3

Elementary Groundwater Flow
and Transport Processes

3.1 Introduction

In Chapter 2, porosity \( n \) was defined as the volume \( V_p \) of the pores of a rock or soil sample divided by the total volume \( V_t \) of both pores and solid material so that \( n = \frac{V_p}{V_t} \). But porosity at a geometrical point cannot be defined since a porous medium is a conglomerate of solid grains and voids. It is then necessary to introduce the concept of representative elementary volume (REV). The REV is sufficiently large to define a space-averaged porosity, but it is small enough that the variation from one REV to the next may be approximated by a continuous function on the scale of the measuring instruments. Thus one could take, for example, an REV of 1 cm\(^3\) for a fine sand, but it could be quite larger for a fractured rock. An alternate approach is to consider the porous medium as a realization of a random process.

The porosity at a geometrical point is then an *ensemble average* of an infinite number of realizations (de Marsily, 1986; Charbeneau, 2000). Either of these constructs allows us to use the infinitesimal calculus and thus to apply the concepts of fluid mechanics.

Essentially all natural groundwater flows are three dimensional (3D). That is, the average velocity of a percolating water particle is represented by a vector that has three components. A simple example is the 3D radial flow toward a well (see Chapters 10 and 11). However, there are many situations in which the velocities are nearly coplanar or there is radial symmetry. In these cases the flow can be analyzed as two dimensional (2D) with sufficient accuracy for many engineering problems. An example is the infiltration of water into a series of long, parallel horizontal tile drains. Away from the pipe extremities, the shape of the water table is independent of the location along the pipe.
drain. In some cases the flow problem can be further reduced to one dimension. For example, the flow in karst conduit can be regarded as approximately one dimensional (1D). The existence of symmetry or special assumptions permits the simplification of many problems. It is important, however, to recognize the size of the errors that such simplifications can entail.

Groundwater flow variables, such as velocity and pressure, can vary in time or can be independent of time. For example, when a well is pumped, the water table drawdown increases with time. This is an unsteady or transient flow problem. If the flow variables do not change with time, the flow is steady.

This chapter is concerned with elementary 1D and 2D steady groundwater flow and transport problems. Advanced 3D flows, transient flows, transport processes, and modeling are treated in subsequent chapters. The geological setting is discussed in Chapter 2. Chapters 2 and 3 can be regarded as an introduction to groundwater hydrogeology and engineering. The following chapters deal with a number of advanced or specialized subjects in more detail.

The notions of saturated zone, confined aquifer, unconfined aquifer, water table, aquitard, aquiclude, perched aquifer, unsaturated zone or vadose zone and the physical properties of aquifers are discussed in Chapter 2. This chapter starts directly with the discussion of the hydraulics of groundwater or the motion of water below ground. The second part of this chapter deals with the transport of contaminants by groundwater.

3.2 Pressure, Suction, Piezometric Head, and Hydraulic Gradient

The water pressures are not only expressed as gage pressures but they are also expressed as absolute pressures. These pressures are related by the equation (Figure 3.1)

\[ h_p = \frac{P - \gamma_w g}{\rho g} \]

where
- \( h_p \) is the pressure head,
- \( P \) is the absolute pressure,
- \( \gamma_w \) is the specific weight of water,
- \( \rho \) is the density of water,
- \( g \) is the gravitational acceleration.

Gage pressures are used in the following discussion. The water table is at the local atmospheric pressure and serves as a datum for gage pressures. Point A in Figure 3.1 is in the saturated zone and the gage pressure is positive and is called the pore pressure. Point B is in the unsaturated zone and the gage pressure is negative. This negative pressure is referred to as a suction or tension. The suction is expressed as a positive number. Thus a positive suction corresponds to a negative gage pressure. The dimensions of pressure are \( F/L^2 \), that is Newton per square meter or pascal (Pa), kiloNewton per square meter or kilopascal (kPa) in SI units and pound per square inch and pound per square feet in U.S. units. The law of hydrostatics states that pressure \( p \) can be expressed in terms of height of liquid \( h_p \) measured from the water table (assuming that groundwater is at rest or moving horizontally). This height is called the pressure head:

\[ h_p = \frac{P - \gamma_w g}{\rho g} \]

For point A the quantity \( h_p \) is positive whereas it is negative for point B. The pressure head is generally expressed in meters of water but it can also be expressed in centimeters of mercury. Some conversion factors for pressure and pressure heads are listed in Table 3.1 and some approximate equivalents of atmospheric pressure are listed in Table 3.2.

![Figure 3.1 Absolute and gage pressures](image-url)
If the medium is saturated, pore pressure, $p$, can be measured by the pressure head, $h_p = \frac{p}{\gamma_w}$, in a piezometer, a nonflowing well. The difference between the altitude of the well, $H$ (Figure 3.2), and the depth to the water inside the well is the total head, $h_t$, at the well.

In fluid mechanics the total head is defined as the sum of the elevation head, $z$, the pressure head, $p/\gamma_w$, and the velocity head, $v^2/2g$, where $v$ is the flow velocity. For groundwater flow the velocity head can generally be ignored because the water moves very slowly. Therefore, the total head at an observation well is taken to be equal to the piezometric head, or the sum of the elevation head and the pressure head. The symbol $\psi$ is sometimes used to designate the pressure head.

$$h_t = z + \frac{p}{\gamma_w} = z + \psi \quad (3.3)$$

The piezometric head is also referred to as the piezometric potential. The change of piezometric head per unit distance in a given direction is the hydraulic gradient. If the direction is not specified it is assumed to be in the direction of the maximum gradient. The hydraulic gradient is a dimensionless quantity ($L/L$) when consistent units are used (see also Chapter 2, Sections 2.4.7 and 2.4.11).

Example 3.1

With the data of Figure 3.2, find the average hydraulic gradient.

Solution. The average hydraulic gradient is $h_t/L = [(100 - 10) - (95 - 15)]/2000 = (90 - 80)/2000 = 0.005$.

If the piezometric head is known at three observation wells A, B, and C that are not in a straight line, then both the direction of the groundwater flow and the hydraulic gradient can be calculated approximately. For this purpose, with reference to Figure 3.3, (1) select observation well C with the intermediate head. (2) By linear interpolation, find the location of the point P having the intermediate head on the line connecting the observation wells A and B with the maximum and minimum heads.

### TABLE 3.1 Conversion Factors for Pressures and Related Terms

<table>
<thead>
<tr>
<th>To Convert from</th>
<th>To</th>
<th>Multiply by</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atmosphere</td>
<td>Pascal</td>
<td>1.013 E+5</td>
</tr>
<tr>
<td>Bar</td>
<td>Pascal</td>
<td>1.000 E+5</td>
</tr>
<tr>
<td>Dyne/cm²</td>
<td>Pascal</td>
<td>0.100</td>
</tr>
<tr>
<td>Feet of water (39.4°F)</td>
<td>Pascal</td>
<td>2.989 E+3</td>
</tr>
<tr>
<td>Inch of mercury (32°F)</td>
<td>Pascal</td>
<td>3.386 E+3</td>
</tr>
<tr>
<td>Inches of water (39.4°F)</td>
<td>Pascal</td>
<td>249.1</td>
</tr>
<tr>
<td>Millibar</td>
<td>Pascal</td>
<td>100</td>
</tr>
<tr>
<td>Millimeters of mercury (0°C)</td>
<td>Pascal</td>
<td>133.3</td>
</tr>
<tr>
<td>Pound per square foot</td>
<td>Pascal</td>
<td>47.88</td>
</tr>
</tbody>
</table>


### TABLE 3.2 Approximate Equivalents of Atmospheric Pressure

<table>
<thead>
<tr>
<th>U.S.</th>
<th>Metric</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.7 psi abs.</td>
<td>101.3 kN/m², abs.</td>
</tr>
<tr>
<td>29.9 in Hg</td>
<td>760 mmHg</td>
</tr>
<tr>
<td>33.9 ft H₂O</td>
<td>10.3 mH₂O</td>
</tr>
</tbody>
</table>

### FIGURE 3.2 Heads and gradients. (From Heath, R.C., Basic Ground-water hydrology, U.S. Geological Survey, Water-Supply Paper 2200, 7th printing, Denver, CO, 1995.)
(3) Connect point P with the intermediate well, this line is a segment of a piezometric contour line. (4) Draw a line perpendicular to this contour that passes either through the maximum head or through the minimum head observation well. This line is in the direction of groundwater movement. (5) The ratio of the head differential between the ends of the perpendicular line and the length of this line is the hydraulic gradient.

