the normalized PSF of the blur. This differs from the blur described by a linear filter, which can be viewed as a redistribution of signal by weights, as described by the convolution integral, while scatter...
must be viewed as a redistribution by probabilities (Figures 14.10 and 14.11). This distinction has been recognized for some time (Dainty and Shaw 1974; Wagner 1977; Barrett and Swindell 1981; Sandrik and Wagner 1982; Metz and Vyborny 1983), but noise transfer relationships were first described by Rabbani et al. (1987), and later using point process theory by Barrett and Myers (2004).

The output of a scatter stage is always a point process such as a quantum image, and transfer properties given by

\[
\overline{q}_{\text{out}}(x) = \overline{q}_{\text{in}}(x) \ast p(x)
\]

where \( \ast \) represents the scatter operator, and \( p(x) \) and \( T(u) \) are the normalized scatter PSF and characteristic function, respectively.

### 14.4.1.4 Linear Filter

Convolution operations are represented as linear filters in the cascaded model, describing deterministic blurring operations. For example, integration of secondary quanta in sensor elements can be represented as convolution with a rectangle, \( h \), with width equal to that of a sensor element. The input can be either a point process, \( \overline{q}(x) \), or an analog signal, but the output is always an analog signal. Transfer relationships are given by

\[
\overline{q}_{\text{out}}(x) = \overline{q}_{\text{in}}(x) \ast p(x)
\]

\[
\overline{q}_{\text{out}} = A\overline{q}_{\text{in}}
\]

\[
W_{\text{out}}(u) = W_{\text{in}}(u)P(u)^2,
\]

where \( p(x) \) is the filter kernel, \( A = \int_{-\infty}^{\infty} p(u)du \) is the kernel area, and \( P(u) \) is the Fourier transform of \( p(x) \).

### 14.5 Detective Quantum Efficiency

As mentioned above, the DQE is our primary metric of detector performance and, thereby, image quality for a specified exposure, given by Equations 14.2 and 14.3:

\[
\text{DQE}(u) = \frac{\text{NEQ}(u)}{\overline{q}_{\text{in}}},
\]

where \( \text{NEQ}(u) \) is the noise equivalent quanta expressed as a function of spatial frequency, and \( \overline{q}_{\text{in}} \) is the number of X-ray quanta incident on the detector per unit area. Shaw (1963) showed that the NEQ could be expressed on an absolute scale by expressing image noise in terms of a number of Poisson-distributed input photons per unit area:

\[
\text{NEQ}(u) = \text{SNR}^2(u) = \frac{\overline{q}^2 G^2 |\text{MTF}(u)|^2}{W_d(u)}
\]

in units of [\( \overline{q} \)], where \( \text{SNR}(u) \) is the image SNR, and \( G \) is the system gain relating \( \overline{q}_{\text{in}} \) to output, \( \overline{q}_{\text{out}} \). Therefore,

\[
\text{DQE}(u) = \frac{\overline{q}^2 G^2 |\text{MTF}(u)|^2}{W_d(u)}
\]

\[
= \frac{\text{MTF}^2(u)}{\overline{q}[W_d(u)/d^2]}
\]

Interpretation of the NEQ provides a great deal of insight regarding the information content of an image. For instance,
Wagner et al. (1974), Wagner (1977), and Wagner and Brown (1985) showed that, for the detection of an object, \( \Delta S(u) \), having frequency components, \( \Delta S(u) \), the NEQ is directly related to the “ideal observer SNR,” \( SNR_\text{I} \), according to (Medical Imaging–the Assessment of Image Quality 1996):

\[
SNR^2 = \int_{-\infty}^{\infty} |\Delta S(u)|^2 NEQ(u) du = \int_{-\infty}^{\infty} qo |\Delta S(u)|^2 DQE(u) du,
\]

(14.68)

describing what can and cannot be seen in a noise-limited image. Maybe more importantly, it indicates that image quality will be maximized by maximizing the DQE at spatial frequencies of importance.

### 14.5.1 Cascaded Models of DQE

#### 14.5.1.1 Model 1: Light Generation in Phosphor

Complex processes can often be represented as a cascade of these simple processes to determine models of the DQE. For example, a radiographic phosphor with quantum efficiency \( \alpha \) converts each interacting X-ray into a large number of optical quanta. The optical quanta are subsequently scattered before they leave the phosphor. If we assume a thin transparent phosphor (ignoring Swank noise and Lubberts effect), we can represent the distribution of light as a cascade of quantum selection, gain, and scatter stages, as illustrated in Figure 14.12. The output random process, \( q_{\text{out}}(x) \) and corresponding transfer relationships are given by

\[
\tilde{q}_{\text{out}}(x) = \alpha g \tilde{q}_o(x) \ast_i p(x),
\]

(14.69)

\[
\tilde{q}_{\text{out}} = \alpha \tilde{q}_o, 
\]

(14.70)

\[
W_{\text{out}}(u) = \alpha g^2 \left[ 1 + \frac{\sigma^2_g}{\bar{g}} - \frac{1}{\bar{g}} \tilde{q}_o |T(u)|^2 \right] + \alpha \tilde{q}_o, 
\]

(14.71)

where \( \tilde{q}_o(x) \) describes the distribution of incident X-ray quanta, \( \tilde{g} \) is the conversion gain relating the number of interacting X-ray quanta to the number of light quanta that will eventually leave the phosphor, and \( p(x) \) describes the scatter of light. The DQE of this simple detector is, therefore, given by

\[
\text{DQE}(u) = \frac{\alpha q |T(u)|^2}{W_{\text{out}}(u)} = \frac{\alpha}{1 + \frac{\sigma^2_g}{\bar{g}} - \frac{1}{\bar{g}} |T(u)|^2}
\]

(14.72)

\[
= \frac{\alpha}{1 + \epsilon^g + \frac{1}{\bar{g}} |T(u)|^2}
\]

(14.73)

where \( \epsilon_g = \sigma^2_g/\bar{g} - 1 \) is sometimes called the gain “Poisson excess,” in that it describes how much the gain variance is greater than a Poisson gain.

