Handbook of Peridynamic Modeling

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Peridynamic Modeling of Fiber-reinforced Composites

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Chapter 12

Peridynamic Modeling of Fiber-reinforced Composites

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12.1 Introduction

Utilization of composite materials is increasing in many different industries, including aerospace, automotive, and marine industries, due to their superiority of specific strength and stiffness properties over those of metals. However, damage initiation and its subsequent propagation in composites are not understood as clearly as they are, for example, for metals because of the presence of stiff fibers embedded into a soft matrix.
material, causing inhomogeneity. In order to better understand failure mechanisms, Hallett and Wisnom [9] conducted experiments on double-edge-notched composite specimens made from E-glass. They reported the occurrence of matrix cracking before the ultimate failure for all specimens representing four different lay-ups when loaded in tension. Furthermore, it was reported that fiber failure initiated at the notch tip. Later, Green et al. [8] investigated the effect of scaling on the tensile strength of notched composites made from unidirectional carbon-fiber/epoxy pre-preg by considering hole diameter and laminate thickness as independent variables. These experiments showed that failure mechanisms in composites are very complex due to matrix cracking, fiber breakage, and delamination.

In order to investigate the behavior of cracks, Wu [19] considered unidirectional fiberglass-reinforced Scotch-plies with center cracks oriented in the direction of the fibers. The plies had fibers in the $0^\circ$ and $45^\circ$ directions and were loaded in tension, pure shear, and combined tension and shear. In all three types of loading, it was observed that the crack propagated in a direction co-linear with the initial crack.

It is, therefore, evident that the inhomogeneous nature of the composites must be retained in the analysis to predict the correct failure modes. Peridynamics offers the ability to predict complex failure patterns in composite structures. The peridynamic (PD) theory was applied successfully by Colavito et al. [5, 6] to predict damage in laminated composites subjected to low-velocity impact and static indentation. Askari et al. [2] and Xu et al. [20] also used peridynamic simulations to predict damage in laminates subjected to low-velocity impact and for notched laminated composites under biaxial loads. Kilic et al. [13] predicted the basic failure modes of fiber breakage, matrix cracking, and delamination in various laminates with a pre-existing central crack under tension by considering the matrix and fiber materials independently. As an application study, Oterkus et al. [15] demonstrated that PD analysis is capable of capturing bearing and shear-out failure modes in bolted composite lap-joints. More recently, Askari et al. [3] considered the effect of both high- and low-energy hail impacts against a toughened-epoxy, intermediate-modulus, carbon-fiber composite. Also, Hu et al. [10] predicted the basic failure modes of fiber breakage, matrix cracking, and delamination in laminates with a pre-existing central crack under tension. A multi-scale approach for modeling of composites by linking of micro- and macroscales was described by Alali and Lipton [1]. The analytical derivation of the PD material parameters, including thermal loading conditions, was recently given by Oterkus and Madenci [16]. They also demonstrated the constraints on material constants due to the assumption of pairwise interactions in bond-based peridynamics. An ordinary state-based formulation that overcomes these constraints was presented by Madenci and Oterkus [14].

This chapter presents a peridynamic formulation of fiber-reinforced composite materials subjected to mechanical and thermal loading conditions. The PD approach to model a lamina is first validated against analytical solutions by considering uniaxial tension and uniform temperature change. Then, damage growth patterns from a pre-existing crack in a lamina for different fiber orientations are computed and compared against experimental observations. This approach is further extended to analyze composite laminates and to predict damage growth patterns from a pre-existing...
crack in two distinct laminate constructions under tension. In the absence of a crack, the PD displacement predictions are compared against those of the classical laminate theory. In the presence of a crack, damage patterns are qualitatively compared against experimental observations.

12.2 Peridynamic analysis of a lamina

Silling [18] reformulated the continuum mechanics equations and replaced the divergence term in the classical equation of motion with an integral term, which makes the new form of the equation of motion applicable whether or not a discontinuity exists in the structure:

\[ \rho (\mathbf{x}) \ddot{\mathbf{u}}(\mathbf{x}, t) = \int_{H} f(\mathbf{x}, \mathbf{x}, \mathbf{u}, \mathbf{u}) \, dH + b(\mathbf{x}, t). \]  

(12.1)

In Eq. (12.1), the horizon, \( H \), includes all the material points that the material point \( \mathbf{x} \) can interact with inside the body. The interaction force or peridynamic force between material points \( \mathbf{x} \) and \( \mathbf{x}' \) can be expressed as \( f(\mathbf{x}, \mathbf{x}, \mathbf{u}, \mathbf{u}) \), and it is a function of the relative position vector, \( \mathbf{x} - \mathbf{x}' \), and relative displacement vector, \( \mathbf{u} - \mathbf{u}' \). The peridynamic force is along the same direction of the relative position of these material points in the deformed configuration, i.e., \( \mathbf{y} = (\mathbf{x} + \mathbf{u}) \) \( (\mathbf{x} + \mathbf{u}) \). For a fiber-reinforced composite lamina, the peridynamic force can be expressed as

\[ \mathbf{f} = c (s, s) \frac{\mathbf{y}}{\mathbf{y}} \frac{\mathbf{y}}{\mathbf{y}} = c \bar{s} \frac{\mathbf{y}}{\mathbf{y}} \frac{\mathbf{y}}{\mathbf{y}}, \]  

(12.2)

where \( c, s, \bar{s}, \) and \( s \) represent the peridynamic material parameter, total stretch, mechanical stretch, and thermal stretch between material points \( \mathbf{x} \) and \( \mathbf{x}' \), respectively. The total stretch, \( s \), and the thermal stretch due to thermal loading, \( s' \), are defined as

\[ s = \frac{\mathbf{y}}{\mathbf{x}} \frac{\mathbf{x}}{\mathbf{y}} \frac{\mathbf{x}}{\mathbf{x}} \]  

(12.3)

and

\[ s = \alpha_{PD} \Delta T, \]  

(12.4)

where \( \alpha_{PD} \) and \( \Delta T \) represent the peridynamic coefficient of thermal expansion and the temperature change, respectively. By using Eqs. (12.3) and (12.4), the mechanical stretch, \( \bar{s} \), can be computed as

\[ \bar{s} = s - s' = \frac{\mathbf{y}}{\mathbf{x}} \frac{\mathbf{x}}{\mathbf{x}} \frac{\mathbf{x}}{\mathbf{x}} \alpha_{PD} \Delta T. \]  

(12.5)

Since the fiber-reinforced composite lamina is an anisotropic material, the directional dependency must be included in the PD formulation. In order to represent
directional dependency, two different PD material parameters can be introduced for a fiber-reinforced composite lamina with a fiber orientation of \( \theta \), as shown in Fig. 12.1. In this figure, the material point \( q \), which is a family member of the material point of interest \( i \), represents material points that interact with material point \( i \) only along the fiber direction. However, the material point \( p \), which is also a family member of the material point \( i \), represents material points that interact with material point \( i \) in any arbitrary direction, including the fiber direction.

Figure 12.1: PD horizon (interaction domain) of material point \( i \) for a lamina with a fiber orientation of \( \theta \) and PD bonds between material point \( i \) and other material points within its horizon.