Example 3.2

Using the data of Figure 3.3, find the flow direction and the hydraulic gradient.

Solution. (1) Select well C with an intermediate head of 25.17 m. (2) Find the distance x from well A to point P: 

\[ x = (25.30 - 25.17) \times 230.5/(25.30 - 25.00) = 99.88 \text{ m} \]

(3) Draw the line BD perpendicular to CP, this is the direction of groundwater movement. Measure the distance BD = 114.29 m. (4) Divide the head differential by the length BD to obtain the hydraulic gradient 

\[ (25.17 - 25.00)/114.29 = 0.0015 \]

If the medium is unsaturated, the negative pressure or suction or tension is measured by a tensiometer. This instrument is composed of a vertical tube closed at the top and a porous ceramic cup at the bottom (Figure 3.4). The tensiometer is initially filled and the ceramic cup is saturated with water. When the instrument is placed in the soil, the water in the tensiometer is generally at atmospheric pressure. The soil water, which is at negative pressure, produces a suction that drains water from the tensiometer causing a pressure drop. When equilibrium is reached, the pressure inside the tensiometer is equal to that in the soil and can be measured with a vacuum gage. The practical suction range of a tensiometer is 0–0.8 bar or approximately 0–8 m of water. Relationships between the soil moisture content \( \theta \) and the pressure head \( \psi \) are presented in Chapter 2 and in Chapters 6 and 7.

The total pressure at a point in a porous medium is the weight per unit area of the overburden above this point. This total pressure is the sum of the pore pressure and the intergranular stress, that is, the stress due to forces transmitted from grain to grain of the rock matrix.

3.3 The Motion of Groundwater

The motion of water requires energy. This energy can be expressed as a head above a datum. The elevation of this datum is arbitrary. This is because the difference in energy or the difference in head is the concern. It is therefore important that the energies be measured with respect to the same datum. In groundwater engineering the mean sea level (MSL) is usually taken as the datum. The hydraulic head is defined as the energy per unit weight measured relative to the datum.

Water can possess several forms of energy. Perhaps the most obvious is the energy that water possesses by virtue of its elevation above the datum. This is the potential energy. A mass \( m \) of water at an elevation \( z \) above the datum has a potential energy \( mgz \), where \( g \) is the acceleration due to gravity. This is the work necessary to move the mass \( m \) from the datum to the elevation \( z \). If \( \rho \) is the density of the water, a unit volume of water has a mass \( \rho \) and a weight \( \rho g \) and a potential energy \( \rho g z \). The potential energy per unit weight, that is the elevation head, is thus \( \rho g z/\rho = z \). Note that the head has the unit of length.

The energy that water possesses by virtue of its motion is the kinetic energy. A mass \( m \) of water that moves with a velocity \( v \) has a kinetic energy \( \frac{1}{2}mv^2 \). Thus the kinetic energy per unit volume
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is $\frac{1}{2} \rho v^2$ and the kinetic energy per unit weight or velocity head is $\frac{1}{2} \rho v^2/\rho g = v^2/2g$. The velocity head has the dimension of length. When groundwater is flowing through the pores of the rock or soil formation, the velocity is very small, perhaps of the order of centimeters per year, and the velocity head is usually negligible with respect to the other forms of energy. One exception is near wells where the velocity increases significantly. Another exception is in certain karst conduits where groundwater can flow fast enough that the velocity head is important.

The energy that water possesses by virtue of its pressure is the pressure energy. The pressure intensity of the fluid, $p$, acting on an area $dA$ produces a force $p \, dA$. If the area is displaced by a distance $ds$, in the flow direction, then the force produces an amount of work $p \, dA \, ds$ known as flow work. The volume $dA \, ds$ has a weight $\rho \, g \, dA \, ds$ and the flow work per unit weight is $p \, dA \, ds/\rho \, g \, dA \, ds = p/\rho g$ known as the pressure head. The sum of the elevation head and the pressure head is known as the piezometric head $h = z + p/\rho g$.

### 3.4 Flow Through Porous Media: Darcy’s Law

The French engineer Henry Darcy performed experiments on the filtration of water through sand columns. His finding that the rate of flow through a sand column is proportional to the loss of head appeared in an appendix to his treatise on the public fountains of the city of Dijon (Darcy, 1856). For further historic details, see Chapter 1. Figure 3.5 shows the original set up utilized by Darcy and Figure 3.6 shows some of his experimental results as plotted by Hubbert (1956) from Darcy’s data. Darcy’s equation is:

$$Q = KA \frac{h_1 - h_2}{l}$$

or

$$\frac{Q}{A} = q = K \frac{h_1 - h_2}{l}$$

![FIGURE 3.5](a) Darcy’s original apparatus with mercury manometer and (b) equivalent apparatus with water manometers. (From Hubbert, M.K., Bull. Am. Assoc. Petrol. Geol., 37, 1954, 1953. With permission.)
The one-dimensional form of Darcy’s law is

\[ q = K \frac{(p_1 - p_2) + z_1 - (p_1 - p_2) + z_2}{L} \]  

(3.5)

where subscripts 1 and 2 refer to the points at which the pressure heads and the elevation heads are considered, respectively. \( L \) is the distance between these points.

**Example 3.3**

Find the hydraulic conductivity of the sands used in Darcy’s first series experiments (Figure 3.6), assuming that the height of the sand column is 3 m and the diameter of the stand pipe is 0.35 m.

**Solution.** Take the flow rate \( Q = 30 \text{ L/min} = 0.03 \text{ m}^3/\text{min} \). The specific discharge is \( q = Q/A = 0.03/ (\pi \times 0.35^2/4) = 0.312 \text{ m/min} \). From Equation 3.5 \( K = q/L\Delta h \). From the graph \( \Delta h = 10.5 \text{ m} \). Thus \( K = 0.312 \times 3.0/10.5 = 0.089 \text{ m/min or 0.0015 m/s} \). This corresponds to a coarse sand.

**Example 3.4**

Find the time it takes for a molecule of water to move from a factory to a bore hole located 4 km away in a homogeneous silty sand unconfined aquifer with a hydraulic conductivity of \( K = 5 \times 10^{-5} \text{ m/s} \) or 4.32 m/day, an effective porosity of 0.4 and observing that the water table drops 12 m from the factory to the bore hole.

**Solution.** As a simple approximation \( v = q/n_e = Kii/n_e \) and the pore velocity is calculated as

\[ v = \frac{4.32 \times (12/4000)}{0.4} = 0.0324 \text{ m/day} \]

It would take 4000/(0.0324 × 365) ≈ 338 year. If instead, the aquifer was a fractured limestone with a porosity of 0.01 and the hydraulic conductivity the same, the pore velocity would be approximately 1.3 m/day and the time to travel the 4 km would be 8.5 year. With a porosity of 0.001 the travel time would reduce to 0.85 year or about 10 months. Pumping at the bore hole will increase the hydraulic gradient and increase the pore velocity and thus decrease the travel time.

### 3.4.1 Similarity of Darcy’s Law and Other Laws of Physics

Darcy’s law is similar to Fourier’s law of heat transfer, Ohm’s law of electricity, and Fick’s law of solute diffusion. Fourier’s law governs the conduction of heat from high temperatures to
low temperatures. It states that the heat flux is proportional to the temperature gradient and the constant of proportionality is the thermal conductivity. A number of flow through porous media problems have been solved using the heat conduction analogy. The Theis equation for transient flow toward wells (see Chapter 29 on “Well Hydraulics and Aquifer Tests”) was obtained using an analogous problem in heat flow. Ohm’s law can be stated as \( I = \frac{V}{R} \), where \( I \) is the electric current, \( R \) is the resistance, and \( V \) is the voltage or potential difference across the resistor. For a cylindrical wire of length \( L \), cross-sectional area \( A \), and conductivity \( c \), the resistance is \( R = L/(cA) \) so that the expression for the current becomes

\[
I = cA \frac{V}{L} \quad (3.6)
\]

The similarity between Equations 3.4 and 3.6 is now obvious. The flow rate is analogous to the current and the drop of head to the voltage drop. This analogy forms the basis for the resistance network models of aquifers (Karplus, 1958). Fick’s law states that diffusion of a solute takes place along the concentration gradient from zones of high concentration to zones of low concentration, and the coefficient of proportionality is the diffusion coefficient. Fick’s law is used in Section 3.13 and in Chapters 18 and 19, which are concerned with pollutant transport.

