Handbook of Peridynamic Modeling

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Nonlocal Calculus of Variations and Well-Posedness of Peridynamics

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Chapter 3

Nonlocal Calculus of Variations
and Well-Posedness of
Peridynamics

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3.1 Introduction

Recent studies of peridynamic models in various applications, as documented in other parts of this handbook, have motivated the development of mathematical theory for nonlocal models [18]. Similar to the mathematical study of local PDE models, a systematic and rigorous mathematical treatment of nonlocal integral equations such as those represented by peridynamic models can offer theoretical insight in the modeling process. It can also provide guidance to numerical simulations and serve as a basis for verification and validation studies.

To gain better mathematical understanding of peridynamic models, there are a number of popular, and at the same time fundamental, mathematical questions to be answered first, such as

What are suitable function spaces and topologies in which to define the solutions?
How to incorporate suitable nonlocal boundary conditions or nonlocal constraints?

Are the peridynamic models well-posed in the sense that solutions exist which are unique and are dependent continuously on data?

What are the relations between nonlocal peridynamic models, differential equation models in classical continuum mechanics, dynamics of interacting particles and/or molecular dynamics?

What are the characteristics of solutions to peridynamic models? What are their regularity properties? What singular behavior may solutions develop?

In the last few years, these questions have received much attention from the mathematics community. There has been considerable progress made towards answering these questions in some specialized cases. Moreover, the new development of mathematical theory for peridynamics has also helped to resolve various issues in other related subject areas such as nonlocal diffusion and stochastic jump processes. In this chapter, our main objective is to review some of the progress that has been made on establishing the well-posedness of nonlocal peridynamic models. We choose to emphasize on discussing results that have been established within the framework of nonlocal vector calculus and nonlocal calculus of variations. Materials are presented here with both general, less-technical descriptions and illustrative examples so as to offer readers a glimpse of the key ingredients of the nonlocal vector calculus framework and the theory of nonlocal calculus of variations, along with a demonstration of how they can be applied to formulate and analyze peridynamic models. Discussions of relations between nonlocal and local models are given in the next chapter, together with relations between their discretizations. Qualitative studies of how solutions to peridynamic models change with respect to materials parameters and external conditions are of great importance in practice. In particular, the characterization of their regularity properties is a much desirable work to be carried out. But other than the minimal regularity guaranteed by the well-posedness theory, no further discussions on the regularities are made here.

With a few exceptions, most of the materials presented in this chapter are associated with linear equations and variational problems. As pointed out by Silling in Section 2.2.8 of the handbook, under the assumption of small deformation, the peridynamic equation of motion may be approximated by a linear integro-differential equation given in (2.25), which is restated below as

\[ \rho \partial_{tt} \mathbf{u}(x,t) = \int C(x,x')(\mathbf{u}(x',t) - \mathbf{u}(x,t))dx' + b(x) \]  

(3.1)

where \( \rho \) represents a constant density and \( b(x) \) denotes the external force.

Factors that affect properties of solutions to the above linearized model include the tensor \( C \), which defines the type of nonlocal interactions in the materials system, and appropriate nonlocal boundary conditions and initial conditions.
For a linear bond-based model, $C$ is a rank-one tensor of the form,

$$C(x,x) = \omega(|x-x'|) \left[ \alpha(x,x) \alpha(x',x) \right]$$

(3.2)

where

$$\alpha(x,x) = \frac{x-x}{|x-x'|^2}$$

denotes a scaled vector along the bond $x$ to $x'$.

For the state-based model [41], $C$ is given by

$$C(x,x) = \eta \omega(|x-x'|) \left[ \alpha(x,x) \alpha(x',x) \right] + \sigma \int \left( \omega(x-z) \omega(z-x) \alpha(z,x) \alpha(x',z) \right. $$

$$+ \omega(x-x') \omega(x-z) \alpha(x,x') \alpha(x',z) $$

$$\left. + \omega(x-z) \omega(x-x') \alpha(x,z) \alpha(x',x) \right) d\mathbf{z}.$$

(3.3)

The additional terms appearing in the tensor above but not in the tensor for the bond-based model signify the indirect interactions of materials points $x$ and $x'$ through an intermediate point $z$ [41].

Assuming that the nonlocal interaction kernel $\omega(|x-x'|)$ is supported in a spherical neighborhood of radius $\delta$ (the so-called peridynamic horizon), then in the bond-based model, we see that the nonlocal interaction exists only when the distance between $x$ and $x'$ is smaller than $\delta$. However, the range of interactions becomes broader in the state-based model due to the indirect interactions [41].

Let us recall some of the comments made on linearized peridynamic models in Section 2.2.8. First, it is pointed out that it is easier to study issues like the well-posedness and convergence of numerical approximations in the context of linearized equations. Moreover, Silling notes that linearized peridynamics also retains some of the complexities of the full nonlinear theory when one is concerned with practical numerical computations. Thus, mathematical analysis of linear equations of the form (3.1) and their corresponding steady-state equations serves as an interesting first step towards a better mathematical understanding of more general nonlinear peridynamic models. This is not to discount the greater practical significance of nonlinear theory. Indeed, extending the linear theory to full nonlinear models will surely be a major undertaking and is currently under active investigation.

The rest of the chapter is organized as follows: we first present in Section 3.2 a brief overview of recent mathematical studies on the well-posedness of peridynamic models. This is followed by an introduction of the notion of nonlocal flux, nonlocal balance laws, and the nonlocal vector calculus in Section 3.3. Then, in Section 3.4, we offer a step-by-step guided tour of the key elements of nonlocal calculus of variations, using a nonlocal constrained value problem associated with a linear bond-based model as the illustrative example. The mathematical framework established here can be used to analyze other nonlocal problems. It also lays the foundation for a rigorous analysis of numerical approximations. The mathematical development mirrors...
the standard calculus of variations theory for local PDEs, which helps to add some familiarity to the discussion. Yet, whenever possible, we point out the differences between the local version and the nonlocal counterpart. We focus on presenting the main ingredients without getting into greater technical details so as to make the exposition more accessible. Only basic variational principles and function spaces are required. Through simple but detailed illustrations, it is hoped that researchers outside the mathematics community can also benefit from a good understanding of the basic mathematical theory which may be important for verification and validation of the peridynamic models and simulations. We then continue with various extensions in section 3.5 and finally offer some remarks in section 3.6.