The peridynamic parameters concerning the interaction of material points in the fiber direction only and in any arbitrary direction can be defined as \( c_F \) and \( c_A \), respectively. Extending the procedure introduced by Gerstle et al. [7] for isotropic materials, the PD parameters \( c_F \) and \( c_A \) can be expressed in terms of the engineering material constants \( E_1, E_2, G_{12}, \) and \( \nu_{12} \) by equating strain energy densities of a material point based on the classical continuum mechanics and PD theory for simple loading conditions.

Based on the classical continuum mechanics, the strain energy density of a material point, \( W^{CM} \), for a two-dimensional composite lamina is expressed as

\[
W^{CM} = \frac{1}{2} \sigma^T (\varepsilon - \varepsilon^*),
\]  

(12.6)

in which the stress, \( \sigma \), total strain, \( \varepsilon \), and thermal strain vectors, \( \varepsilon^* \), are defined as

\[
\sigma^T = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{12} \end{bmatrix},
\]  

(12.7)

\[
\varepsilon^T = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \gamma_{12} \end{bmatrix},
\]  

(12.8)
The stress and strain components given in Eq. (12.7)–(12.9) are defined with respect to the principal (material) coordinate system, \((x_1, x_2)\). For a composite lamina with a fiber orientation of \(\theta\), the stress and strain components are related through the constitutive relation as

\[
\sigma = \mathbf{Q} (\varepsilon - \varepsilon^*) ,
\]

in which the stiffness matrix \(\mathbf{Q}\) is defined as

\[
\mathbf{Q} = \begin{bmatrix}
    Q_{11} & Q_{12} & 0 \\
    Q_{12} & Q_{22} & 0 \\
    0 & 0 & Q_{66}
\end{bmatrix}
\]

(12.11)

where

\[
Q_{11} = \frac{E_{11}}{1 - \nu_{12} \nu_{21}} , \quad Q_{12} = \frac{\nu_{12} E_{22}}{1 - \nu_{12} \nu_{21}} , \quad Q_{22} = \frac{E_{22}}{1 - \nu_{12} \nu_{21}} , \quad Q_{66} = G_{12}
\]

(12.12)

with \(\nu_{12}/E_{11} = \nu_{21}/E_{22}\). Four independent material constants in Eq. (12.12) correspond to the elastic modulus in the fiber direction, \(E_{11}\), elastic modulus in the transverse direction, \(E_{22}\), the in-plane shear modulus, \(G_{12}\), and the in-plane Poisson’s ratio, \(\nu_{12}\). Furthermore, the thermal strain vector can be expressed in terms of thermal expansion coefficients in the fiber and transverse directions, \(\alpha_{11}\) and \(\alpha_{22}\), respectively, for a temperature change, \(\Delta T\), as

\[
\varepsilon^* T = \{ \alpha_{11} \Delta T \quad \alpha_{22} \Delta T \quad 0 \} .
\]

(12.13)

Alternatively, the strain energy density of the same material point in PD theory, \(W^{PD}\), can be obtained by integrating the micropotential of each PD interaction,

\[
W^{PD} = \frac{1}{2} \sum_{q=1}^{Q} w_q V_q + \frac{1}{2} \int w dH,
\]

(12.14)

in which \(Q\) is the number of fiber bonds within the horizon of material point \(i\). The micropotential of each PD interaction can be expressed as

\[
w = \frac{1}{2} c(\phi) \xi^2(\phi) \xi.
\]

(12.15)

Note that the stretch of the bond between the material points \(i\) and \(p\), \(\xi(\phi)\), is referenced to a polar coordinate system \((\xi, \phi)\) (see Fig. 12.2). Furthermore, for each peridynamic interaction, the peridynamic parameter \(c(\phi)\) can be defined as

\[
c(\phi) = \left\{ \begin{array}{ll}
    c_F + c_A & \phi = 0 \\
    c_A & \phi = 0
\end{array} \right.
\]

(12.16)
The peridynamic coefficient of thermal expansion, $\alpha_{PD} (\phi)$, which is used to calculate the thermal stretch term for a lamina, is dependent on the bond orientation between material points $i$ and $p$. It can be expressed in terms of the coefficients of thermal expansion, $\alpha_{11}$ and $\alpha_{22}$, in the form

$$\alpha_{PD} (\phi) = \alpha_{11} \cos^2 (\phi) + \alpha_{22} \sin^2 (\phi). \quad (12.17)$$

After substituting Eq. (12.15) into Eq. (12.14), Eq. (12.14) can be rewritten as

$$W^{PD} = \frac{1}{2} \sum_{q=1}^{Q} \frac{c_F \bar{s}^2 q_i \bar{s} q_i}{2} V_q + \frac{1}{2} \int_{H} \frac{c_A \bar{s}^2 \bar{\xi}}{2} \, dH, \quad (12.18)$$

in which $Q$ is the number of fiber bonds within the horizon of material point $i$. The initial length of the bond in the fiber direction and its mechanical stretch after deformation between material points $q$ and $i$ are denoted by $\bar{s} q_i$ and $\bar{s} q_i$, respectively. The volume of the material point $q$ that interacts with material point $i$ is denoted by $V_q$ and can be approximately calculated as

$$V_q = \frac{\pi h \delta^2}{N}, \quad (12.19)$$

in which $N$ is the number of material points within the horizon of material point $i$, and $h$ is the thickness of the lamina.

In order to determine the bond constants $c_F$ and $c_A$ in terms of the engineering constants $E_{11}$, $E_{22}$, $G_{12}$, and $\nu_{12}$, four different simple loading conditions can be considered, as explained in Appendix A. Equating the strain energy densities from PD theory and classical continuum mechanics for these loading conditions results in explicit expressions for $c_F$ and $c_A$ as

$$c_F = \frac{2 E_1 (E_1 - E_2)}{(E_1 + \frac{1}{3} E_2) \left( \sum_{q=1}^{Q} \bar{s} q_i V_q \right)} \quad (12.20)$$

Figure 12.2: Components of the initial bond length between material points $i$ and $p$. 

Diagram illustrating the components of the initial bond length between material points $i$ and $p$. The peridynamic coefficient of thermal expansion, $\alpha_{PD} (\phi)$, is shown as a function of bond orientation $\phi$. The components are labeled with $\xi_1$, $\xi_2$, and $\xi_3$.
and
\[ c_A = \frac{8E_1E_2}{(E_1 - \frac{1}{9}E_2)\pi h\delta^3}, \]  
(12.21)
along with constraints on material constants \( G_{12} \) and \( v_{12} \) as
\[ G_{12} = \frac{v_1E_2}{1 - v_{12}v_2} = \frac{E_1E_2}{3(E_1 - \frac{1}{9}E_2)} \]
(12.22)
and
\[ v_{12} = \frac{1}{3}. \]
(12.23)

As presented in Oterkus et al. [17], the constitutive or force-stretch relationships for the in-plane interactions (bonds) of two material points are shown in Fig. 12.3. The critical parameters that define the failure of these bonds under tension and compression are \((s_{Ft}, s_{At})\) and \((s_{Fc}, s_{Ac})\), respectively, which can be determined based on the experimental measurements. At least four different uniaxial tension test cases of an unnotched laminate should be performed by considering different lay-up configurations. As shown in Fig. 12.4, these four lay-up configurations can be chosen as \((25/0/75)\), \((50/0/50)\), \((75/0/25)\), and \((100/0/0)\), where each value in the lay-up configuration indicates the percentage of \(0^\circ\), \(\pm 45^\circ\), and \(90^\circ\) plies in the laminate, respectively. Tensile failure strain of the \((100/0/0)\) laminate corresponds to the critical stretch of a bond in fiber direction under tension, \( s_{Ft} \). Linear extrapolation of the tensile failure strains for different lay-up configurations leads to the failure strain for the \((0/0/100)\) laminate, i.e., the critical stretch of a bond in arbitrary direction under tension, \( s_{At} \).