### 3.4.2 Limitations of Darcy’s Law

Darcy’s law implies that the flow is laminar, as is generally the case in porous media. The limit of validity can be stated in terms of the Reynolds number, \( N_R \),

\[
N_R = \frac{D}{\nu} \quad (3.7)
\]

where

- \( q \) is the velocity
- \( \nu \) is the kinematic viscosity of the fluid, defined as its dynamic viscosity \( \mu \) divided by its density \( \rho \)
- \( D \) is a representative length

For flow in porous media, \( q \) is taken equal to the specific discharge, and the representative length, \( D \), is often taken equal to the pore size or the effective grain diameter, \( d_{eq} \) (the grain size such that 10% of the material is larger by weight). The Reynolds number measures the importance of the inertia forces relative to the viscous forces. It depends on the viscosity that varies with temperature as shown in Table 3.3. As a result \( N_R \) also varies with temperature. Likewise the hydraulic conductivity \( K = k \rho g / \mu \), where \( k \) is the intrinsic permeability of the porous medium (see Chapter 2), also varies with temperature through \( \mu \). Schneebeli (1955), using spheres of uniform diameter, found that deviations from Darcy’s law start at \( N_R \approx 5 \) as inertia forces become effective and that turbulent flow started around \( N_R \approx 60 \). For flows in which the dimension \( D \) is large, such as in rocks with large fractures (see Chapter 20) or in karstic limestones (see Chapter 21), the flow can be turbulent and Darcy’s law does not apply.

Darcy’s law, as given above, applies to isotropic media, that is, where the hydraulic conductivity is independent of direction. It also applies to flows where the direction of the hydraulic conductivity corresponds to the direction of the hydraulic gradient. In anisotropic, media the hydraulic conductivity depends upon the direction of measurement. Then a hydraulic conductivity tensor is used and Darcy’s law is expressed as a tensorial equation (see Chapter 4 on “Two- and Three-Dimensional Flow of Groundwater”).

### 3.4.3 Laboratory Measurement of Hydraulic Conductivity

Samples of the aquifer material must be obtained and returned to the laboratory in undisturbed condition. This is generally possible for consolidated materials but usually impossible for unconsolidated material and rarely possible for fissured aquifers. The samples must be representative of the aquifer. Where the aquifer has horizontally bedded strata, the samples can be collected from bore holes that intersect the several strata. They can also be obtained from cliffs or quarry faces. Where the strata are inclined the samples likewise can be obtained from bore holes as well as from the outcrop. Bore holes samples are preferred because the outcrop material may be weathered and consequently unrepresentative of the aquifer. Consolidated aquifer samples typically are cylinders or cubes with diameter and length of 25–50 mm. Cubes have the advantage that they permit the determination of the permeability in three directions.

**Permeameters** are used for the laboratory determination of hydraulic conductivity making use of Darcy’s law. It is best to use groundwater from the formation in the permeameter test.

### Table 3.3 Density and Viscosity of Water as a Function of Temperature

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Density (( \rho ) kg/cm³)</th>
<th>Viscosity (( \mu \times 10^6 ) Ns/m²)</th>
<th>Kinematic Viscosity (( \nu \times 10^6 ) m²/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>999.8</td>
<td>1.781</td>
<td>1.785</td>
</tr>
<tr>
<td>5</td>
<td>1000.0</td>
<td>1.518</td>
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<td>10</td>
<td>999.7</td>
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</tr>
<tr>
<td>20</td>
<td>998.2</td>
<td>1.002</td>
<td>1.003</td>
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<tr>
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<td>997.0</td>
<td>0.890</td>
<td>0.893</td>
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<tr>
<td>30</td>
<td>995.7</td>
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<td>0.800</td>
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<tr>
<td>40</td>
<td>992.2</td>
<td>0.653</td>
<td>0.658</td>
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<tr>
<td>50</td>
<td>988.0</td>
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<tr>
<td>100</td>
<td>958.4</td>
<td>0.282</td>
<td>0.294</td>
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</table>


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**Density and Viscosity of Water as a Function of Temperature**

- **Density** (\( \rho \) kg/cm³)
- **Viscosity** (\( \mu \times 10^6 \) Ns/m²)
- **Kinematic Viscosity** (\( \nu \times 10^6 \) m²/s)
because this water will be in chemical equilibrium with the aquifer material. In particular, clays can swell or shrink with changes in the water chemistry. There are two types of permeameter: the constant head permeameter (Figure 3.7a) used for noncohesive soils such as sands and gravels and the falling head permeameter (Figure 3.7b) used for materials with lower hydraulic conductivity. For the constant head permeameter, the hydraulic conductivity obtained from Darcy’s law is

\[ K = \frac{\mathcal{V} L}{A t h} \]  (3.8)

where

\( \mathcal{V} \) is the volume of water collected in time \( t \)

\( A \) and \( L \) are the cross-section area and length of the sample, respectively

\( h \) is the constant head

For the falling head permeameter, the hydraulic conductivity is obtained by equating the flow rate through the sample to the flow rate obtained from the observed head drop as

\[ K = \frac{r^2 L}{R^2 t} \ln \left( \frac{h_1}{h_2} \right) \]  (3.9)

where

\( R \) and \( r \) are the radii of the sample and of the tube, respectively

\( L \) is the length of the sample

\( h_1 \) and \( h_2 \) are the heads at the beginning and at a later time \( t \)

### 3.4.4 Field Measurement of the Hydraulic Conductivity

A dependable method of field determination of hydraulic conductivity is by pumping tests. The hydraulic conductivity is obtained from observations of the water levels near pumping wells. It yields an integrated value of \( K \) (on a scale of 10 to 10³ m) rather than the almost punctual information (on a scale of 10⁻² to 10⁻¹ m) obtained by laboratory tests. It also has the advantages that the aquifer is not disturbed and formation water is used. The pumping tests are described in detail in Chapter 29 on “Hydraulics of Wells and Aquifer Tests.”

Tracer tests using a dye such as fluorescein or a salt such as calcium chloride can also be used. If there is a drop of water table \( h \) in a distance \( L \) between the injection test hole and the observation bore hole and \( t \) is the observed travel time between the two bore holes, the hydraulic conductivity is obtained by equating the pore velocity obtained by Darcy’s law and that obtained by dividing the distance by the time. This results in

\[ K = \frac{n L^2}{h t} \]  (3.10)

where \( n \) is the porosity of the material. In practice this test is difficult to accomplish because the flow direction must be known exactly, the distance between the bore holes should be small enough that the travel time does not become too long, and it is assumed that the aquifer is not stratified.

Other tests such as the slug test, the auger-hole test, and the piezometer test are based on the observation of the rate of recovery of the water level in a bore hole after the water level has been abruptly raised or lowered by addition or removal of water with a bailer or bucket or by insertion or withdrawal of a solid piece of pipe or slug into the well. These tests tend to give a more localized value of the hydraulic conductivity than the well tests but are less expensive to conduct. For more detail about these tests, the reader is referred to Bouwer (1978), Schwartz and Zhang (2003), or Butler (1997). Chapters 14 and 15 on aquifer characterization give a summary of several methods employed for the determination of aquifer properties.

### 3.5 One-Dimensional Hydraulics

The two basic laws of hydraulics are the continuity equation and the energy equation. They are discussed here for the case of steady incompressible flow. The continuity equation is a statement of the conservation of mass. For an incompressible fluid such as water, the equation becomes a conservation of volumes:

\[ Q = A_1 V_1 = A_2 V_2 \]  (3.11)

where

\( Q \) is the flow rate (L·T⁻¹)

\( A_1 \) and \( A_2 \) are the cross-sectional areas

\( V_1 \) and \( V_2 \) are the mean velocities at Sections 1 and 2, respectively

This assumes that there is no inflow or outflow between Sections 1 and 2. Figure 3.8 illustrates the continuity in a karst conduit.
The conservation of energy for steady flow is

\[
z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + h_L \tag{3.12}
\]

where
- \( z \) is the elevation head or the elevation above a datum or reference plane
- \( p/\gamma \) is the pressure head (where \( \gamma \) is the specific weight of the liquid)
- \( V^2/2g \) is the velocity head
- \( h_L \) is the head loss

Each term has the dimension of length (L), and represents a form of energy per unit weight. As discussed earlier, the elevation head is the potential energy per unit weight, the pressure head is the work per unit weight, the velocity head is the kinetic energy per unit weight, and the head loss is the energy loss due to friction or other causes per unit weight. The equation can also be written as energy per unit mass

\[
gz_1 + \frac{p_1}{\rho} + \frac{1}{2} V_1^2 = gz_2 + \frac{p_2}{\rho} + \frac{1}{2} V_2^2 + gh_L \tag{3.13}
\]

where \( \rho \) is the fluid density. If the head loss is negligible and is made equal to zero, then the equation is known as the Bernoulli equation. It states that in an unbranched conduit the sum of the potential, pressure, and kinetic energies remains constant if the losses are neglected. In flow through porous media, the velocity is very small and the velocity head is generally neglected but the friction losses due to viscosity, however, are important. Figure 3.8 illustrates the terms of Equation 3.12 in a karst system.