3.2 A brief review of well-posedness results

One of the first theoretical questions asked about a mathematical equation is often the existence and uniqueness of its solutions. We mention in this section a few selected works in the literature concerning the well-posedness of peridynamic models. This is not intended to be a comprehensive review since there are a large number of studies on related nonlocal models which are not discussed here. We instead focus on several earlier works in this direction that help set the stage for more detailed discussions given in the next section.

As with the development of peridynamic models, rigorous mathematical studies of the well-posedness of peridynamics have started with bond-based peridynamic models that are special cases of state-based peridynamic models. The existence and uniqueness of solutions to the initial value problem associated with a linear bond-based peridynamic equation of motion were first demonstrated by Emmrich and Weckner in [23]. They studied a one-dimensional infinite peridynamic bar with an integrable nonlocal interaction kernel. The resulting solution space is given by the standard $L^p$ Lebesgue function space, a space of functions whose $p$th powers are integrable ($p \geq 1$). The well-posedness results were extended by Du and Zhou [20] to allow more general kernels that have finite second moments but may not be integrable. These more general kernels lead to a wider class of solution spaces.

For nonlinear initial value problems, well-posedness studies include the work of Erbay, Erkip and Muslu [24] in one space dimension, and the work of Du, Kamm, Lehoucq, and Parks [17] for one-dimensional scalar nonlocal conservation laws. More recently, Emmrich and Puhst analyzed nonlinear bond-based models [22] with globally Lipschitz continuous pairwise force functions. For nonlinear variational problems, a recent existence result was given in [5]. A study related to nonlinear variational problems associated with the more general state-based models was made in [36].

Zhou and Du [47] extended the analysis of initial value problems to study variational problems associated with nonlocal boundary conditions and considered two-dimensional bond-based peridynamic systems as special cases. Analysis of a scalar
nonlocal variational problem in multiple dimensions has also been studied by Aksoylu and Parks in [2].

For further review of early literatures on the subject of well-posedness of peridynamic models, we refer to the paper by Silling and Lehoucq [39] and the paper by Lehoucq, Emmrich and Puhst [21].

A more systematic effort to develop a mathematical theory for nonlocal problems in parallel to that for local PDEs has been made in a series of works by Du, Gunzburger, Lehoucq and Zhou [13, 14, 15] resulting in the framework of nonlocal vector calculus developed in [15]. It is then applied to nonlocal and fractional or anomalous diffusion, nonlocal transport, and nonlocal mechanics problems such as peridynamics [13]. The paper [13] provides demonstrations of how nonlocal equations are well-posed on bounded domains with various nonlocal boundary conditions (constraints on domains of nonzero volume). It also shows a compelling analogy that can be drawn between nonlocal models and classical PDEs, including that of a conforming finite element discretization. In [14], a rigorous variational formulation of a linear nonlocal state-based PD system, which is called a peridynamic Navier equation, is presented using basic operators of the nonlocal vector calculus. The paper also demonstrates that, in free space, the local limit of the nonlocal peridynamic Navier operator recovers the conventional Navier operator of linear elasticity with the entire range of allowable elastic constants, e.g., the full range of Poisson ratio in the interval \((-1, 1/2)\). Moreover, the linear peridynamic Navier equation is shown to be well-posed over the space of square integrable functions for a locally integrable tensor kernel. A constraint on the displacement field over a volume of nonzero measure is applied to avoid using the trace operator that is not well defined on this space.

A series of follow-up works by Mengesha and Du further developed the framework of nonlocal calculus of variations that allows one to study the well-posedness of linear and nonlinear peridynamic models under more general assumptions. The papers [33, 34, 35, 36] generalized the analysis of linear nonlocal models to include a nonintegrable tensor kernel and to cases where the kernel may change signs. This latter generalization, suggested by Silling, means that both repulsive and attractive forces can be present which is necessary for dealing with material instability within the peridynamic mechanics theory [40, 44]. Nonlinear variational problems associated with a variety of volumetric constraints are also studied. In addition, [33, 34, 35, 36] documented basic ingredients for the functional analysis of nonlocal energy spaces and nonlocal variational problems with general kernels for both scalar equations and for systems.

3.3 Nonlocal balance laws and nonlocal vector calculus

In continuum mechanics, physical balance laws are often written mathematically as systems of partial differential equations. On the other hand, peridynamic models introduced by Silling in [40] can be viewed as nonlocal balance laws by defining a
nonlocal flux in terms of interactions between regions having positive measures and possibly not sharing a common boundary.

**Nonlocal flux** For a typical peridynamic equation of motion, such as the equation (2.25) with a kernel given by (3.2), a nonlocal flux functional can be defined for any pair of domains \( \Omega_1 \) and \( \Omega_2 \) by

\[
F(\Omega_1, \Omega_2) = \int_{\Omega_1} \int_{\Omega_2} C(x', x) (u(x') - u(x)) \, dV_x \, dV_{x'}
\]

for a tensor \( C(x', x) \) which is symmetric with respect to \( x' \) and \( x \).

For more general state-based peridynamic models [42] of the form:

\[
\rho \frac{\partial \mathbf{y}(x,t)}{\partial t} = \int \left\{ T[x] < x > - T[x'] < x - x' > \right\} dV_{x'} + b(x),
\]

with \( T[x] < x > \) and \( T[x'] < x - x' > \) being the peridynamic force states, a similar notion of nonlocal flux can be defined, as in [15], by

\[
F(\Omega_1, \Omega_2) = \int_{\Omega_1} \int_{\Omega_2} T[x] < x > - T[x'] < x - x' > \, dV_x \, dV_{x'}.
\]

A defining characteristic of the nonlocal flux is that the integrands in (3.4) and (3.6) are anti-symmetric with respect to an interchange of \( x \) and \( x' \). This is a necessary and sufficient condition for the resulting balance law to satisfy an action-reaction principle. The nonlocal flux is an additive form, in the sense that

\[
F(\Omega_1, \Omega_2) + F(\Omega_1, \Omega_3) = F(\Omega_1, \Omega_2 \cup \Omega_3), \quad \Omega_2 \cap \Omega_3 = \emptyset,
\]

and an alternating form in the sense that

\[
F(\Omega_1, \Omega_2) = F(\Omega_2, \Omega_1), \quad \Omega_1, \Omega_2.
\]

Moreover, nonlocal flux can be well-defined for two domains \( \Omega_1 \) and \( \Omega_2 \) without letting \( \Omega_1 \) and \( \Omega_2 \) be in direct contact. Thus, it broadens the notion of local fluxes. These discussions and axiomatic derivations of various properties of nonlocal flux functionals can be found in [15]. The study of nonlocal flux there also influenced how nonlocal operators are introduced in the nonlocal vector calculus framework.