Figure 12.3: Force-stretch relation for PD bonds in a lamina.

The dominant failure mechanism for the fiber-reinforced composites under compression loading is microbuckling. Therefore, a simultaneous failure of bonds in fiber
and arbitrary directions can be assumed under compression loading. Hence, a uniaxial compression test for an unnotched quasi-isotropic laminate with a (25/50/25) lay-up can be performed to determine the critical stretch of bonds in fiber and arbitrary directions under compression, \( s_{F_t} \) and \( s_{A_t} \), respectively.

Termination of the interaction between material points can be associated with the failure of material by modifying the peridynamic force relation given in Eq. (12.2) by introducing the failure parameter \( \mu(x, x', t) \):

\[
f = \mu(x, x', t) c \left( s - s^* \right) \frac{y - y'}{y} \cdot \frac{y - y'}{|y - y'|}, \tag{12.24}
\]

where the failure parameter can be defined as

\[
\mu(x, x', t) = \begin{cases} 
1 & \text{if } s(x, x', t) < s_c \text{ for all } 0 < t < t_0 \\
0 & \text{otherwise} 
\end{cases} \tag{12.25}
\]

The inexplicit nature of local damage at a material point, \( x \), arising from the introduction of failure in the constitutive model, is removed by defining the local damage as

\[
D(x, t) = 1 - \frac{\int_H \mu(x, x', t) dH}{\int_H dH}. \tag{12.26}
\]

Surface correction is an important concept in PD theory. The peridynamic parameters given in Eqs. (12.20) and (12.21) are derived under the assumption that the material point located at \( x \) is in a single material with its complete neighborhood entirely embedded within its horizon. However, this assumption becomes invalid when the material point is close to free surfaces (Fig. 12.5). It results in a reduction in material stiffness near the free surfaces, and this stiffness reduction must be corrected. After determining the surface correction factor for each bond, the PD force in that bond is modified based on the associated surface correction factor. A detailed derivation of the surface correction factors is given in Appendix B.

Figure 12.4: Determination of critical stretch values \( s_{F_t} \) and \( s_{A_t} \).
12.3 Peridynamic analysis of a laminate

The PD formulation for a composite lamina can be extended to consider a composite laminate. In order to capture the deformation behavior of a laminate in the thickness direction and define the interaction between neighboring plies, two additional bond constants between neighboring plies are introduced, as shown in Fig. 12.6.

Transverse normal and shear deformations between material points located on adjacent (neighboring) layers are related through the bond constants $c_N$ and $c_S$, respectively. As shown in Fig. 12.6, interlayer bonds only exist in the normal direction, whereas shear bonds exist in all directions between the neighboring plies. Hence, a
material point can interact with two other material points via interlayer bonds that have the same in-plane coordinates.

As in the case of in-plane deformation in the case of a lamina, the peridynamic parameters associated with transverse normal and shear deformations, $c_N$ and $c_S$, can be derived in the form

$$c_N = \frac{E_m}{h \bar{V}}$$

and

$$c_S = \frac{2G_m}{\pi h} \left( \frac{1}{6^2 + h^2 \ln \left( \frac{h^2}{\delta^2 + h^2} \right)} \right),$$

(12.28)

where $E_m$ and $G_m$ are the elastic and shear moduli of the matrix material, respectively, and $\bar{V}$ is the volume of a material point. Detailed derivations of these expressions are given in Appendix C.

Note that the shear bonds have a different characteristic than bonds in fiber and arbitrary directions, and interlayer bonds, because the shear bond constant relates the body force density, $f$, to the change in angle of the bond from its original orientation (shear angle), $\varphi$. Therefore, the force density and micropotential expressions for a shear bond can be written as

$$f = c_S \varphi \left( \Delta x \right)^2 \frac{y}{y} - \frac{y}{y}$$

(12.29)

and

$$w = \frac{1}{2} c_S \varphi^2,$$

(12.30)

where $(\Delta x)$ is the spacing between material points on the plane of the lamina.

Failures of the interlayer and shear bonds correspond to mode-I and mode-II, respectively. Interlayer damage represents the breakage of (interlayer) bonds between a layer and its adjacent layers above and below. Hence, it provides the extent of delamination between the adjacent layers. Therefore, the interlayer bonds are assumed to fail only in tension. The critical stretch value for the interlayer bonds, $s_{Nc}$, can be obtained analytically by equating the energy consumed by an advancing mode-I crack to the work required to break all interlayer bonds as

$$s_{Nc} = \sqrt{\frac{2G_{IC}}{hE_m}},$$

(12.31)

where $G_{IC}$ is the mode-I critical energy release rate of the matrix material.

The shear bonds can fail if the shear angle of the bonds exceeds the critical shear angle value, $\varphi_{Sc}$. It can also be obtained analytically by equating the energy consumed by an advancing Mode-II crack to the work required to break all shear bonds as

$$\varphi_{Sc} = \sqrt{\frac{G_{IIc}}{hG_m}},$$

(12.32)
where $G_{IIc}$ is the mode-II critical energy release rate of the matrix material.

Derivations of the relationships between the critical stretch value for the interlayer bonds, $s_{Nc}$, and the mode-I critical energy release rate, and between the critical shear angle value, $\phi_{Sc}$, and the mode-II critical energy release rate, respectively, are given in Appendix D.

### 12.4 Numerical results

**A lamina under uniaxial tension and uniform temperature change.** A unidirectional thin lamina with a fiber orientation of $\theta = 0$ is considered, as shown in Fig. 12.7. The length and width of the lamina are specified as $L = 152.4$ mm and $W = 76.2$ mm, respectively. It has a thickness of $h = 0.1651$ mm. Its elastic moduli in the fiber and transverse directions are $E_{11} = 159.96$ GPa and $E_{22} = 8.96$ GPa, respectively. The thermal expansion coefficients in the fiber and transverse directions are $\alpha_{11} = -1.52$ ppm $/ ^\circ$C and $\alpha_{22} = 34.3$ ppm $/ ^\circ$C, respectively. The PD model is generated using a single layer of material points with a grid size of $\Delta x = 6.35 \times 10^{-4}$ m. The horizon radius is specified as $\delta = 3.015 \Delta x$. The peridynamic parameters are computed using Eqs. (12.20) and (12.21) as $c_F = 5.72 \times 10^{23}$ N/m$^6$ and $c_A = 1.86 \times 10^{22}$ N/m$^6$. The quasi-static solution is obtained using the adaptive dynamic relaxation technique with a time increment of 1 and a stable mass density value of $7.005 \times 10^{18}$ kg/m$^3$ [12]. Failure is not allowed in order to verify the solution against analytical predictions based on classical continuum mechanics.