In a karst terrain, flow usually occurs in larger conduits that hydraulically, act as pipes or fissures. If the Reynolds number is sufficiently small, the flow is laminar, otherwise it is turbulent. For circular pipes the Reynolds number is defined as

\[
N_R = \frac{\rho V D}{\mu} = \frac{V D}{v} \tag{3.14}
\]

where
- \( D \) is the diameter
- \( V = Q/A \) is the average flow velocity, that is, the flow rate \( Q \) divided by the cross-sectional area \( A \)
- \( \rho \) is the density of the fluid
- \( \mu \) is the absolute viscosity
- \( v \) is the kinematic viscosity

With a consistent set of units, the Reynolds number is a dimensionless quantity. The flow is laminar if the Reynolds number is less than 2000 and it is generally turbulent if the Reynolds number is larger than 4000. For values between 2000 and 4000 the flow is in a transitional regime.

For noncircular conduits such as fractures, the Reynolds number is defined as

\[
N_R = \frac{\rho V (4R_h)}{\mu} \tag{3.15}
\]

where \( R_h = A/P \) is the hydraulic radius which is the ratio of the cross-sectional area, \( A \), to the wetted perimeter, \( P \). For a circular cross-section, \( D = 4R_h \).

Friction head losses in conduits are calculated using the Darcy-Weisbach formula that is generally written as

\[
h_L = f L \frac{V^2}{D 2g} \tag{3.16}
\]

in which \( f \) is a dimensionless quantity called the friction coefficient. For laminar flow in circular conduits (also known as Poiseuille flow), \( f = 64/N_R \) and the flow rate is thus expressed as

\[
Q = \frac{\rho g D^2}{32\mu} \frac{A}{L} h_L \tag{3.17}
\]

This equation is seen to be of the same form as Darcy’s law (Equation 3.4) with \( K = \rho g D^2/(32\mu) \) or \( k = D^2/32 \) and \( i = h_L/L \).

The Darcy–Weisbach formula (Equation 3.16) also holds for turbulent flows. Methods of calculation of the friction coefficient \( f \) for turbulent flows are described in fluid mechanics textbooks such as Finnemore and Franzini (2002).

Free surface open channel flow can occur in large karst cavities. Free surface streams in karst aquifers are discussed in Chapter 21.

### 3.6 One-Dimensional Flow through Porous Media—Leaky Aquifers

Simple 1D flows in porous media can be analyzed using Darcy’s law as expressed in Equation 3.5. As an example, consider the case of a leaky aquifer. Many aquifers are either not fully confined or unconfined. One aquifer may be overlain by another and the stratum separating them is not fully impervious: its hydraulic...
conductivity is much less than that of either top or bottom aquifer but it is not zero. The rate of leakage \( q \) (volume per unit area per unit time) can be calculated using Darcy’s law as

\[
q = K' \frac{(h' - h)}{b}
\]

(3.18)

where

- \( K' \) and \( b \) are the vertical hydraulic conductivity and the thickness of the aquitard
- \( h' \) and \( h \) are either the heads at the top of the aquitard (bottom of upper aquifer) and at the top of the lower aquifer, just below the aquitard (Figure 3.9)

Observe that the level in the right piezometer is slightly lower than the water table due to the downward motion of the flow. This can be verified by writing Darcy’s law between point A at the water table and point B at the bottom of the unconfined aquifer: \( q = K(H - h')/(y + z) \), from which \( H - h' = q(y + z)/K \).

**Example 3.5**

As an example of a 1D flow problem, consider the case of the confined aquifer of Figure 3.9 that is recharged from an unconfined aquifer through an aquitard. The recharge rate is 0.3 m/year or \( 8.22 \times 10^{-4} \) m/day. The water table is at \( H = 30 \) m above the datum. The aquitard is 2 m thick and its vertical hydraulic conductivity is \( K' = 10^{-3} \) m/day. The unconfined aquifer is 20 m thick and has a hydraulic conductivity \( K = 10^{-1} \) m/day. Find the piezometric head \( h' \) at the bottom of the unconfined aquifer and the difference in elevation between the water table and the piezometric surface of the confined aquifer.

**Solution.** Let \( y \) be the height of the piezometric surface over the top of the aquitard and \( z \) the difference in elevation between the water table and the piezometric surface (Figure 3.9). Applying Darcy’s law (Equation 3.5) between points A and B yields \( 8.22 \times 10^{-4} = 10^{-3} (H - h')/(y + z) \). Thus \( h' = H - 8.22 \times 10^{-1}(y + z) = 30 - 8.22 \times 10^{-1}(20) = 29.84 \) m. Writing Darcy’s equation between the top and the bottom of the aquitard yields \( 8.22 \times 10^{-4} = 10^{-3} (h' - h)/b \), thus \( h = h' - 8.22 \times 10^{-1} b = 29.84 - 8.22 \times 10^{-1} \times 2 = 28.20 \). Hence, \( z = H - h = 30 - 28.20 = 1.80 \) m.

When a well discharges from a leaky confined aquifer, the piezometric surface is lowered throughout a wide circular area. This lowering, called drawdown, is largest near the well and decreases outward. This variable drawdown changes the head differential between the confined and unconfined aquifers and alters the rate of leakage through the aquitard. From Darcy’s law it follows that, at any point, the downward flow is proportional to the difference of elevation between the water table of the unconfined aquifer and the piezometric surface of the confined aquifer. Steady and unsteady flows toward wells in leaky aquifers are discussed in Chapter 10.

### 3.7 Dupuit–Forchheimer Assumptions

For some of the 2D flow problems, one component of the flow can be neglected with respect to the other. In particular, in some unconfined flows with a free surface, the vertical component of the flow can be neglected. This approximation pioneered by Dupuit (1863) and utilized later by Forchheimer (1930) is known as the Dupuit–Forchheimer assumption. It gives reasonable results when the depth of the unconfined flow is shallow and the slope of the free surface is small. These assumptions are summarized as follows:

1. The flow is horizontal at any vertical cross-section.
2. The velocity is constant over the depth.
3. The velocity is calculated using the slope of the free surface as the hydraulic gradient.
4. The slope of the water table is relatively small.

#### 3.7.1 Steady Flow over a Horizontal Aquiclude

A simple application of the Dupuit–Forchheimer approximation is the analysis of steady flow through an unconfined aquifer overlying an impervious horizontal aquiclude (Figure 3.10). The discharge per unit width is \( q_x = qh \), where \( q \) is the Darcy velocity or specific discharge and \( h \) is the depth of flow. From Darcy’s law,

\[
q_x = -K h \frac{dh}{dx}
\]

(3.19)
3.7.2 Seepage from Open Channels

The Dupuit–Forchheimer assumptions can be used to analyze the seepage from an open channel embedded in a homogeneous soil underlain by a material of much lower hydraulic conductivity, assumed impervious in the analysis (Figure 3.11). The average slope of the water table is \( D_w/(L – 0.5 W) \). The specific discharge

\[
K \left( h_1^2 - h_2^2 \right) = R_x L^2
\]

Example 3.6

For an unconfined aquifer with a hydraulic conductivity \( K = 1.75 \) m/day, an effective porosity of 0.3, and water depths of 10 and 8 m at two observation wells 200 m apart, calculate the discharge per unit width, the specific discharge, and the pore velocity.