**Nonlocal balance laws** By integrating the equation of motion (3.5) over a spatial domain \( \Omega \subset \mathbb{R}^n \), the nonlocal balance law corresponding to the state-based peridynamic model (3.5) takes on the form:

\[
\frac{d}{dt} \int_{\Omega} \rho \mathbf{y} \, dV_x = F(\Omega, \mathbb{R}^n, \Omega) + \int_{\Omega} \mathbf{b} \, dV_x.
\]

In other words, peridynamic equations may be seen as continuum models that represent nonlocal balance laws involving nonlocal fluxes and nonlocal operators; they...
avoid classical notions like the local flux and deformation gradient, thus offering the opportunity to work with more singular solutions to better characterize materials defects.

The nonlocal vector calculus developed in [15] introduces some basic nonlocal operators to systematically represent nonlocal flux and nonlocal balance laws. The theory also helps to elucidate the often puzzling issue of boundary conditions, or, more properly speaking, nonlocal solution constraints for nonlocal equations.

**Nonlocal operators** Various simple nonlocal operators have been defined as the basic building blocks of nonlocal vector calculus. For example, for any two points \( \mathbf{x} \) and \( \mathbf{y} \) in \( \mathbb{R}^n \), we pick \( \alpha = \alpha(\mathbf{x}, \mathbf{y}) \) as a given two-point vector such as the unit vector \( \alpha(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})/\|\mathbf{x} - \mathbf{y}\| \) or, to be used here, the rescaled unit vector \( \alpha(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})/\|\mathbf{x} - \mathbf{y}\|^2 \). With respect to \( \alpha \), we have the nonlocal divergence operator \( \mathcal{D} \) defined by

\[
(\mathcal{D}\psi)(\mathbf{x}) = \int_{\mathbb{R}^n} \left( \psi(\mathbf{x}, \mathbf{x}) + \psi(\mathbf{x}, \mathbf{y}) \right) \alpha(\mathbf{x}, \mathbf{y}) d\mathbf{x}
\]

for any 2-point tensor-valued function \( \psi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^m \). Its adjoint operator \( \mathcal{D}^* \) is defined by

\[
(\mathcal{D}^* \mathbf{v})(\mathbf{x}, \mathbf{y}) = (\mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{y})) \alpha(\mathbf{x}, \mathbf{y})
\]

for any vector-valued function \( \mathbf{v}: \mathbb{R}^n \rightarrow \mathbb{R}^m \). We note the matrix and vector product identities that

\[
(\beta \alpha) \alpha = \beta \alpha^T \alpha = \alpha^2 \beta, \quad (\alpha \beta) = \alpha^T \beta \alpha, \quad \beta \in \mathbb{R}^m. \tag{3.7}
\]

Unlike classical divergence operators that map between vector and scalar fields of the same variable, operators \( \mathcal{D} \) and \( \mathcal{D}^* \) are maps between vector and tensor fields with different numbers of variables. This can be better understood through the exterior calculus interpretation of differential operators. Viewing vector and tensor fields with different numbers of variables as nonlocal forms of different orders, the nonlocal operators mimic exterior differentiations of local differential forms having different orders.

\( \mathcal{D} \) and \( \mathcal{D}^* \) are adjoint operators to each other in the sense that

\[
\int_{\mathbb{R}^n} \mathbf{v}(\mathbf{x}) \cdot (\mathcal{D}\psi)(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mathcal{D}^* \mathbf{v})(\mathbf{x}, \mathbf{y}) \psi(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}.
\]

This duality property may also be seen as an instance of the Fubini theorem if the integrands satisfy suitable regularity assumptions. Symbolically speaking, and to draw analogy to classical vector calculus, it can also be interpreted as a nonlocal integration by parts formula.

For notational convenience, we introduce as in [15] that

\[
(\mathcal{D}\varphi)(\mathbf{x}) = \int_{\mathbb{R}^n} \left( \varphi(\mathbf{x}, \mathbf{x}) + \varphi(\mathbf{x}, \mathbf{y}) \right) \alpha(\mathbf{x}, \mathbf{y}) d\mathbf{x}, \quad \varphi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}.
\]

Its adjoint is then given by

\[
(\mathcal{D}^* \mathbf{v})(\mathbf{x}, \mathbf{y}) = (\mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{y})) \alpha(\mathbf{x}, \mathbf{y}), \quad \mathbf{v}: \mathbb{R}^n \rightarrow \mathbb{R}.
\]
Note that $\mathcal{D} v$ is a symmetric 2-point tensor-valued function, that is $(\mathcal{D} v)(x,x) = (\mathcal{D} v)(x,x)$. Meanwhile, for any 2-point tensor function $\psi = \psi(x,y)$, $\mathcal{D}\psi$ depends only on the symmetric part of $\psi$ with respect to its two variables $x$ and $y$. In addition, we see that the integrand in the definition of $\mathcal{D}\psi$ is an anti-symmetric 2-point function so that the identity

$$\int_{\Omega} (\mathcal{D}\psi)(x) dx = -\int_{\Omega} (\mathcal{D}\psi)(x) dx$$

always holds. This can be interpreted as a nonlocal version of the classical divergence theorem [15], since the term on the right-hand side can be precisely characterized as a nonlocal flux

$$\mathcal{F}(\Omega^c, \Omega) = -\mathcal{F}(\Omega, \Omega^c)$$

that replaces the classical local flux defined on the boundary of $\Omega$.

Nonlocal extensions to Stokes theorem and other integral identities can also be established with the definition of nonlocal curl operators.

We note that in the definitions of nonlocal operators, while the space dimensions $m$ and $n$ can be different in general, the cases most relevant to peridynamics are $m = 1$ and $m = n$.

**Formulating nonlocal models via nonlocal operator** The nonlocal vector calculus allows us to reformulate linear peridynamic models in ways similar to local linear PDE models expressed in terms of basic divergence, gradient, and curl operators.

For example, with a given kernel $\omega_\delta = \omega_\delta(x - x)$, for any function $u = u(x)$ defined in $\mathbb{R}^n$, using (3.7) and the notation $\omega_\delta(y - x) = \omega_\delta(y - x) \alpha(x,y)^2$, a composition of above defined operators $\mathcal{D}$ and $\mathcal{D}^*$ gives

$$\mathcal{L}_\delta(u)(x) = \mathcal{D}(\omega_\delta \mathcal{D} u)(x) = \int \omega_\delta(y - x)(u(y) - u(x)) dy \quad (3.8)$$

which is often called a nonlocal Laplacian or a nonlocal diffusion operator.