![Figure 12.7: Loading and geometry of the unidirectional lamina under uniaxial tension and uniform temperature change.](image)

First, a uniaxial tension loading of $P = 159.96$ MPa is applied as a body load of $b_x = 5.95 \times 10^{18}$ N/m$^3$ along the edges of the lamina through a volumetric region with a depth of $b = 2.54 \times 10^{-3}$ m. The variation of the horizontal and vertical displacement components along the central axes in the $x$- and $y$-directions, respectively,
is computed at the end of 8000 time steps and compared against analytical results, as shown in Figs. 12.8 and 12.9. Analytical results are computed by using the relations

\[ u_x = \frac{P}{E_1} x \]  
(12.33)

and

\[ u_y = \nu_{12} \frac{P}{E_1} y. \]  
(12.34)

For both displacement components, there is a remarkable agreement between analytical and PD results.

Figure 12.8: Horizontal displacement along the central axis at the end of 8000 time steps.

Next, the lamina is only subjected to a uniform temperature change of \( \Delta T = 50 \) °C. For this loading condition, the analytical horizontal and vertical displacements along the central axes are computed by

\[ u_x = \alpha_{11} \Delta T x \]  
(12.35)

and

\[ u_y = \alpha_{22} \Delta T y. \]  
(12.36)

Comparisons of horizontal and vertical displacements obtained analytically and from PD analysis, shown in Figs. 12.10 and 12.11, indicate a remarkably close agreement.
Laminates under uniaxial tension. The validation is continued by considering two different three-ply laminates with stacking sequences of \([0/90/0]\) and \([0/45/0]\) subjected to uniform tension loading, as shown in Fig. 12.12. The geometrical and material properties are the same as those of the lamina. The interlayer and shear bond constants are computed by using Eqs. (12.27) and (12.28) as \(c_N = 3.45 \times 10^{23} \text{N/m}^6\) and \(c_S = 1.55 \times 10^{18} \text{N/m}^5\), respectively. The uniaxial tension loading is applied as a body load of \(b_x = 5.95 \times 10^{-10} \text{N/m}^3\) through a volumetric region with a depth of \(b = 2.54 \times 10^{-3} \text{m}\). It corresponds to a stress resultant value of \(N = 79228.2 \text{N/m}\) along the edges of the laminate. During the process, failure is not allowed in order to compare the PD solution to the analytical solution based on the classical laminate theory.

The variation of the horizontal and vertical displacement components along the central axes in the \(x\)- and \(y\)-directions, respectively, for the \(90^\circ\) ply of the \([0/90/0]\) lay-up are computed at the end of 8000 time steps and compared against analytical results, as shown in Figs. 12.13 and 12.14. Comparisons of the displacement components in the \(45^\circ\) ply of the \([0/45/0]\) lay-up are shown in Figs. 12.15 and 12.16. For both laminates, the agreement between the analytical and PD displacements is remarkably close.

A lamina with a pre-existing central crack under tension. In order to demonstrate the failure prediction capability of the PD approach, the same lamina with a pre-existing central crack is considered for three different fiber orientations, \(\theta = 0\), \(90\), and \(45\). As shown in Fig. 12.17, the crack is aligned with the \(y\)-axis and has a length of \(2a = 0.01778 \text{m}\). The lamina is subjected to a velocity boundary condition of \(v_0 = 2.02 \times 10^{-7} \text{m/s}\) along the edges of the lamina through a volumetric region with a depth of \(b = 2.54 \times 10^{-3} \text{m}\). The failure is only allowed in tension for bonds in fiber and arbitrary directions. The critical stretch for the bond in arbitrary direction is specified as \(s_{At} = 0.0135\). For the bond in fiber direction, it is assumed that
Figure 12.10: Variation of horizontal displacement along the central axis at the end of 8000 time steps when no failure is allowed.

Figure 12.11: Variation of vertical displacement along the central axis at the end of 8000 time steps when no failure is allowed.

Figure 12.12: Loading and geometry of a composite laminate under uniaxial tension.
its critical stretch value is twice the critical stretch for the bond in arbitrary direction, i.e., $s_{F_1} = 0.027$.

As shown in Fig. 12.18, the crack propagates in the fiber direction, referred to as the splitting mode in all cases. Similar experimental observations confirm that the current PD model accurately captures the failure modes.

**Laminates with a pre-existing central crack under tension.** The failure prediction capability of the PD theory is further demonstrated by introducing a central crack in the two laminate lay-ups of $[0/90/0]$ and $[0/45/0]$. As shown in Fig. 12.19, the crack is aligned with the $y$-axis and has a length of $2a = 0.01778$ m. The laminates are subjected to a velocity boundary condition of $v_0 = 2.02 \times 10^{-7}$ m/s. The critical stretch parameters specified for the bonds in fiber and arbitrary directions are $s_{F_1} = 0.027$ and $s_{A_1} = 0.0135$, respectively. The critical stretch and angle parameters for the interlayer and shear bonds are specified as $s_{N_1} = \phi_{S_1} = 0.0135$. The procedure for computing these critical stretch and angle values is explained in Appendix D.

For the $[0/90/0]$ laminate, an “H” type splitting failure mode is observed for all plies, as shown in Fig. 12.20. In this case, $0$ plies are dominant in the loading direction; therefore, $0$ plies determine the failure behavior of the laminate. However, in the $[0/45/0]$ laminate, a “Z” type failure mode is obtained in all plies due to the presence of a $45$ ply, as shown in Fig. 12.21. Also observed is the delamination failure mode between the plies due to the breakage of shear bonds around crack tip regions, as shown in Figs. 12.22 and 12.23. These damage patterns are consistent with those observed in a study by Bogert et al. [4].
12.5 Conclusions

Based on the numerical results, the PD approach successfully predicts the damage growth patterns in fiber-reinforced laminates with a pre-existing crack while considering the distinct properties of the fiber and matrix, as well as the interlayer material between the plies. The predictions capture the correct failure mechanisms of the splitting failure mode and delamination without resorting to any special treatments, and agree with the experimental observations published in the literature. The simulations also capture failure modes among each ply, which are usually distinct; they heavily depend on fiber direction, which is realistically exhibited in the current results. It can be concluded that the peridynamic theory is a powerful method that can be employed for failure analysis of composite materials.

12.6 Appendix A: PD material constants of a lamina

As shown in Fig. 12.24, a lamina is discretized with a single layer of material points in the thickness direction. The PD horizon, $H$, has the shape of a disk with radius $\delta$ and thickness $h$. The displacements of material points $i$ and $p$ are represented by $u^{(i)}$ and $u^{(p)}$, respectively. The initial relative position vector between these material points is denoted by $\chi = x^{(p)} - x^{(i)}$.

Similar to the determination of a PD material constant for an isotropic material [7], equating the strain energy density of a material point in a lamina computed by using the PD theory and classical continuum mechanics yields the relationships be-
Figure 12.15: Horizontal displacement along the central axis in the 45° ply of the [0°/45°/0°] lay-up at the end of 8000 time steps.

tween PD material constants $c_F$ and $c_A$ in terms of the engineering constants $E_1$, $E_2$, $G_{12}$, and $\nu_{12}$.