**Solution.** From Equation 3.20, the discharge per unit width is

\[
q_x = \frac{K (h_1^2 - h_0^2)}{2L}
\]

If there is a uniform recharge with a rate \( R_o \), then \( q_x = R_o x \) with \( x = 0 \) at the groundwater divide and

\[
R_o x = K h \frac{dh}{dx}
\]

from which

\[
K \left( h_1^2 - h_0^2 \right) = R_o L^2
\]

Example 3.7

Estimate the seepage from a canal with a depth \( H_w = 1 \) m, dug in a soil with a hydraulic conductivity \( K = 2 \) m/day so that the distance from the bottom of the channel to the impervious stratum is \( D_w = 10 \) m, given a drop in the water table \( D_w = 0.5 \) m is observed at a distance \( L = 0.5 \) m from the shore. Is this seepage a significant portion of the flow?

**Solution.** Replacement in Equation 3.23 yields

\[
Q = 2KD_w \frac{D_i + H_w - 0.5D_w}{L - 0.5W_w}
\]

Bouwer (1969) states that this type of analysis gives reasonable results for \( D_i < 3W \). General solutions of the canal seepage problem have been given by Bouwer (1969, 1978), Harr (1962), Polubarinova-Kochina (1962), and Yussuff et al. (1994). The case of partially lined channels has been examined by Subramanya et al. (1973). Integrated groundwater-surface water models have been developed and applied to the Imperial County in southern California by Saquib et al. (1995) and Taghavi et al. (1995).
3.7.3 Recharge Basins
A third application of the Dupuit–Forchheimer assumptions is the analysis of the recharge of an unconfined aquifer from a recharge basin. A long rectangular basin of width \( W \) is considered with the assumption that the flow is horizontal and steady (Figure 3.12). Thus the depth of the aquifer, \( H \), should not be large compared with the width, \( W \), of the recharge basin, that is, \( H \leq W \), otherwise a more detailed analysis, such as a digital model, should be used. Letting \( R_e \) be the infiltration rate from the recharge area, \( x \) the horizontal distance from the centerline of the recharge area, \( h \) the height of the groundwater mound above the static water table, and approximating the average transmissivity \( T = K(H + h/2) \) by \( KH \) (Figure 3.12), the Dupuit–Forchheimer assumptions yield

\[
R_e x = -T \frac{dh}{dx} \tag{3.24}
\]

Integration of Equation 3.24 yields \( h_e - h_c = R_e W^2/(8T) \), where \( h_e \) and \( h_c \) are the height of the groundwater mound at the center and at the edge of the recharge basin, respectively. If \( h_e \) and \( h_c \) are measured in the field, then the aquifer transmissivity can be calculated.

Further discussion can be found in Bouwer (1970, 1978). An analytical solution has been given for rectangular recharging areas by Marínó (1975). Bouwer et al. (1999), and Bouwer (2002) developed additional equations to estimate heights of perched groundwater mounds above restricting layers in the vadose zone for narrow rectangular and circular recharge areas and gave design procedures for the design of recharge projects.

Example 3.8
A long recharge basin has a width \( W = 70 \) m and recharge rate \( R_e = 0.6 \) m/day. The observed heights of the mound are \( h_e = 1.2 \) m and \( h_c = 0.7 \) m. Find the aquifer transmissivity.

**Solution.** From the integration of Equation 3.24, \( T = 0.6(70)^2/[8(1.2 - 0.7)] = 735 \text{ m}^2/\text{day} \). Note that if \( H > 70 \) m, the transmissivity obtained is an "effective" transmissivity, as the lower layers of the aquifer are not fully contributing to the flow.

3.7.4 Steady Flow toward a Well in Confined and Unconfined Aquifers
Consider a well that fully penetrates an isotropic confined aquifer of hydraulic conductivity \( K \). The initial piezometric level is assumed to be horizontal so that originally there is no motion of the groundwater. As water is being pumped, it flows from the aquifer toward the well lowering the piezometric surface and creating a drawdown (Figure 3.13).

For a constant pumping rate, a steady state is eventually achieved. Consider two imaginary cylinders around the well with radii \( r_1 \) and \( r_2 \). The flows through each of these cylinders of height \( b \) are horizontal. Furthermore the flows must be equal to the discharge at the well for steady state conditions. Thus, from Darcy’s law

\[
Q = 2\pi r b K_i = 2\pi r b K_i \tag{3.25}
\]

where \( i_1 \) and \( i_2 \) are the values of the hydraulic gradient at radii \( r_1 \) and \( r_2 \), respectively. Since \( 2\pi b K \) is constant, and since \( r_1 < r_2 \), then \( i_1 > i_2 \). Thus the hydraulic gradient becomes steeper as the water approaches the well creating the cone of depression. As the same flow occurs through the two cylinders, the gross velocity, or specific discharge, increases as the well is approached. Writing Equation 3.25 as \( Q = 2\pi r b K \frac{dh}{dr} \) and integrating yields the *Thiem equation* for confined aquifers:

\[
Q = \frac{2\pi Kb (h_2 - h_1)}{\ln(r_2/r_1)} = \frac{2\pi T (h_2 - h_1)}{\ln(r_2/r_1)} \tag{3.26}
\]

where \( T = Kb \) is the *transmissivity* of the aquifer. This solution is applicable between \( r_\alpha \leq r < r_2 < R \), where \( R \) is the *radius of influence*. This radius delineates the region inside of which the effect of pumping is felt and outside of which the drawdown is taken to be zero.

In the case of an *unconfined aquifer*, the saturated depth \( b \) is not constant and decreases toward the well. Perhaps the most important application of the Dupuit assumption is the computation of steady flow toward a well in an unconfined aquifer as
done by Forchheimer. Assuming essentially horizontal flow, the discharge may be expressed as

\[ Q = \frac{\pi K (h_2^2 - h_1^2)}{\ln(r_2/r_1)} \]  

(3.27)

where \( h_2 \) and \( h_1 \) are interpreted as the saturated thicknesses at locations \( r_1 \) and \( r_2 \) for \( r_w \leq r_1, r_2 < R \). Equation 3.27 is known as the Dupuit–Forchheimer formula. Close to the well the vertical component of the flow comes into effect and the Dupuit–Forchheimer assumptions are not fully satisfied. Equation 3.26 can still be used with reasonable results for unconfined aquifers if \( T \) is interpreted as the average transmissivity \( K(h_1 + h_2)/2 \) and the drawdowns are small compared with the saturated thickness of the aquifer. Equations 3.26 and 3.27 can also be used with one observation well and considering the pumping well as the other observation point. Both can be solved for the hydraulic conductivity \( K \) given the discharge and the heads \( h_1 \) and \( h_2 \) at radii \( r_1 \) and \( r_2 \).

**Example 3.9**

A well with a radius of 50 cm completely penetrates an unconfined aquifer. It has been pumped for a long time at the rate of 15,000 m³/day. The drawdown in the well is 10 m. Find the hydraulic conductivity of the formation given that the well essentially does not affect the water table at a distance of 600 m = \( R \), where the depth of the water table is 50 m. Well losses are neglected.

**Solution.** Using Equation 3.27 with \( T = K(h_1 + h_2)/2 \) and \( h_1 = 50 - 10 = 40 \) m, and solving for \( K \) gives

\[ K = \frac{Q \ln(r_2/r_1)}{\pi(h_2^2 - h_1^2)} = \frac{15,000 \ln(600/0.50)}{\pi(50^2 - 40^2)} = 37.6 \text{ m/day} \]

Solving the well equation for the aquifer properties is called the **inverse problem**. The method employed in Example 3.9 using the Dupuit–Forchheimer equation only gives a rough approximation, as in practice steady state rarely exists and the transient flow formulas must be used instead. The transient flow methods (discussed in Chapter 29 on “Hydraulics of Wells and Aquifer Tests”) also yield the storage coefficient \( S \) in addition to the hydraulic conductivity \( K \).

### 3.8 Velocity Potential, Force Potential, and Flow Nets

The piezometric head \( h = z + p/\gamma \) is interpreted as an energy or potential per unit weight. The quantity

\[ \Phi = Kh + C = K\left(z + \frac{p}{\gamma}\right) + C \]  

(3.28)

where \( C \) is an arbitrary constant, is defined as the **velocity potential**, \( \Phi \). By virtue of Darcy’s law, the negative of its derivative in
the flow direction, for constant $K$, is the Darcy velocity, $q$. As a more general term, the negative of the gradient of the velocity potential is a velocity. This is the definition of the velocity potential in classical hydrodynamics. (Note. In Chapter 5 a minus sign is introduced in the right-hand side of Equation 3.28, then the velocity is the derivative of the velocity potential.)