In the context of peridynamic models, the kernel function $\omega_\delta$ has a compact support over $y - x \leq \delta$ with $\delta > 0$ being referred to as the horizon parameter. The kernel is suitably scaled so that

$$\mathcal{D}(\omega_\delta) \longrightarrow (K), \quad \text{as} \quad \delta \to 0$$

for a coefficient matrix denoted by the tensor $K$. More detailed discussions are given later; an intuitive example of such a limit is when $\omega_\delta(\cdot)$ approaches a constant multiple of the Dirac delta function at $x = 0$. This is why $\mathcal{L}_\delta$ may be viewed as an nonlocal analog of classical diffusion (Laplacian) operator.

**Peridynamic Navier operator** More closely related to peridynamic models is the so-called peridynamic Navier operator

$$\mathcal{L}_\delta = \eta \mathcal{D}(\omega_\delta \mathcal{D}) + \sigma \mathcal{D}(\omega_\delta \mathcal{D}) \quad (3.9)$$
where $\mathbf{I}$ is the identity tensor. Moreover, given
\[
\omega_\delta(x, y) = \frac{n}{m(x)} \omega_\delta(x - y) \quad \text{with} \quad m(x) = \int_{\Omega} \omega_\delta(x - y) \, dy,
\]
the operators $\mathcal{D}_{\omega_\delta}$ and $\mathcal{D}^*_{\omega_\delta}$ that map (1-point) functions to themselves are defined by
\[
\mathcal{D}_{\omega_\delta}(v)(x) = \mathcal{D} \left( \omega_\delta(x, y) v(y) \right)(x), \quad v: \mathbb{R}^n \rightarrow \mathbb{R},
\]
\[
\mathcal{D}^*_{\omega_\delta}(u)(x) = \int_{\mathbb{R}^n} \mathcal{D}^* \left( u(x, y) \omega_\delta(x - y) \right) \, dy, \quad u: \mathbb{R}^n \rightarrow \mathbb{R}^n.
\]

The expression given in (3.9) offers a concise formulation of the integral operator associated with the tensor kernel given in (3.3). Moreover, since the nonlocal operators $\mathcal{D}_{\omega_\delta}$ and $\mathcal{D}^*_{\omega_\delta}$ have close correspondence to the classical divergence and gradient operator in the local limit [15, 37, 36], we have as $\delta \rightarrow 0$,
\[
\eta \mathcal{D}(\omega_\delta \mathcal{D}) \quad \sigma \mathcal{D}^*_{\omega_\delta} \left( \mathcal{D}_{\omega_\delta} \right) \quad \mu \quad \mu \quad \lambda,
\]
where the limit is the classical, local Navier operator of linear elasticity [10].

For the special case of $\sigma = 0$, we get the nonlocal vector calculus formulation of the bond-based linear peridynamic operator
\[
\mathcal{L}_b = \mathcal{D}(\omega_\delta \mathcal{D})
\]
whose local limit is given by
\[
\mu + \mu,
\]
a special form of the Navier operator for homogeneous materials with a Poisson ratio $1/4$. More discussions will be provided in Chapter 4.

The weighted nonlocal operators The weighted nonlocal operators $\mathcal{D}_{\omega_\delta}$ and $\mathcal{D}^*_{\omega_\delta}$ defined in (3.10) and (3.11) have been introduced in [15], as they are more closely aligned with conventional differential operators defined on vector fields than operators $\mathcal{D}, \mathcal{D}^*, \mathcal{D}$, and $\mathcal{D}^*$. Indeed, it has been shown via Fourier analysis that, in the whole space, the local limits as $\delta \rightarrow 0$ of $\mathcal{D}_{\omega_\delta}$ and $\mathcal{D}^*_{\omega_\delta}$ correspond to classical gradient and divergence operators [15], subject to suitable choices of the kernels $\omega_\delta$. Recently, the analysis has been further extended in [36, 37] to offer more detailed characterizations of the related function spaces and topologies on bounded domains.

Weighted nonlocal operators also have natural physical interpretations. In fact, for a displacement field $u = u(x)$, the quantity $\mathcal{D}_{\omega_\delta}(u)(x)$ gives the linearized nonlocal dilatation and
\[
\mathcal{D} \left( u(x, x) \right) \frac{1}{n} \mathcal{D}_{\omega_\delta}(u)(x)
\]
gives the deviatoric part of the nonlocal elastic strain.

Symbolically, one may recover $\mathcal{D}$ and $\mathcal{D}^*$ from $\mathcal{D}_{\omega_\delta}$ by picking a special weight $\omega_\delta$ in the form of a Dirac delta measure.
Nonlocal vector calculus and Green’s identities

In [15], more extensive discussions have been given on $\mathcal{D}$, $\mathcal{D}^*$, $\mathcal{D}_\omega$, $\mathcal{D}^*_\omega$, along with nonlocal Green’s identities such as

$$
\int_{\mathbb{R}^n} u \left( \mathcal{D} \left( \omega \left( \mathcal{D}^* v \right) \right) \right) dx - \int_{\mathbb{R}^n} \left( \mathcal{D} \left( \omega \left( \mathcal{D}^* u \right) \right) \right) v dx = 0,
$$

$$
\int_{\mathbb{R}^n} u \mathcal{D}_{\omega} \left( \mathcal{D}_{\omega} v \right) dx = \int_{\mathbb{R}^n} \left( \mathcal{D}_{\omega} u \mathcal{D}_{\omega} v \right) dx = \int_{\mathbb{R}^n} \mathcal{D}_{\omega} \left( \mathcal{D}_{\omega} u \right) v dx,
$$

and

$$
\int_{\mathbb{R}^n} u \mathcal{D}_{\omega} \left( \mathcal{D}^*_{\omega} v \right) dx = \int_{\mathbb{R}^n} \left( \mathcal{D}_{\omega} u \mathcal{D}^*_{\omega} v \right) dx = \int_{\mathbb{R}^n} \mathcal{D}_{\omega} \left( \mathcal{D}^*_{\omega} u \right) v dx.
$$

These identities are nonlocal analogs of the classical counterparts that are stated formally in [15] and they can be rigorously justified in suitably defined function spaces. The latter are discussed in the next section.