The strain energy densities are calculated by considering four different simple loading conditions:

1. Simple shear, $\gamma_{12} = \zeta, \epsilon_{11} = \epsilon_{22} = 0$
2. Uniaxial stretch in the fiber direction, $\epsilon_{11} = \zeta, \epsilon_{22} = \gamma_{12} = 0$
3. Uniaxial stretch in the transverse direction, $\epsilon_{22} = \zeta, \epsilon_{11} = \gamma_{12} = 0$
4. Biaxial stretch, $\epsilon_{11} = \epsilon_{22} = \zeta, \gamma_{12} = 0$

12.6.1 **Simple shear**

As shown in Fig. 12.25, the length of the relative position of material points $y$ and $y'$ in the deformed state becomes

$$ |y - y'| = |1 + (\sin(\phi)\cos(\phi))\zeta| \cdot |x - x'|. \quad (12.37) $$

The mechanical stretch, $\bar{s}$, in the absence of temperature change can be expressed in terms of the relative displacement of the material points $i$ and $p$, arising from the mechanical loading, as

$$ \bar{s} = (\sin(\phi)\cos(\phi))\zeta. \quad (12.38) $$

By using Eq. (12.18), the strain energy density based on the PD theory at a material point in a composite lamina can be evaluated as

$$ W_{PD} = c_F (0) + \frac{1}{2} \left( \int_{H} c_A \left( \frac{\sin(\phi)\cos(\phi)}{2} \right)^2 \zeta^2 dH \right) = \frac{\pi c_A h \delta^3}{48} \zeta^2. \quad (12.39) $$
Figure 12.16: Vertical displacement along the central axis in the 45° ply of the \[0/45/0\] lay-up at the end of 8000 time steps.

Figure 12.17: Loading and geometry of the unidirectional lamina with a crack under tension loading.

By using Eq. (12.6) in conjunction with Eq. (12.10), the strain energy density of a material point based on classical continuum mechanics can be written as

\[ W_{CM}^{\text{CM}} = \frac{1}{2} Q_{66} \zeta^2. \]  

(12.40)

By equating strain energies obtained from PD theory and classical continuum mechanics, Eq. (12.39) and Eq. (12.40), respectively, lead to

\[ c_A = \frac{24 Q_{66}}{\pi h \delta^3}. \]  

(12.41)

12.6.2 Uniaxial stretch in the fiber direction

As shown in Fig. 12.26, the length of the relative position of material points \( \mathbf{y} \) and \( \mathbf{y} \) in the deformed state for this loading condition becomes

\[ |\mathbf{y} - \mathbf{y}| = \left[ 1 + \left( \cos^2(\phi) \right) \zeta \right] |\mathbf{x} - \mathbf{x}|. \]  

(12.42)
Figure 12.18: Damage plots for laminae having a central crack with fiber orientations of (a) $\theta = 0^\circ$, (b) $\theta = 90^\circ$, and (c) $\theta = 45^\circ$.

The mechanical stretch, $\tilde{s}$, in the absence of temperature change can be expressed in terms of the relative displacement of the material points $i$ and $p$, arising from the mechanical loading, as

$$
\tilde{s} = (\cos^2(\phi)) \xi.
$$

(12.43)

By using Eq. (12.18), the strain energy density based on the PD theory at a material point in a composite lamina can be evaluated as

$$
W^{PD} = \frac{1}{2} \sum_{q=1}^{Q} \frac{c \xi}{2} \delta_{q} V_{q} + \frac{1}{2} \left( \int_{H} \frac{c \xi (\cos^2(\phi)) \xi}{2} dH \right)
$$

$$
= \frac{c \xi}{4} \left( \sum_{q=1}^{Q} \delta_{q} V_{q} \right) + \frac{\pi c A h \delta^3}{16} \xi^2.
$$

(12.44)

By using Eq. (12.6), the strain energy density of a material point based on classical continuum mechanics can be written as

$$
W^{CM} = \frac{1}{2} Q_{11} \xi^2.
$$

(12.45)

By equating strain energies obtained from PD theory and classical continuum
mechanics, Eq. (12.44) and Eq. (12.45), respectively, lead to

\[ c_F = \frac{2 \left( Q_{11} - Q_{22} \right)}{\left( \sum_{q=1}^{Q} \xi_q V_q \right)} . \]  

(12.46)

12.6.3 Uniaxial stretch in the transverse direction

As shown in Fig. 12.27, the length of the relative position of material points \( y \) and \( x \) in the deformed state for this loading condition becomes

\[ |y' - y| = \left| 1 + \left( \sin^2(\phi) \right) \zeta \right| x - x| . \]  

(12.47)

The mechanical stretch, \( \bar{s} \), in the absence of temperature change can be expressed in terms of the relative displacement of the material points \( i \) and \( p \), arising from the mechanical loading, as

\[ \bar{s} = \left( \sin^2(\phi) \right) \zeta . \]  

(12.48)

By using Eq. (12.18), the strain energy density based on the PD theory at a material point in a composite lamina can be evaluated as

\[ W_{PD} = c_F (0) + \frac{1}{2} \left( \int_{H} c_A \left( \frac{\sin^2(\phi) \xi}{2} \right)^2 \xi dH \right) = \frac{\pi c_A h \delta^3}{16} \zeta^2 . \]  

(12.49)

By using Eq. (12.6), the strain energy density of a material point based on classical continuum mechanics can be written as

\[ W_{CM} = \frac{1}{2} Q_{22} \zeta^2 . \]  

(12.50)
By equating strain energies obtained from PD theory and classical continuum mechanics, Eq. (12.49) and Eq. (12.50), respectively, lead to

\[ c_A = \frac{8Q_{22}}{\pi h\delta^3}. \]  

(12.51)

Comparing Eqs. (12.41) and (12.51) yields

\[ Q_{22} = 3Q_{66}. \]  

(12.52)

12.6.4 Biaxial stretch

As shown in Fig. 12.28, the length of the relative position of material points \( y \) and \( y \) in the deformed state for this loading condition becomes

\[ |y - y| = [1 + \zeta]|x - x|. \]  

(12.53)

The mechanical stretch, \( \tilde{s} \), in the absence of temperature change can be expressed in terms of the relative displacement of the material points \( i \) and \( p \), arising from the mechanical loading, as

\[ \tilde{s} = \zeta. \]  

(12.54)
By using Eq. (12.18), the strain energy density based on the PD theory at a material point in a composite lamina can be evaluated as

$$W_{PD} = \frac{1}{2} \sum_{q=1}^{Q} c_F \xi_q^2 \xi_{\mu} \delta \frac{\zeta_2^2}{2} V_q + \frac{1}{2} \left( \int_H \frac{c_A \xi_2 \xi_1^2}{2} dH \right) $$

(12.55)

By using Eq. (12.6), the strain energy density of a material point based on classical continuum mechanics can be written as

$$W_{CM} = \frac{1}{2} (Q_{11} + 2Q_{12} + Q_{22}) \xi^2. $$

(12.56)

By equating strain energies obtained from PD theory and classical continuum mechanics, Eq. (12.55) and Eq. (12.56), respectively, lead to

$$Q_{22} = 3Q_{12}. $$

(12.57)

The expressions for the bond constants $c_F$ and $c_A$ given by Eqs. (12.41) and
Figure 12.22: Shear bond damage plots for a laminate of \([0^\circ/90^\circ/0^\circ]\) with a pre-existing crack: (a) bottom ply, 0\(^\circ\); (b) center ply, 90\(^\circ\); and (c) top ply, 0\(^\circ\).