Hubbert (1953, 1987) introduced the concept of the force potential $\psi$ (with $C = 0$ in Equation 3.28)

$$\psi = gh = \frac{g}{K} \phi$$

(Note. In Chapter 5 the symbol $\Psi$ is used for the “stream function,” Equation 5.16.) The force potential at a point is the work that is required to move a unit mass of fluid from a reference elevation and pressure to the elevation and pressure at the given point. In vector form, for $K$ constant,

$$q = -\text{grad} \Phi$$

or

$$q = -\frac{K}{g} \text{grad} \psi$$

The gradient of the force potential is the force per unit of mass acting upon the water at a given point. Formula 3.29b is a generalized Darcy’s law in three dimensions (Hubbert, 1953, 1987).

From Equation 3.31 $q$ is seen to have the same direction as $-\text{grad} \psi$ as long as $K$ is constant. Thus, the streamlines, which are lines everywhere tangent to the velocity vector, are perpendicular to lines of $\psi = \text{constant}$ or equipotential lines. The streamlines and the equipotential lines are orthogonal. A network of streamlines and equipotential lines form the flow net, which is a useful tool in the analysis of 2D flows. When the hydraulic conductivity is not constant, then Equation 3.30 must be used. With some practice, good flow nets can be drawn by hand. The equipotential lines are drawn so that the drop of head $\Delta h$ or potential drop $\Delta \Phi$ between adjacent lines is the same. The streamlines are drawn so that the same fraction of the total flow $\Delta Q$ takes place between adjacent streamlines. They are normal to the equipotential lines (Figure 3.14) forming “square” shapes. The flow between two streamlines, $\Delta Q$, is obtained from Darcy’s equation as

$$\Delta Q = Kbw \frac{\Delta h}{\Delta L}$$

FIGURE 3.14 Streamlines and equipotential lines in a flow net.

FIGURE 3.15 Flow net of a recharge area and gaining stream. (a) Plan view and (b) cross-section. (From Heath, R.C., Basic ground-water hydrology, U.S. Geological Survey, Water-Supply Paper 2200, 7th printing, Denver, CO, 1995.)
where

- \( b \) is the average depth of flow
- \( \Delta h \) and \( \Delta L \) are the difference in head and the distance between adjacent equipotential lines, respectively
- \( w \) is the width between adjacent streamlines for the square considered

The total flow through a group of \( n \) flow paths is

\[
Q = n \Delta Q
\]

Figure 3.15 shows a flow net for a recharge area in an unconfined aquifer.

The mathematics, construction, and application of flow nets are discussed in Chapter 5 on “Groundwater and Seepage.”

### 3.9 Laplace’s Equation

Laplace’s equation is fundamental to the analysis of many groundwater flow problems. It arises from the combination of Darcy’s law and the equation of continuity or conservation of mass for a homogeneous isotropic aquifer. Consider a steady flow of an incompressible fluid through an elementary cube (Figure 3.16) of a porous medium of porosity \( n \). Let \( u, v, w \) be the velocity components in the \( x, y, z \) directions, respectively. The inflow through the vertical face near the origin is \( nu \, dy \, dz \). The outflow through the vertical face away from the origin is \( nu \, dy \, dz \) plus \( (nu) \, dx \). The net change of volume in the \( x \)-direction (inflow rate–outflow rate) is

\[
- \frac{\partial}{\partial x} (nu) \, dx \, dy \, dz.
\]

The sum of the net changes of volume in the \( x, y, z \) directions must be equal to zero. Thus

\[
- \frac{\partial}{\partial x} (nu) + \frac{\partial}{\partial y} (nv) + \frac{\partial}{\partial z} (nw) = 0
\]

For an incompressible isotropic homogeneous porous medium, the equation of continuity for steady incompressible flow is thus

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]  \hspace{1cm} (3.33)

Using Darcy’s law for an isotropic homogeneous medium

\[
u = -K \frac{\partial h}{\partial x}, \quad v = -K \frac{\partial h}{\partial y}, \quad w = -K \frac{\partial h}{\partial z}
\]

Laplace’s equation is obtained in terms of the head \( h \):

\[
\nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0
\]  \hspace{1cm} (3.34)

If in Equation 3.33 the velocity components are expressed in terms of the velocity potential (see Equation 3.30)

\[
u = -\frac{\partial \Phi}{\partial x}, \quad v = -\frac{\partial \Phi}{\partial y}, \quad w = -\frac{\partial \Phi}{\partial z}
\]

then Laplace’s equation is expressed in terms of the velocity potential \( \Phi \)

\[
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0
\]  \hspace{1cm} (3.35)

Equation 3.35 is fundamental to the analysis of flow nets as developed in Chapter 5.

If the porous medium of a confined aquifer is assumed to be compressible with a compressibility \( \alpha \) (the reciprocal of the modulus of elasticity), and the water with a compressibility \( \beta \), then a specific storage coefficient \( S_s \) is introduced as

\[
S_s = \rho g (\alpha + n \beta)
\]  \hspace{1cm} (3.36)

\( S_s \) has the dimension \( L^{-1} \). The compressibility \( \alpha \) has a range of values from \( 10^{-6} \) to \( 10^{-4} \) \( \text{Pa}^{-1} \) for clays and a range of \( 10^{-9} \) to \( 10^{-11} \) \( \text{Pa}^{-1} \) for sound rock with intermediate values for sand, gravel, and jointed rock, and the compressibility of water, \( \beta \), is \( 4.4 \times 10^{-10} \) \( \text{Pa}^{-1} \) (Freeze and Cherry, 1979). The term \( \rho g \alpha \) represents the water yield from storage due to the compression of the porous medium and \( \beta \rho g n \) is the water yield resulting from the expansion of water storage. Integrating \( S_s \) over the thickness of the confined aquifer one obtains the storativity or storage coefficient \( S \). Thus, \( S \approx S_s \, b \), where \( b \) is the thickness of the aquifer. The storativity \( S \) is
dimensionless and for confined aquifers it is of the order of $5 \times 10^{-2}$ to $10^{-5}$ (de Marsily, 1986). For a horizontal confined aquifer, the storage coefficient is interpreted as the volume of water that the aquifer releases per unit surface area of aquifer per unit drop of the piezometric surface.

For an unconfined aquifer, $S$ is the drainage porosity or specific yield, which is the volume of water released per unit drop of the water table per unit horizontal area. The storage coefficient can be determined in the field from pumping tests as described in Chapter 10.

Since, for the elementary cube of Figure 3.16, the difference between the inflow and the outflow is now equal to the rate of change of storage, the flow is now unsteady, thus requiring the time derivative in the right-hand side of Equation 3.34. For 2D flow in a horizontal confined aquifer, Equation 3.34 becomes

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t} \tag{3.37}$$

where $S (\approx S \beta)$ is the aquifer storativity, $T = Kb$ is the transmissivity of the aquifer, and $b$ is the depth for a confined aquifer and is approximated as the average depth for an unconfined aquifer. If there is a leakage or inflow rate $q$ in the aquifer per unit area, then Equation 3.37 becomes

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{q}{T} = \frac{S}{T} \frac{\partial h}{\partial t} \tag{3.38}$$

The leakage rate can be calculated by Darcy’s law (see Equation 3.18).

The Dupuit–Forchheimer assumption that was applied to 1D unconfined flow (see Equations 3.19 and 3.20) can be generalized to the 2D case. In this case the inflows and outflows into a small parallelepiped of unconfined aquifer can be calculated by Darcy's law and their sum must be equal to zero for a steady incompressible flow. This yields

$$\frac{\partial^2 (h^2)}{\partial x^2} + \frac{\partial^2 (h^2)}{\partial y^2} = 0 \tag{3.39}$$

It is seen that for the Dupuit–Forchheimer assumptions, the square of the head must satisfy Laplace’s equation (Harr, 1962). If there is recharge at the rate $R_c$, then Equation 3.39 is replaced by (Fetter, 2004)

$$-K \left(\frac{\partial^2 (h^2)}{\partial x^2} + \frac{\partial^2 (h^2)}{\partial y^2}\right) = 2R_c \tag{3.40}$$

### Example 3.10

A river and a canal run parallel to each other $L = 500$ m apart (Figure 3.17). They fully penetrate an unconfined aquifer with a hydraulic conductivity of 0.3 m/day. The elevation of the water surface in the river is 1.25 m lower than in the canal where the depth is 5 m. Assuming no recharge, find the water table elevation midway between the river and the canal and find the discharge into the river.