In [15], other nonlocal operators such as nonlocal curls have also been introduced. In the context of peridynamics, nonlocal curls have not been used in the analysis or formulation but it is conceivable that there is some potential value when electromagnetic interactions are coupled to mechanical and thermodynamic models.

3.4 Nonlocal calculus of variations — an illustration

Building upon the foundation of the nonlocal vector calculus, we may further develop the theory of nonlocal calculus of variations that can be used to analyze nonlocal steady-state variational problems and time-dependent problems associated with peridynamic equations. To make the presentation accessible to a broader audience, some of the key ingredients are presented here through an illustrative example, namely a variational problem associated with a linear bond-based peridynamic model with a homogeneous condition imposed on the displacement field on part of the material domain. We assume only minimal knowledge of the variational principles for PDEs and function spaces for most of the discussion.

A steady-state problem

We consider a linear steady-state problem related to the nonlocal bond-based peridynamic model

$$
\mathcal{L}_b u(x) = b(x), \quad x \in \Omega
$$

(3.12)
where $\Omega$ is a bounded domain in $\mathbb{R}^n$ and the nonlocal bond-based peridynamic operator $\mathcal{L}_{\delta}$ is defined by

$$\mathcal{L}_{\delta} u(x) = \mathcal{D}(\omega_{\delta} \mathcal{D}(u))(x)$$

$$= \int_{\Omega} \omega_{\delta}(y, x) \mathbf{\alpha}(y, x) \cdot \mathbf{\alpha}(y, x) \cdot \left( u(y) - u(x) \right) dy$$

$$= \int_{\Omega} \omega_{\delta}(y, x) \frac{y - x}{|y - x|^2} \cdot \left( u(y) - u(x) \right) dy \quad (3.13)$$

where $\Omega_I$ denotes the interaction domain on which the solution may be constrained. Without loss of generality, $\Omega$ and $\Omega_I$ are assumed to have no interior intersection, and we use $\tilde{\Omega} = \Omega \cup \Omega_I$ to denote the whole domain. The above bond-based model corresponds to the steady-state equation of (3.1) with the tensor $C$ being a scalar multiple of the outer product of the unit vector along the bond direction, so that the nonlocal force is a linear Hookean spring force aligned with the bond direction.

Based on the definitions of the nonlocal operator $\mathcal{L}_{\delta}$, we see that the definition of $\mathcal{L}_{\delta} u(x)$ in $\Omega$ is related to how $u$ is defined on $\Omega_I$. This leads to the issue of nonlocal constraints.

**Nonlocal boundary conditions and nonlocal constraints** For nonlocal models such as peridynamics, one important issue is the proper understanding of the nonlocal analog of boundary conditions as those specified for PDEs. In [13], the notion of volume constrained problems is discussed. Some remarks along this direction are given here.

First of all, one may define a notion of essential constraints and natural or variational conditions for nonlocal problems by drawing analogy to the classical variational problems for PDEs that are defined with respect to suitable boundary conditions. At the same time, for nonlocal models there is a conceptual change to the notion of boundary. Indeed, the underlying problem is dictated by the form and the domain of nonlocal interactions so that when a materials point $x$ is located close to a material boundary, it is perceivable that a new physical principle must be adopted to account for the presence of the boundary. Thus, either additional constraints are imposed or changes to laws of nonlocal interactions that offer similar effects to that of the local boundary conditions on PDEs should be applied. It is possible to distinguish these different types of constraints or changes through the framework of nonlocal calculus of variations.

For example, if $\Omega_I$ is an empty set, then $\mathcal{L}_{\delta}$ can be well defined all over $\tilde{\Omega} = \Omega$ without the need to impose additional conditions on $u$. In this case, the term $b$ may represent either soft or hard loading with the distinction only showing up through the dependence on the horizon parameter. This case may be viewed as a nonlocal analog of natural boundary conditions for PDEs. In our earlier works, we also have tried to give a more symbolically similar formulation by taking a non-empty $\Omega_I$ in the definitions of the nonlocal operators as that given by (3.13), and imposing the Equation (3.12) on both $\Omega$ that may represent the nonlocal equation, and $\Omega_I$ that may represent the domain of nonlocal constraints or nonlocal boundary conditions.
Again, viewed in terms of \( \bar{\Omega} \), such a separation of the equation domain and the
constraint domain becomes superfluous mathematically, though they can be subject
to different physical interpretations.

On the other hand, \( \Omega_I \) can be a constraint set of nonzero measure (i.e., it is not
a lower-dimensional boundary) on which essential, rather than variational or natural,
conditions on the solutions are imposed. For example, we may impose the condition
\( u = 0 \) on \( \Omega_I \). This is an essential condition that we must build explicitly into the
variational principle. A typical example is
\[
\Omega_I = \Omega_\delta = \{ x \in \Omega^C : \text{dist}(x, \partial \Omega) < \delta \},
\]
which represents a \( \delta \)-layer surrounding \( \Omega \) with \( \partial \Omega \) being its inner boundary.

As seen in later discussions, for interaction kernels that may have stronger singularities
than those with infinite first-order moments, the solutions of nonlocal problems
may enjoy better spatial regularities so that constraints on a lower-dimensional
boundary could be well defined. For such cases, essential conditions on the boundary
can also be imposed, though we do not intend to pursue further discussions here.

**Nonlocal constrained value problem**  Related to Equation (3.12), we consider the
following nonlocal constrained value problem.

\[
\begin{aligned}
L_\delta u(x) &= b(x), & x &\in \Omega, \\
u(x) &= 0, & x &\in \Omega_\delta.
\end{aligned}
\]  

(3.14)

The rest of the discussion provides a systematic study of the well-posedness of
the above problem and it largely follows from the materials presented in [34].

**Energy functional**  A step towards a variational formulation of the nonlocal problem
is to establish the corresponding energy functional corresponding to (3.12) that
can be defined as
\[ E(u) = \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \frac{1}{2} \omega_\delta(y - x) \left( D^* u(x,y) \right)^2 dy dx - \int_{\tilde{\Omega}} b(x) u(x) dx. \] (3.15)

The energy includes the nonlocal elasticity energy and the contribution from the external force \( b = b(x) \).