(12.46), as well as the relations given in Eqs. (12.52) and (12.57), can be rewritten in terms of the engineering constants as

\[
G_{12} = \nu_{12} E_2 \frac{1}{1 - \nu_{21} \nu_{12}} ,
\]

and

\[
\nu_{12} = \frac{1}{3}.
\]
Figure 12.23: Shear bond damage plots for a laminate of \([0/45/0]\) with a pre-existing crack: (a) bottom ply, 0°; (b) center ply, 45°; and (c) top ply, 0°.
12.7 Appendix B: Surface correction factors for a composite lamina

The surface correction factors for bonds in fiber and arbitrary directions are determined by computing the strain energy density at two distinct material points under uniaxial strain conditions in the $x_1$ and $x_2$ directions, i.e., $\varepsilon_{11} = 0, \varepsilon_{22} = \gamma_{12} = 0$ and $\varepsilon_{22} = 0, \varepsilon_{11} = \gamma_{12} = 0$. The first material point located near an external surface has a truncated horizon, as shown in Fig. 12.29. The second material point is located far away from an external boundary and is completely embedded in a single lamina, as shown in Fig. 12.30.

The strain energy density of a material point at $x$ is decomposed as

$$ W(x) = W_{(F)}(x) + W_{(A)}(x), $$ (12.62)

where $W_{(F)}(x)$ and $W_{(A)}(x)$ represent the contribution of bonds in fiber and arbitrary directions, respectively.

Uniaxial strain loading is applied in the $x_1$- and $x_2$-directions, respectively, and the resulting displacement field can be expressed at material point $x$ as

$$ u^T(x) = \begin{cases} \frac{\partial u^*_1}{\partial x_1} x_1 & 0 \\ 0 & \frac{\partial u^*_2}{\partial x_2} x_2 \end{cases} $$ (12.63)

and

$$ u^T(x) = \begin{cases} 0 & \frac{\partial u^*_1}{\partial x_1} x_1 \\ \frac{\partial u^*_2}{\partial x_2} x_2 & 0 \end{cases} $$ (12.64)

in which $\frac{\partial u_\beta}{\partial x_\beta}$ with $\beta = 1, 2$ is the applied constant displacement gradient. The
Figure 12.25: Relative displacement between material points $i$ and $p$. 

strain energy density, $W_B(x)$, due to the applied displacement gradient is expressed as

$$W_{PD}^B(x) = \int_H w(u \cdot u, x') \, dH,$$  \hfill (12.65)

where $H$ represents the horizon of the material point at $x$ and $w$ represents the strain energy density of the PD bond between material points at $x$ and $x'$. The subscripts $1$ and $2$ denote uniaxial strain loading condition in the $x_1$- and $x_2$-directions, respectively.

In accordance with Eq. (12.62), the strain energy density at material point $x$ due to the applied uniaxial strain loading in the $x_1$- and $x_2$-directions can be decomposed as

$$W_1 = W_{(F)}_1 + W_{(A)}_1 \hfill (12.66)$$

and

$$W_2 = W_{(F)}_2 + W_{(A)}_2. \hfill (12.67)$$

With this decomposition, the strain energy density vectors $W_{(F)}(x)$ and $W_{(A)}(x)$ can be formed as

$$W_{(F)}(x) = \{ W_{(F)}_1 \ W_{(F)}_2 \} \hfill (12.68)$$

and

$$W_{(A)}(x) = \{ W_{(A)}_1 \ W_{(A)}_2 \}; \hfill (12.69)$$

where $W_{(F)}(x)$ and $W_{(A)}(x)$ represent the contribution of bonds in fiber and arbitrary directions to the strain energy density of the material point at $x$, respectively.

For both bonds in fiber and arbitrary directions, the correction factors corresponding to the two loading directions can be defined as the ratio of the strain energy density of the material point embedded far away from an external surface in a lamina,
Figure 12.26: Relative displacement between material points \(i\) and \(p\).

The strain energy density for a material point near an external surface with a truncated horizon, \(W_{PD}^{\gamma}\), with \(\beta = 1, 2\) and \(\gamma = F, A\). For a material point whose horizon is completely embedded in a single lamina, the strain energy densities for the uniaxial strain loading condition in the \(x_1\)- and \(x_2\)-directions can be computed by using classical continuum mechanics as

\[
W_{CM}^{\beta} = \frac{1}{2} Q_{11} \, \zeta^2 
\]

(12.70)

and

\[
W_{CM}^{F} = \frac{1}{2} Q_{22} \, \zeta^2 
\]

(12.71)

where \(Q_{11}\) and \(Q_{22}\) are the coefficients of the reduced stiffness matrix \(Q\) [11]. The strain energy densities given by Eqs. (12.70) and (12.71) can be decomposed into two parts, which are associated with the deformations of bonds in fiber and arbitrary directions, i.e., \(W_{CM}^{\beta}\), with \(\beta = 1, 2\) and \(\gamma = F, A\), as

\[
W_{CM}^{1} = W_{CM}^{F} + W_{CM}^{A} 
\]

(12.72)

and

\[
W_{CM}^{2} = W_{CM}^{F} + W_{CM}^{A} 
\]

(12.73)

However, the explicit form of this decomposition is not obvious because each lamina is treated as homogenous and orthotropic within the realm of classical continuum mechanics. The decomposition can be obtained by utilizing Eq. (12.14) in conjunction with the bond constant expressions given in Eqs. (12.20) and (12.21). This assumption leads to the following decomposition of strain energies given by Eqs. (12.70) and (12.71) in the form

\[
W_{CM}^{F} = \frac{1}{2} \left( Q_{11} \, Q_{22} \right) \, \zeta^2 
\]

(12.74)
Figure 12.27: Relative displacement between material points $i$ and $p$.

$$W_{CM}^{(A)1} = \frac{1}{2} Q_{22} \zeta^2,$$  \hspace{1cm} (12.75)

$$W_{CM}^{(F)2} = 0,$$  \hspace{1cm} (12.76)

and

$$W_{CM}^{(A)2} = \frac{1}{2} Q_{22} \zeta^2.$$  \hspace{1cm} (12.77)

With these values, a vector of correction factors for bonds in fiber and arbitrary directions at material point $x$ can be formed as

$$g(\gamma)(x) = \begin{bmatrix} g(\gamma)_1 \newline g(\gamma)_2 \end{bmatrix} = \begin{bmatrix} W_{CM}^{(\gamma)1}/W_{PD}^{(\gamma)1} \newline W_{CM}^{(\gamma)2}/W_{PD}^{(\gamma)2} \end{bmatrix}^T,$$ with $\gamma = F, A.$

These correction factors are only based on loading in the $x_1$- and $x_2$-directions. However, they can be used as the principal values of an ellipse in order to approximate the surface correction factor in an arbitrary direction of unit vector, $n$ (Fig. 12.31).