Figure 14.12 shows sample input and output functions passing through this model. On the left is a distribution of incident X-ray quanta. On the right is a sample output describing the distribution of detected secondary light quanta exiting the screen.

The cascaded approach can be used to describe more sophisticated models. See Hajdok et al. (2008) for a model of Swank noise, Yao and Cunningham (2001), and Yun et al. (2013b) for a model that includes thick detectors, additive readout noise, broad X-ray spectra, and escape and reabsorption of scatter photons from photoelectric, Compton, and coherent scatter in the detector, and Friedman and Cunningham (2010) for the effects of converter lag.

#### 14.5.1.2 Model 2: Ideal Sensor Array

A second cascaded model is shown in Figure 14.13, consisting of a distribution of quanta incident on an ideal sensor array. The DQE of this detector can be predicted by examining the transfer of signal and noise from input to output.

The input to the model at step 0 is \( \tilde{q}_o(x) \), a random distribution of X-ray quanta incident on the detector. We must assume shift invariance and WSS noise processes requiring \( \tilde{q}_o(x) \) be uniformly distributed over an infinite detector, which we represent as having width, \( L \), in the limit \( L \to \infty \).

Therefore, \( \tilde{N}_o = \tilde{q}_o L \). The Wiener NPS of \( \tilde{q}_o(x) \) is \( W_{\tilde{q}}(u) = \tilde{q}_o^2 \).

As illustrated in step 1, this detector counts all X-ray photons incident in each detector element. The number of photons interacting in the \( n \)th element of width, \( a \), and scaled by a constant, \( k \), representing detector gain, is given by

![Figure 14.12](image-url) Conversion of incident quanta into a cluster of secondary quanta can be represented by cascading quantum selection, gain, and scatter stages. Input quanta (left) are uncorrelated in this example, but the resulting secondary quanta are not (right).
The Wiener NPS is, therefore, given by

$$W_d(u) = \frac{1}{a^2} \left[ W_d(u) + \sum_n W_s \left( \frac{u \pm n}{a} \right) \right]$$  \hspace{1cm} (14.77)

and

$$W_d(u) = k^2 q_o^2 a^2 \text{sinc}^2(au).$$  \hspace{1cm} (14.75)

resulting in a series of \( \delta \)-functions scaled by element values, \( \tilde{d}_n \).

The Wiener NPS of \( \tilde{d}^*(x) \) is determined by noting that \( \tilde{d}^*(x) \) is a wide-sense cyclostationary random process, since the mean and autocovariance are periodic with shifts of \( na \). Thus, while signal aliasing is described as a convolution of \( D(u) \) with

$$\frac{1}{a} \sum_n \delta(u - na)$$

to get \( \tilde{D}_a(u) \) at step 2, in the spatial frequency domain, noise aliasing is described as a convolution of \( W_d(u) \) with

$$\frac{1}{a^2} \sum_n \delta(u - nla).$$

The final result comes from the property of sinc-functions that an infinite sequence of sinc(\( au \)) functions, shifted by integer multiples of \( au \), sums to unity.

The MTF and DQE performance of this detector design can be determined from these results. For example, the pre-sampling MTF is determined as the ratio of the output to input mean signal in the spatial frequency domain, while ignoring aliased terms, normalized to unity at \( u = 0 \):

$$\text{MTF}(u) = \text{sinc}(au).$$  \hspace{1cm} (14.80)
The corresponding DQE is determined as:

$$\text{DQE}(u) = \frac{\text{MTF}^2(u)}{\langle |\hat{\delta}(\nu)|^2 \rangle}.$$  \hspace{1cm} (14.81)

The average of scaled $\delta$-functions in the angle brackets is

$$\langle |\hat{\delta}(\nu)|^2 \rangle = \frac{1}{a} \delta,$$  \hspace{1cm} giving

$$\text{DQE}(u) = \text{sinc}^2(ua).$$  \hspace{1cm} (14.82)

This result shows that a simple idealized detector is not necessarily a perfect detector for producing high SNR images. In this example, noise aliasing causes the DQE to follow a sinc$^2$ shape, dropping to the value sinc$^2(1/2) = (4/\pi^2) \approx 0.41$ at the sampling cutoff frequency, $u = 0.5a$. Image noise would appear to have a high-frequency structure, something that is common to selenium-based detectors that typically suffer from noise aliasing, as described by this model.

### 14.5.2 Experimental DQE Measurements

Equation 14.67 provides a convenient form for practical measurements of the DQE, requiring a measurement of: (i) MTF; (ii) $\bar{q}$; and (iii) $W(u)/\delta^2$.

Guidelines for the measurement of each have been described by IEC 62220-1 series of standards by the International Electrotechnical Commission. Figure 14.14 shows DQE curves obtained on five new radiographic detectors. At low spatial frequencies, important for visualizing large image structures, the DQE differs by a factor of approximately $2\times$. This means the system with the lowest DQE requires twice the patient exposure to produce the same image quality at low frequencies. Of greater concern is at high spatial frequencies, where the same system requires $20\times$ the exposure for visualizing small structures and fine detail. These results illustrate the wide range of image quality that may be achieved with different imaging systems, and the importance of making an informed purchase decision and to ensure imaging systems are maintained to achieve the best possible DQE values over their useful life.

**REFERENCES**