Another step following the definition of the energy functional is the computation of the energy variation
\[
\frac{d}{dt} E(u + tv) \bigg|_{t=0} = \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \omega_\delta(\phi \mathcal{D} u)(\mathcal{D} v) dy dx - \int_{\tilde{\Omega}} b(x) v(x) dx
\]
\[
= \int_{\tilde{\Omega}} \mathcal{D} \left( \omega_\delta(\phi \mathcal{D} u) \right) v(x) dx - \int_{\tilde{\Omega}} b(x) v(x) dx
\]
\[
= \int_{\tilde{\Omega}} \left( -L_\delta u(x) - b(x) \right) v(x) dx,
\]
where the second equation is obtained formally through the duality or the definition of the nonlocal operators for functions that make the operations well defined. This calculation is reminiscent of that in the principle of virtual works of classical mechanics and it can be used to establish the relation between the strong form and the weak form of the variational problem when the functions are chosen from suitable spaces.

**Assumptions on the nonlocal interaction kernel** The properties of the linear nonlocal model are very much dependent on the nonlocal interaction kernel \( \omega_\delta \) used to define the nonlocal operator. It is therefore helpful to state clearly the type of kernels that are amendable to the nonlocal calculus of variations frameworks. We note that \( \omega_\delta \) is often assumed to satisfy
\[
1. \quad \omega_\delta(r) \geq 0, \quad r \in [0, \delta), \tag{3.16}
\]
\[
2. \quad \text{supp}(\omega_\delta) \subseteq B_\delta(0), \quad \text{i.e.,} \quad \omega_\delta(r) = 0, \quad r \geq \delta, \tag{3.17}
\]
\[
3. \quad \int_{\mathbb{R}^n} \omega_\delta(x) dx = m < \infty \tag{3.18}
\]
where the last normalization condition on \( \omega_\delta \) implies that \( \omega_\delta \) has finite second-order moments (which is necessary for well-defined elastic moduli [40]). The latter is also equivalent to the requirement that the energy is finite for a linear displacement field. A consequence is that for any square integrable vector-valued function that has square integrable first-order partial derivatives, the energy remains finite.

We note that the non-negativity assumption \( \omega_\delta(r) \geq 0 \) can be relaxed [33]. It is also possible to consider interaction kernels that are not radial nor translational invariant. More discussions are given in the next section.

**Solution space** The canonical function space in which to look for a solution of the variational problem is one that includes functions that have well-defined energy and satisfy specified constraints.
For the problem under consideration, we first define the so-called energy space which contains square integrable functions (i.e., functions in $L^2(\tilde{\Omega})$) with a finite energy

$$ V = \{u \in L^2(\tilde{\Omega}) | \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \omega_{\delta}(x-y)(D u)^2 \, dy \, dx < \infty \} . $$

The space $V$ is an inner product space corresponding to the norm

$$ \|u\|_V = \left( \|u\|^2_{L^2(\tilde{\Omega})} + \|u\|_{L^2(\tilde{\Omega})}^2 \right)^{1/2} , $$

where the semi-norm $\|u\|_V$ on $V$ is defined as

$$ \|u\|_V^2 = \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \omega_{\delta}(x-y)(D u)^2 \, dy \, dx . $$

For the nonlocal interaction kernels satisfying the assumptions made earlier, we note that in general $V$ contains the space $H^1(\tilde{\Omega})$, which is the subspace of $L^2(\tilde{\Omega})$ with functions having all of their first-order partial derivatives also in $L^2(\tilde{\Omega})$. Meanwhile, for a kernel $\omega_{\delta}$ that is integrable, it has been shown that $V = L^2(\tilde{\Omega})$.

The solution space of the variational problem is simply a subspace of $V$ with functions satisfying the imposed constraints. That is, for the problem under consideration, we have

$$ V_{c,\delta} = \{u \in V | u = 0 \text{ in } \Omega \} . $$

(3.19)

Various properties of the function spaces $V$ and $V_{c,\delta}$ can be studied in both specific spaces identified with either $L^2$ spaces or fractional Sobolev spaces for specific kernels [15] and general abstract solution spaces [34]. These studies lead to results like

- $V$, equipped with the norm $\|u\|^2_{L^2} + \|u\|_V^2$, is a Hilbert space.

- The embeddings $H^1 \hookrightarrow V \hookrightarrow L^2$ are continuous.

- The solution space $V_{c,\delta}$ is a subspace of $V$ that is closed and dense in $L^2$ (thus separable), containing a dense subspace that is made of all functions in $H^1_0(\Omega)$ with zero extensions on $\Omega$.

Additional properties of $V$ and $V_{c,\delta}$ can be established, such as compact embedding properties and extension theorems [34], which require more technical discussions and are beyond the scope of our presentation here.

We note that the discussion of the function spaces associated with the nonlocal models is not only of interest for mathematical analysis and well-posedness studies but also important for designing numerical approximation schemes, in particular, conforming finite element approximations. For example, when the kernel becomes integrable, then $V = L^2$ so that discontinuous finite elements are in fact conforming approximations. This is in sharp contrast to the conforming approximation of conventional second-order elliptic equations that only allow continuous finite element spaces.
Variational principle  We can now consider the constrained minimization problem
\[
\min_{u \in V_c, \delta} E(u).
\]
Based on the calculation of the variations of the energy functional given earlier, we see that a necessary and sufficient condition for the solution \( u \in V_c, \delta \) of the variational problem, if it exists, is that
\[
\frac{d}{dt} E(u + tv) \bigg|_{t=0} = 0
\]
for any \( v \) also in \( V_c, \delta \). The condition (3.20) is commonly referred to as the Euler–Lagrange equation for the minimization problem.

Bilinear form and weak form  Let us first define
\[
B_\delta(u, v) = \int_{\tilde{\Omega}} \int_{\tilde{\Omega}} \omega_\delta(\|x - y\|) (D^* u)(D^* v) dy dx, \quad (3.21)
\]
which is a symmetric continuous bilinear form defined in \( V \times V \). The symmetry of \( B_\delta(., .) \) means simply that
\[
B_\delta(u, v) = B_\delta(v, u)
\]
for any \( u, v \in V \).

Let \( (., .) \) denote the standard \( L^2 \) inner product on \( \tilde{\Omega} \). The weak form of the variational problem is then given by: for \( b \in V_{c, \delta} \) (the dual space of \( V_{c, \delta} \), which is the space of all bounded linear functionals on \( V_{c, \delta} \)), find \( u \in V_{c, \delta} \) such that
\[
B_\delta(u, v) = (b, v), \quad (3.22)
\]
which is equivalent to (3.20) or rather \( u \) is the minimizer of the energy functional in the solution space \( V_{c, \delta} \). This can also be viewed as a consequence of the nonlocal version of the principle of virtual work.