In the case of a surface correction factor for a PD bond between material points $x_{(i)}$ and $x_{(j)}$ under general loading conditions, shown in Fig. 12.32(a), the correction factors in the direction of the relative position vector, $\mathbf{n} = \xi / \zeta = n_1, n_2 \quad T$, in the undeformed configuration between these two material points, can be obtained in a similar manner.

A vector of correction factors at material points $x_{(i)}$ and $x_{(j)}$ can be formed as

$$g(\gamma)(x_{(i)}) = \begin{bmatrix} g(\gamma)_{1(i)} \newline g(\gamma)_{2(i)} \end{bmatrix}^T = \begin{bmatrix} W_{CM}^{(\gamma)1}_{(i)}/W_{PD}^{(\gamma)1}_{(i)} \newline W_{CM}^{(\gamma)2}_{(i)}/W_{PD}^{(\gamma)2}_{(i)} \end{bmatrix}^T,$$  \hspace{1cm} (12.78)

and

$$g(\gamma)(x_{(j)}) = \begin{bmatrix} g(\gamma)_{1(j)} \newline g(\gamma)_{2(j)} \end{bmatrix}^T = \begin{bmatrix} W_{CM}^{(\gamma)1}_{(j)}/W_{PD}^{(\gamma)1}_{(j)} \newline W_{CM}^{(\gamma)2}_{(j)}/W_{PD}^{(\gamma)2}_{(j)} \end{bmatrix}^T.$$  \hspace{1cm} (12.79)
These correction factors are, in general, different at material points \( x_{(i)} \) and \( x_{(j)} \). Therefore, the correction factor for a PD bond between material points \( x_{(i)} \) and \( x_{(j)} \) can be obtained by their mean values as

\[
\bar{g}_{(\gamma)(i)(j)} = \left\{ \bar{g}_{(\gamma)(i)(j)1}, \bar{g}_{(\gamma)(i)(j)2} \right\}^T = \frac{\left( g_{(\gamma)(i)} + g_{(\gamma)(j)} \right)}{2},
\]

which can be used as the principal values of an ellipse, as shown in Fig. 12.32. The intersection of the ellipse and a relative position vector of material points \( x_{(i)} \) and \( x_{(j)} \), \( \mathbf{n} \), provides the correction factors as

\[
G_{(\gamma)(i)(j)} = \left( \left[ n_1 / \bar{g}_{(\gamma)(i)(j)1} \right]^2 + \left[ n_2 / \bar{g}_{(\gamma)(i)(j)2} \right]^2 \right)^{1/2}. \tag{12.81}
\]

After considering the surface effects, the discrete form of the equations of motion given in Eq. (12.1) can be expressed as
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Figure 12.30: Material point $x$ far away from external surfaces of a lamina.

Figure 12.31: Construction of an ellipse for surface correction factors.

\[ \rho (x_{\|}) \ddot{u} (x_{\|}, t) = \sum_{j=1}^{M} \left( a_{\|j} G_{(F)}(x_{\|}, x_{\|}, t) \right. \left. f (u (x_{\|}, t), u (x_{\|}, t), x_{\|}, x_{\|}) \right) + V_{\|} + b (x_{\|}, t), \]

where the coefficient $a_{\|j}$ takes a value of either 1 or 0 if the interaction between material points $x_{\|}$ and $x_{\|}$ is in the fiber direction or not, respectively.

12.8 Appendix C: PD interlayer and shear bond constants of a laminate

The interlayer bond constant, $c_N$, and the shear bond constant, $c_S$, shown in Fig. 12.33, can be expressed in terms of engineering constants based on the transverse normal and shear deformation response of the isotropic matrix material by equating the total strain energy density of interlayer and shear bonds calculated from peridynamic theory and classical continuum mechanics. The strain energy density of the
interlayer bonds associated with the material point \( a \) can be computed by summing up the strain energy density of the two interlayer bonds between material points \( d \) and \( e \) and the material point \( a \) (Fig. 12.33). The total strain energy density due to the interlayer and shear bonds can be computed as

\[
W_{PD}^{PD} = \frac{1}{2} \sum_{j=d,e} \frac{C_N}{2} \frac{x_{ja}^2}{2} V_j + \frac{1}{2} \int_{H} \frac{C_S \phi^2}{2} dH. \tag{12.83}
\]

Figure 12.33: Interlayer and shear bonds between neighboring plies (only some of the interactions are depicted explicitly for clarity).
The expression for the shear angle in Eq. (12.83) is obtained by determining the average shear angle inside the quadrilateral formed by material points \( a, b, c, \) and \( d \), as shown in Fig. 12.34. Averaging is achieved by computing the shear angles along the lines between material points \( a \) and \( d \), and \( b \) and \( c \), which are defined as \( \phi_{da} \) and \( \phi_{bc} \), respectively.

![Figure 12.34: Shear bonds between material points \( b \) and \( a \) and between material points \( d \) and \( c \) in both undeformed and deformed configurations.](image)

These shear angles are obtained from the ratio of the displacements \( u_{da} \) and \( u_{bc} \) of material points \( d \) and \( b \) with respect to \( a \) and \( c \), respectively, to the ply thickness, \( h \), as

\[
\phi_{da} = \frac{u_{da}}{h} \tag{12.84}
\]

and

\[
\phi_{bc} = \frac{u_{bc}}{h}. \tag{12.85}
\]

The relative displacements \( u_{da} \) and \( u_{bc} \) are approximated as the change in length of the bonds between material points \( d \) and \( c \), and \( b \) and \( a \), respectively,

\[
u_{da} = \begin{pmatrix} \xi_{dc} + \eta_{dc} \\ \xi_{dc} \end{pmatrix} \tag{12.86}
\]

and

\[
u_{bc} = \begin{pmatrix} \xi_{ba} + \eta_{ba} \\ \xi_{ba} \end{pmatrix}, \tag{12.87}
\]

where \( \xi_{ba} \) and \( \xi_{dc} \) correspond to the bond vectors between material points \( b \) and \( a \), and between material points \( d \) and \( c \), respectively. Similarly, the vectors \( \eta_{ba} \) and \( \eta_{dc} \) are the relative displacement vectors between material points \( b \) and \( a \), and between material points \( d \) and \( c \), respectively.

Note that the minus sign in Eq. (12.86) arises due to the contraction of the shear
bond between material points \( d \) and \( c \), whereas the bond between material points \( b \) and \( a \) extends. The average value of the shear strains \( \phi_{da} \) and \( \phi_{bc} \) results in

\[
\phi = \frac{\phi_{da} + \phi_{bc}}{2} = \frac{(\xi_{ba} + \eta_{ba}) \xi_{ba}}{2h} = \frac{(\xi_{dc} + \eta_{dc}) \xi_{dc}}{2h}.
\] (12.88)

In order to obtain the interlayer bond constant, the laminate is subjected to an isotropic expansion loading of \( s = \zeta \), as shown in Fig. 12.35(a).