**Solution.** The 1D form of Equation 3.39 is $d^2(h^2)/dx^2 = 0$, which integrates as $h^2 = c_1 x + c_2$. Applying the boundary conditions at $x = 0$, $h = h_1$ and $x = L$, $h = h_0$, one obtains $h^2 = h_1^2 - (h_1^2 - h_0^2) x/L$. Thus, with $h_1 = 5$ m, $h_2 = 3.75$ m, and $L = 500$ m, $h = [5^2 - (5^2 - 3.75^2) \times 0.5]^1/2 = 4.42$ m. The seepage into the river is given by

$$q = -K(h/dx) = K \left(h_1^2 - h_2^2\right)/2L = 0.3(5^2 - 3.75^2)/1000 = 0.0033 \text{ m}^3/(\text{m}/\text{day})$$

### 3.10 Land Subsidence

Pumping large volumes of water from confined aquifers at rates substantially larger than the natural recharge causes a contraction of the aquifer that can result into a downward movement of the land surface. In unconfined aquifers, groundwater pumping causes a downward movement of the water table that likewise can lead to a downward movement of the land surface. This downward movement of the land surface is called subsidence or consolidation. This movement can be a few centimeters to several meters. If the subsidence is not uniform, the differential settlement can produce severe damage to structures. Important subsidence has occurred in the San Joaquin Valley in California, in Mexico City, in Venice, around Shanghai, and in southern Taiwan.

Galloway et al. (1999) give a detailed description of subsidence in the United States where 17,000 square miles in 45 states have been affected. Man-induced subsidence is caused by three basic mechanisms: (1) compaction of aquifer systems due to groundwater mining, (2) dewatering of organic soils, and (3) mass wasting through dissolution and collapse of rocks that are relatively soluble in water such as limestone, dolomite, and evaporites. Examples of these mechanisms are (1) the subsidence near Mendota in the San Joaquin Valley in California that in places, exceed 28 ft, (2) the extensive land subsidence caused by drainage and oxidation of peat soils in the Everglades in Florida, and (3) the dissolution and erosion of carbonate rocks.
(limestone and dolomite) in Missouri and solution of evaporites (salt and gypsum) in the Permian basin of Texas, New Mexico, Oklahoma, and Kansas.

Large subsidence that results from excessive pumping tends to occur in aquifers containing interbedded layers of silt and clay. As the drainage of the aquitard squeezes fluid from its interior, the internal stress increases. This causes the compressibility to increase by a factor of 20–100 resulting in compaction that is mostly not recoverable. Decrease of the groundwater pumpage can reduce and sometimes reverse the subsidence as is currently the case in the Harris-Galveston area in south central Texas (Hibbs, 1997). UNESCO has published a guidebook to studies of land subsidence due to groundwater withdrawals (Poland, 1984). The next section describes the calculation of subsidence by this mechanism.

Dewatering of organic soils accelerates the aerobic decomposition of the plant litter as compared to the anaerobic decomposition that occurs under undrained conditions. The natural rate of accumulation of organic soil is of the order of a few inches per century, whereas the rate of loss of drained organic soil can be 100 times greater.

Limestone dissolution and groundwater flow result in a geomorphology known as karst. It includes sinkholes, springs, caves, subsurface drainage networks. The frequency of occurrence of sinkholes can be accelerated by groundwater springs, caves, subsurface drainage networks. The next section describes the calculation of subsidence by this mechanism.

**3.10.1 Calculation of Subsidence**

Consider a unit area of a horizontal plane at a depth Z below the ground surface. The total downward pressure \( P_t \) due to the weight of the overburden on the plane is resisted partly by the upward hydrostatic pressure \( P_h \) and partly by the intergranular pressure \( P_i \) exerted between the grains of the material: \( P_t = P_h + P_i \) or \( P_t = P_i - P_h \).

A lowering of the water table results in a decrease of the hydrostatic pressure and a corresponding increase of the intergranular pressure. If \( P_{i1} \) and \( P_{i2} \) denote the intergranular pressures before and after a drop in the water table or piezometric surface, the vertical subsidence can be calculated as

\[
S_v = Z \frac{P_{i2} - P_{i1}}{E} \quad (3.41)
\]

where \( Z \) is the thickness of the soil layer and \( E \) is the modulus of elasticity of the soil. Typical ranges of values of \( E \) are given in Table 3.4. The reciprocal of the modulus of elasticity is the **compressibility**, generally designated by the symbol \( \alpha \) and given in inverse Pascal or square meter per Newton. In general, the modulus of elasticity increases nonlinearly with the intergranular pressure. If there are layers of different soil types, the subsidences are calculated separately for each layer and added to obtain the total subsidence. As the modulus of elasticity of clayey materials is much less than that of sand or gravel, most of the settlement occurs in the clayey layers.

The previous equation can also be used to calculate the rebound when the intergranular pressure decreases. Caution must be exercised because the modulus of elasticity is not the same for decompression as for compression. This is particularly the case for clays. For Boston blue clay, the rebound modulus of elasticity is only about 50% of that for compression (Bouwer, 1978, p. 323). If subsidence has occurred for a long time, complete rebound is unlikely.

### Example 3.11

Consider a 60 m thick sand layer. The water table is located at a depth of 10 m below the ground surface. Calculate the total and the intergranular pressures at 10 m depth and at the bottom of the sand layer, given that the porosity of the sand is \( n = 0.35 \), its volumetric water content above the water table is \( \theta = 0.08 \), the specific weight of the solids is \( \gamma_s = 25.5 \) kN/m\(^3\), and the specific weight of the water is \( \gamma_w = 9.81 \) kN/m\(^3\).

**Solution.** At the water table the intergranular pressure, which is also the total pressure, is \( P_i = 10(1 - 0.35)25.5 + 0.08 \times 9.81) = 173.6 \) kPa. The total pressure at the bottom of the sand layer is \( P_t = 173.6 + 50(1 - 0.35)25.5 + 0.35 \times 9.81) = 1174.0 \) kPa. The hydrostatic pressure at the bottom of the sand layer is \( P_h = 9.81 \times 50 = 490.5 \) kPa. The intergranular pressure is thus 1174.0 - 490.5 = 683.5 kPa.

### Example 3.12

If in Example 3.11 the water table drops 40 m, what is the change in intergranular pressure at the bottom of the sand layer? See Figure 3.18.

**Solution.** The depth to the water table would then be 50 m. The total pressure at the bottom of the sand layer...
3.10.2 Seepage Force

When water flows horizontally through an aquifer, the flow undergoes a reduction of pressure head because of friction. Thus the pressure on the upstream side of a small element is larger than on the downstream side. The water then exerts a net force on the aquifer element. The net force in the flow direction is the seepage force. This force can cause lateral displacements. If the drop of the water table in a length $L$ is $\Delta h$, the horizontal movement is $S_h = \gamma_w \Delta h L / E_h$, where $E_h$ is the modulus of elasticity in the horizontal direction. Letting $i$ be the slope of the water table and $\Delta h = iL$, the horizontal movement, $S_h$, is calculated as

$$S_h = \gamma_w i L^2 / E_h$$

(3.42)

If there is an upward vertical flow, the head loss due to friction, as the water flows into the pores, results in an increase in the hydrostatic pressure. This in turn results in a decrease of the intergranular pressure. A point can be reached when the upward seepage force is large enough to carry the weight of the sand grains so that the sand or silt behaves like a liquid. It has no strength to support any weight on it. This condition is known a quicksand. It is reached when the intergranular pressure vanishes and sand loses its bearing capacity. It can be shown that the upward hydraulic gradient necessary to produce quicksand is very close to one (Harr, 1962; Bauwer, 1978).

3.11 Salt Water Interfaces

The fresh and saline groundwaters have densities $\rho_f$ and $\rho_s$. In coastal aquifers under natural conditions, the lighter fresh water lies over the heavier saline water and the flow is usually from the aquifer to the sea. Mixing of fresh water and salt water occurs only by molecular diffusion. Turbulent diffusion, the most effective mixing mechanism, is absent in aquifers. As a result, the mixing zone between salt water and fresh water is small compared with the thickness of the aquifer and an abrupt well-defined interface is usually assumed. At a point on the interface between the fresh and saline waters, the pressure of the fresh water, $\rho_f g h_f$, usually exceeds the pressure of the saline water, $\rho_s g h_s$, causing the flow from land to sea (Figure 3.19). But when pumping takes place in excess of replenishment, the drawdown of the water table creates a piezometric head in the fresh water that becomes less than in the adjacent salt water wedge. Then the saline water moves inland causing a salt water intrusion. The salt water may reach the well that becomes contaminated. Salt water intrusions have occurred in many coastal aquifers, for example, in the coastal aquifers of Florida, California, the Netherlands, Israel, and the South coast of England, mostly because of excessive pumping.