Let \( \Pi \) denote the set of infinitesimally rigid displacements given by
\[
\Pi = \{ v : v(x) = Qx + p, Q \in \mathbb{R}^d, Q^T = -Q, p \in \mathbb{R}^d \}.
\]
An important property that ensures the unique solvability of the variational problem is that
\[
Z = \{ v : v^2 = B_\delta(v, v) = 0 = \Pi \},
\]
that is, the set \( Z \) contains only rigid body displacement. This particular result has been verified in [14] for integrable kernels and in more general solution spaces for more general kernels [34].

Consequently, we have
\[
V_{c, \delta} \setminus Z = V_{c, \delta} \setminus \Pi = 0. \quad (3.23)
\]
This equation (3.23) is important not only mathematically but also physically as it refers to the fact that there is no nontrivial zero energy mode associated with the energy functional defined by (3.15) in the absence of the external force, that is, if \( b(x) = 0 \) in \( \Omega \).
Nonlocal Poincare's inequality For the well-posedness of the variational problem, we need to show that the bilinear form is coercive in the solution space $V_{c,\delta}$.

Similar to the local counterpart for PDEs, the coercivity of the bilinear form is a consequence of a nonlocal Poincare's inequality. That is, there is a constant $c > 0$, such that for any $v \in V_{c,\delta}$,

$$v \rightarrow c \|v\|_{L^2}.$$  \hspace{1cm} (3.24)

Inequalities like (3.24) have been established for both scalar models and systems with various kernels in, for example, [20, 13, 15, 33]. In general, one can first establish the result for kernels that are integrable, that is, kernels like $\omega_{\delta}(x \ x) = |x' - x|^2 \omega_{\delta}(x \ x)$, and then get the desired inequality for $\omega_{\delta}$ itself by noticing that

$$4 \text{diam}^2(\Omega) \omega_{\delta}(x \ x) \geq \omega_{\delta}(x \ x)$$

where $\text{diam}(\Omega)$ denotes the diameter of the domain $\Omega$.

For an integrable kernel $\omega_{\delta}(x \ x)$, the nonlocal Poincare's inequality can be established by utilizing certain compactness of sequence of functions in $L^2$ with unit $L^2$ norms but vanishing semi-norms. Such a sequence must have a weakly convergent subsequence in $L^2$, with the weak limit in the kernel space $Z$. This in turn implies that the weak limit has to be 0 by (3.23).

Given that $\omega_{\delta}$ is integrable and that convolution operators associated with such kernels are compact, the vanishing seminorm in $v$ then implies the convergence in $L^2$ to 0 with the help of the fact

$$\int_{\Omega} \int_{\Omega} \omega_{\delta}(x \ x) dx \ dx \beta > 0$$

for some positive constant $\beta > 0$. The convergence of a subsequence in $L^2$ norm to 0 contradicts with the fact that all the functions in the sequence have unit $L^2$ norms. This contradiction shows the validity of the nonlocal Poincare's inequality. We refer to [32, 34] for more detailed discussions.

The nonlocal Poincare's inequality implies that the semi-norm $v$ defines a norm on the solution space $V_{c,\delta}$ and the bilinear form $B_{\delta}(\ , \ )$ is coercive, that is

$$B_{\delta}(v, v) \geq \frac{c^2}{1 + c_2} \left( v^2_{L^2} + v^2_{L^2} \right) v \in V_{c,\delta}.$$  \hspace{1cm} (3.25)

In addition, we have that the symmetric bilinear form $B_{\delta}(u, v)$ defines an equivalent inner product on the solution space.

Well-posedness of the variational problem By the properties established for the function spaces and the bilinear form, we are led to the well-posedness of the variational problem or rather the existence, uniqueness, and stability of the weak solution in the sense that for any given $b \in V_{c,\delta}$, there exists a unique solution $u \in V_{c,\delta}$, such that

$$B_{\delta}(u, v) = (b, v) v \in V_{c,\delta},$$
and
\[ \mathbf{u}_v \cdot \mathbf{b} \cdot \]
Consequently, by the nonlocal Poincare’s inequality (3.24), we have
\[ \mathbf{u}_v = (\mathbf{u}^2 + \mathbf{u}^2_{L^2})^{1/2} \frac{1 + c^2}{c} \mathbf{b} \]
with \( \cdot \) being the norm in the dual space \( V_{c, \delta} \).

The above well-posedness result also implies that the nonlocal operator \( L_{\delta} \) is a continuous and self-adjoint operator from \( V_{c, \delta} \) to its dual space with a bounded inverse.

The well-posedness also provides support to the formal derivation of the nonlocal Green identity and the calculation of the variations of the energy that can in fact be rigorous for functions in the desired function spaces. We refer to [33, 35] for more details on suitable interpretations of the operators and identities especially for cases invoking non-integrable interaction kernels.

Note that in the special case that we have an integrable nonlocal interaction kernel, the space \( V \) and its dual \( V^* \) are both equivalent to the conventional Hilbert space \( L^2(\tilde{\Omega}) \). In this case, the weak solutions are also functions in the same space so that the nonlocal equation holds also in \( L^2 \). For nonlocal interaction kernels that are not integrable, the fact that \( \mathbf{b} \in V_{c, \delta} \) can belong to a weaker space than \( L^2 \) implies that it is possible to have the weak solution of the nonlocal model even for external forces that are not integrable functions. Moreover, the solutions are expected to have better regularities than the data in such a context, a property that is common for standard elliptic PDEs, but not shared by the case associated with an integrable kernel [20, 13].

3.5 Nonlocal calculus of variations — further discussions

The framework outlined in the above section is presented for a linear steady-state bond-based peridynamic system. Yet, as demonstrated in [33, 34, 35, 36], various extensions can be made.

For example, well-posedness has been shown for linear scalar equations with kernels with changing signs [33]. In [34], the theory has been applied to perturbations of the linear bond-based models that allow both repulsive and attractive nonlocal interactions. Results have also been presented on the well-posedness of variational problems involving nonlinear perturbations, that is, the energy functional can take on the form
\[ E(u) = \int_\Omega \int_\tilde{\Omega} \frac{1}{2} \omega_\delta(\mathbf{y} - \mathbf{x})(\nabla \mathbf{u}(\mathbf{y},\mathbf{x}))^2 d\mathbf{y} d\mathbf{x} + \int_\Omega \Psi(\mathbf{u})(\mathbf{x}) d\mathbf{x} \int_\Omega \mathbf{b}(\mathbf{x}) \mathbf{u}(\mathbf{x}) d\mathbf{x}, \]
which contains an additional term in comparison to (3.15). In [34], the cases where
\( \Psi = \Psi(u) \) is either a convex map or a more general continuous map are considered. For the latter case, extra conditions on the \( \omega_\delta \) are imposed.