For a material point, \( a \), located in the \( k \)th ply of the laminate (Fig. 12.35(b)), the contributions of the interlayer and shear bonds to the strain energy density of the material point due to isotropic expansion loading can be calculated by using Eq. (12.83),

\[
W^{PD} = \frac{1}{2} \sum_{j=d,e} C_N \xi_{ja} \overline{\xi}_{ja} V_j.
\] (12.89)

Note that the shear strain, \( \phi \), defined in Eq. (12.88) has a zero value for this loading condition because the relative displacements \( u_{da} \) and \( u_{bc} \) given in Eqs. (12.86) and (12.87) are equal in magnitude with opposite signs. Therefore, shear bonds do not have any contribution to the strain energy density for this loading condition. Both bond lengths \( \xi_{da} \) and \( \xi_{ea} \) are equivalent to the ply thickness, \( h \). Therefore, for this loading condition, Eq. (12.89) can be evaluated as

\[
W^{PD} = \frac{C_N \xi^2 h \overline{V}}{2},
\] (12.90)

where \( \overline{V} \) is equal to the volume of material points \( d \) and \( e \), i.e., \( \overline{V} = V_d = V_e \).

The corresponding strain energy density of the material point for the same load-
The energy condition can be calculated by using classical continuum mechanics as

\[ W^{CM} = \frac{1}{2} E_m \zeta^2, \tag{12.91} \]

with \( E_m \) representing the elastic modulus of matrix material. Equating strain energy densities from Eqs. (12.90) and (12.91) yields the expression of the interlayer bond constant, \( c_N \), as

\[ c_N = \frac{E_m}{hV}. \tag{12.92} \]

The shear bond constant, \( c_S \), can be evaluated similarly. In this case, the laminate is subjected to a simple shear loading of \( \gamma = \zeta \), as shown in Fig. 12.36(a). For this loading condition, the interlayer bonds do not extend (Fig. 12.36(b)). Therefore, their stretch values are zero. Hence, the interlayer bonds do not contribute to the strain energy density of the laminate.

As shown in Fig. 12.37, the original and deformed lengths of the shear bond between material points \( b \) and \( a \) can be expressed as

\[ \xi_{ba} = \sqrt{\ell^2 + h^2} \tag{12.93} \]

and

\[ \xi_{ba} + \eta_{ba} = \sqrt{\ell^2 + h^2}, \tag{12.94} \]

where \( h \) is the ply thickness.
Figure 12.37: Shear bonds between material points $b$ and $a$ and between material points $d$ and $c$ in both undeformed and deformed configurations.

For the triangle depicted in Fig. 12.37, by utilizing the law of cosines, the length of the radial component of the deformed bond vector, $\ell$, can be written in terms of the length of the radial component of the original bond vector, $\ell$, and magnitude of displacement vector, $\zeta h$, as

$$\bar{\ell} = \ell^2 + (\zeta h)^2 - 2\ell \zeta h \cos(\theta).$$  \hspace{1cm} (12.95)

After substituting Eq. (12.95) into Eq. (12.94), the deformed bond length can be rewritten as

$$|\xi_{ba} + \eta_{ba}| = \sqrt{\ell^2 + h^2 + \ell \zeta h \cos(\theta)}.$$  \hspace{1cm} (12.96)

In deriving Eq. (12.96), the $(\zeta h)^2$ term is neglected with respect to $h^2$ because $\zeta$ is much less than unity. The square root term on the right-hand side of Eq. (12.96) can be further simplified by using the square root approximation

$$\sqrt{N^2 + d} = N + \frac{d}{2N},$$  \hspace{1cm} (12.97)

where $d << N$. Therefore, the deformed bond length expression given in Eq. (12.96) can be rewritten as

$$|\xi_{ba} + \eta_{ba}| = \sqrt{\ell^2 + h^2} + \frac{\ell \zeta h \cos(\theta)}{\ell^2 + h^2}.$$  \hspace{1cm} (12.98)

The original and deformed bond lengths between material points $d$ and $c$ can be computed similarly as

$$\xi_{dc} = \sqrt{\ell^2 + h^2}$$  \hspace{1cm} (12.99)

and

$$|\xi_{dc} + \eta_{dc}| = \sqrt{\ell^2 + h^2} + \frac{\ell \zeta h \cos(\theta)}{\ell^2 + h^2}.$$  \hspace{1cm} (12.100)

Therefore, the shear angle for this loading condition can be computed by using Eq. (12.88) as

$$\phi = \frac{\ell \zeta \cos(\theta)}{\ell^2 + h^2}.$$  \hspace{1cm} (12.101)
After substituting the shear angle expression given in Eq. (12.101) into the strain energy density expression given in Eq. (12.83) and performing the integration,

$$W^{PD} = \left\{ \frac{\pi S^2 h}{4} \left( \delta^2 + h^2 \ln \frac{h^2}{h^2 + \delta^2} \right) \right\} \xi^2. \quad (12.102)$$

The corresponding strain energy density based on classical continuum mechanics can be computed as

$$W^{CM} = \frac{1}{2} G_m \xi^2. \quad (12.103)$$

Equating the strain energy densities calculated from PD theory and classical continuum mechanics, i.e., Eqs. (12.102) and (12.103), leads to the explicit form of the shear bond constant in terms of the shear modulus of the matrix material, $G_m$,

$$c_S = \frac{2 G_m}{\pi h} \frac{1}{\left( \delta^2 + h^2 \ln \left( \frac{h^2}{\delta^2 + h^2} \right) \right)}. \quad (12.104)$$

### 12.9 Appendix D: Critical Stretch Values for Bond Constants

The critical stretch for the interlayer bond, $s_{Nc}$, can be computed by equating the energy required to break an interlayer bond between material point $x$ located at the $k$th ply and the material point $x'$ located at the $(k+1)$th ply (see Fig. 12.38) to the mode-I critical energy release rate of the material, $G_{IC}$, as

$$h \left( \frac{c_S S_{Nc} h}{2} \right) \bar{V} = G_{IC}, \quad (12.105)$$

where $h$ and $\bar{V}$ represent the thickness of the ply and the volume of the material point $x$, respectively.
As opposed to interlayer bonds, multiple shear bonds exist between the material point \( x \) and other material points in the \((k+1)\)th ply, as shown in Fig. 12.38. The failure of these shear bonds corresponds to a mode-II type of failure. Therefore, the energy required to break all of these shear bonds can be equated to the mode-II critical energy release rate of the material, \( G_{IIc} \), as

\[
\int_{H}^{h c} \frac{c_s \phi_{Sc}^2}{2} dH = G_{IIc},
\]

where \( \phi_{Sc} \) is the critical shear angle. This equation can be rewritten after splitting the domain of integration as

\[
h \left( h \frac{c_s \phi_{Sc}^2}{2} \int_{0}^{\delta} \int_{0}^{2\pi} d\ell \right) = G_{IIc}.
\]

Performing the integrations results in the critical shear angle expression as

\[
\phi_{Sc} = \sqrt{\frac{G_{IIc}}{hG_m}}.
\]

By using the relation given by Eq. (12.105) in conjunction with the bond constant expression given by Eq. (12.27), the critical stretch expression for the interlayer bond can be expressed as

\[
s_{Nc} = \sqrt{\frac{2G_{IC}}{hE_m}}.
\]

Figure 12.38: Interlayer and shear bonds between material point \( x \) and other material points located at the \((k+1)\)th ply.
References


References