The theory or nonlocal calculus of variations is also applicable to the more general state-based systems. A convenient reformulation of the nonlocal energy in this case is given by [14, 35]

\[
E(u) = \int_\Omega \int_\Omega \frac{\eta}{2} \omega_\delta( y \cdot x ) ( \mathcal{D} u(x,y) \cdot \mathcal{D} u(x,y) )^2 \, dy \, dx + \int_\Omega \frac{\tau}{2} ( \mathcal{D}_\omega u(x) )^2 \, dx \int_\Omega b(x) u(x) \, dx.
\]

By computing the variations of the above energy, we can get the linear peridynamic Navier operator given in (3.9), which also matches with the nonlocal operator in (3.1) with a tensor \( C \) given in (3.3).

The study in [35] has shown, in particular, that the same energy space can be used for both bond-based and state-based systems associated with the same nonlocal interaction kernel. In other words, the spaces \( V \) and \( V_{c,\delta} \) defined in the previous section can be used to define energy and solution spaces for the linear state-based system as well.

In a recent study [36], we have also established some basic theory for nonlinear variational problems associated with the following more general energy functional:

\[
E(u) = \int_\Omega \int_\Omega \omega_\delta( y \cdot x ) \zeta ( \mathcal{D} u(x,y) \cdot \mathcal{D} u(x,y) )^2 \, dy \, dx + \int_\Omega \Phi ( \mathcal{D}_\omega u(x) ) \, dx \int_\Omega b(x) u(x) \, dx,
\]

where \( \zeta \) and \( \Phi \) are functions that are assumed to have suitable growth and convexity properties. The results cover variational problems associated with a variety of volumetric conditions.

Time-dependent peridynamic models are of greater interests in practice. Naturally, studies of steady-state models and nonlocal operators lead also to results on the well-posedness of time-dependent peridynamic equation of motion [33] such as the one given by

\[
\rho u_{tt}(x,t) + B(t) u_t(x,t) + \mathcal{L}_\delta u(x,t) = b(x,t), \quad (0, T), x \quad \Omega \quad (3.26)
\]

with initial value \( u( ., 0) = u_0 \) and \( u_t( ., 0) = v_0 \) and nonlocal constraint \( u( ., t) = 0 \) on \( \Omega_t \). Here, \( \rho \) is mass density and for each \( t \in [0, T] \) and \( B(t) \) is a bounded linear operator on \( L^2(\Omega) \) that represents the damping effect. An example of such an operator used is

\[
B(t) w(x) = \int_\Omega \frac{h(x \cdot x)}{x \cdot x} (w(x) \cdot w(x)) \, dx
\]

which appears in [11]. Another form of \( B(t) \) would be to take the local operator

\[
B(t) w(x) = \beta(x,t) w(x)
\]
for a smooth and nonnegative function $\beta$. The existence, uniqueness, and continuous dependence of the time-dependent solution to (3.26) are established for initial data $u_0$ in the constrained solution space similar to $V_{\epsilon,\beta}$ and $v_0 \in L^2$.

Similar analysis can also be applied to recently studied models of nonlocal diffusion and convection presented in [16] where the relation between time-dependent nonlocal models and stochastic jump processes has also been explored.

### 3.6 Summary

With increasing attention being given to peridynamics in materials modeling and simulations, the development of rigorous mathematical theory becomes more and more important. Peridynamics is proposed as an alternative to classical PDE-based continuum models [40]. By avoiding the explicit use of spatial derivatives, it adopts an integral formulation to incorporate nonlocal interactions and to allow singular solutions that potentially can describe the morphology of cracks and fractures more effectively. These advantages in modeling come with some added cost: nonlocal interactions are more difficult to efficiently evaluate and singular solutions are harder to resolve numerically. Given such complexities, it is essential to offer sound verification and validation of modeling and simulations based on peridynamics. Rigorous mathematical analysis, in combination with physical experiments, can play a central role in verification and validation processes.

In recent years, new mathematical results about peridynamics have started to appear [18]. A systematic and rigorous mathematical framework for nonlocal models is also emerging. This chapter presented an overview of the well-posedness study of peridynamic equations, mostly through the framework of nonlocal calculus of variations developed for nonlocal models [13, 14, 33, 34, 35, 36]. Such a framework was built upon the basic elements of the nonlocal vector calculus [13, 15]. A step-by-step illustration was provided here for a variational problem associated with a linear bond-based peridynamic model with minimal background in advanced mathematics.

The development of mathematical theory for peridynamics is related to many studies of other nonlocal models. One may consult, for example, additional references on nonlocal diffusion [3], papers on nonlocal operators in image analysis [7, 25, 28], fractional PDEs [9, 12, 31, 29, 30], nonlocal dispersal [27], and stochastic processes [4, 8]. The basic tool sets developed for the mathematical analysis of peridynamics will likely find broader applications.

There are obviously many more mathematical questions than what have been discussed here. Going back to the questions raised in the introduction section, we see that our focus has been on addressing the first three questions. Nonlocal calculus of variations provides us a rigorous basis with which to study the mathematical properties of simplified linearized models and more general nonlinear peridynamic models. Besides the application to well-posedness theory, it can also be applied to study many other interesting mathematical questions. For example, the framework...
can help us explore the consistency between the nonlocal models and their local limits when the latter are valid. This addresses the fourth question raised in the introduction, which we will revisit in Chapter 4. Another important application is to analyze numerical approximations of peridynamic models such as the Asymptotically Compatible schemes developed in [45] and [46]. Detailed discussions are also given in Chapter 4. Concerning the last question posed in the introduction, we note that characterizing the regularity properties of solutions to peridynamic models is a very important and challenging task. Other than the minimal regularity guaranteed by the well-posedness theory, we do not address the subject here. More future research efforts in this direction are naturally called for. Coupling the analysis closely with constitutive modeling involving nonlinear interactions should also be given high priority.

In summary, deeper mathematical understanding of peridynamic models in their full generality and in broad applications is very much in need. It is important to have modelers, simulation code developers, mathematicians, and experimentalists working together to formulate the important mathematical questions and find their effective solutions.

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