Assuming static conditions, the seepage zone is reduced to a point. The pressure at point A on the interface must be the same...
on the salt water side and on the fresh water side. Thus, the depth $h_i$ of the interface below sea level is (Figure 3.19)

$$h_i = \frac{\rho_t}{\rho_s - \rho_t} h$$

(3.43)

where $h$ is the height of the water table above sea level. With $\rho_s/\rho_t = 1.025$, the fraction in Equation 3.43 is equal to 40. For a confined aquifer, $h$ is the fresh water piezometric head. The wedge of fresh water is known as the Ghyben–Hertzberg lens after the Dutch and German scientists who first obtained Equation 3.43.

When fresh water is pumped from an aquifer overlying a body of salt water, the drawdown of the fresh water table around the well causes a pressure reduction on the interface. This in turn causes the interface to rise below the well. This is called upconing. If the salt water cone reaches the well, it will discharge a mixture of salt and fresh water. For a water table well, assuming hydrostatic conditions, a rough approximation of the height $\delta$ of the cone is (Figure 3.20)

$$\delta = \frac{\rho_t}{\rho_s - \rho_t} s_w$$

(3.44)

where $s_w$ is the drawdown at the well. More exact relationships are given in Chapter 12 on “Sea Water Intrusion into Coastal Aquifers” and, among others, in Bear and Dagan (1964), Dagan and Bear (1968), and Bear (1979).

### 3.12 Groundwater Quality

The quality of groundwater is determined by the dissolved elements and gases and by the presence of suspended solids, bacteria, and viruses. The quality of the groundwater depends upon its natural and physical state and on the changes due to human activity. In its natural state, the dissolved elements and their concentrations depend on the chemical composition of the aquifer and on the travel time of the water through the rock formation. If the rock minerals are relatively soluble, slow water velocity and the ensuing long travel time result in a chemical equilibrium between the water and the rock medium. Because of the large range of flow velocities and of chemical compositions of the aquifers, there is a very wide range of compositions of the groundwater. Figure 3.21 shows the comparison of the range of concentrations of several constituents in groundwater to the concentrations of a 2.7 g sugar cube dissolved in a 2.7 l bottle, in a gasoline tank and in an oil tanker.

If the groundwater is no longer fit for a specific use, such as drinking, the water is said to be contaminated. If the water becomes heavily contaminated it is said to be polluted. Chapter 17 discusses in detail the types of groundwater contaminants, and a table of drinking water standards can be found in Table 32.2 in Chapter 32 on the legal framework for groundwater in the United States. Groundwater monitoring for water quality is discussed in Chapter 35.

Figure 3.22 illustrates the principal groundwater contaminant sources. These sources can be classified as point, line, or nonpoint (areal) sources. The geometry of the source affects the geometry of the contaminant plume. Contaminants can reach the groundwater in several manners. The contaminant may be miscible, that is, it can be dissolved in water, immiscible, that is, the water and the contaminant are in separate phases: lighter or heavier than water and the contaminant can be adsorbed on fine particles that are transported in suspension by the water. The miscible liquid and the water form a single phase. The movement of this single phase flow is governed by Darcy’s law, discussed earlier in this chapter. Solutes are further classified as conservative and nonconservative. Conservative solutes remain stable in the groundwater: they do not react with the rock medium. Chloride solutions fall in this category. Tracers are conservative constituents that do not affect the viscosity and density of the water, such as fluoresceine in low concentration, for example. Conservative and nonconservative transport processes are discussed in detail in Chapters 18 and 19, respectively. Multiphase flow occurs primarily in the unsaturated zone, for example in the case of spills of hydrocarbons resulting in hydrocarbon and water phases. Multiphase flow is discussed in detail in Chapter 8 entitled “Contaminant Transport in the Unsaturated Zone—Theory and Modeling.”
3.13 Transport Mechanisms of Dissolved Contaminants

The principal fate and transport mechanisms are advection, diffusion, dispersion, sorption, and decay. These processes are discussed in their simplest 1D form in this section and in more detail in Chapter 18 and Chapter 19. Other processes such as hydrolysis, volatilization, and biotransformation are not discussed in this section. Biotransformation is discussed in Chapter 31. The combined effects of advection, dispersion, and biodegradation on the transport of contaminants were visualized in the Cape Cod experiments described in LeBlanc et al. (1991) (see also Section 18.10.2) as well as in the Experiments in the Columbus Air Force Base in Mississippi that are discussed in Chapter 26.

### 3.13.1 Advection

Advection is the transport of solute by the bulk groundwater flow. The average pore velocity, $v$, is obtained by dividing the Darcy flux $q$ (see Example 3.4) by the effective porosity $n_e$

$$v = \frac{q}{n_e} \quad (3.45)$$

The 1D mass flux due to advection, $F$, is the product of the quantity of water flowing and the concentration of dissolved solids

$$F = vn_eC \quad (3.46)$$

The change of mass of contaminant over time in a control volume

$$n_e \frac{\partial C}{\partial t} \, dx \, dy \, dz$$

is equal to the balance between the mass inflow and outflow of contaminant

$$Fdz \, dy - \left( F + \frac{\partial F}{\partial x} \right) dx \, dy \, dz = - n_e \frac{\partial}{\partial x} (vC) \, dx \, dy \, dz$$

Thus, for a conservative solute in a homogeneous aquifer, the 1D advective transport equation is

$$\frac{\partial C}{\partial t} = - v \frac{\partial C}{\partial x} \quad (3.47)$$

where

- $x$ is distance in the flow direction (L)
- $t$ is time (T)
- $C$ is the concentration (ML$^{-3}$),
- $v$ is the advective transport velocity (LT$^{-1}$)

It is observed that a precise estimation of the flow velocity is needed for an accurate estimation of the transport. In sand/gravel aquifers with significant groundwater, the plume movement is dominated by advection. However, it must be recalled that Equation 3.47 is not valid in karstic aquifers for which Darcy’s law is not applicable.

### 3.13.2 Diffusion

Diffusion is the flux of solute from a zone of higher concentration to one of lower concentration due to the Brownian motion
of ionic and molecular species. Under steady-state condition, the
diffusion flux $F$ is described by Fick’s first law

$$ F = -D \frac{\partial C}{\partial x} \quad (3.48) $$

where $D$ is the diffusion coefficient ($L^2T^{-1}$). For diffusion in water,
$D$ for different cations and anions ranges approximately from $6 \times 10^{-10}$ to $9 \times 10^{-9} \text{m}^2/\text{s}$ (Fetter, 1999). For diffusion in porous media,
Freeze and Cherry (1979) suggest taking an effective diffusion coefficient
$D^* = \omega D$ to account for the tortuosity of the flow paths with
$\omega$ ranging from 0.5 to 0.01 for laboratory studies of nonadsorbed ions in porous geological materials. A value of $\omega = 0.7$ was
obtained by Perkins and Johnson (1963) for uniform sand columns. The change of concentration over time inside a control volume
subject to diffusion is given by Fick’s second law

$$ \frac{\partial C}{\partial t} = D^* \frac{\partial^2 C}{\partial x^2} \quad (3.49) $$

Consider the case of two adjacent saturated strata. The first one
is initially with zero concentration throughout its length
$C(x, 0) = 0$ and the second stratum maintains constant concentration
at the interface $C(0, t) = C_0$. Because of the concentration
step at the interface, diffusion takes place from the interface into
the first stratum. Far away in the first stratum, where the effect
of diffusion has not yet reached, the concentration is still zero,
namely $C(\infty, t) = 0$. Crank (1956) gives the following solution of
Equation 3.49 subject to the above boundary and initial conditions
for the evolution of the concentration as a function of time and
distance from the interface:

$$ C(x, t) = C_0 \text{erfc} \left( \frac{x}{2 \sqrt{D^* t}} \right) \quad (3.50) $$

where erfc is the complementary error function

$$ \text{erfc}(u) = 1 - \text{erf}(u) \quad (3.51) $$

and erf is the error function

$$ \text{erf}(u) = \frac{2}{\pi} \int_0^u e^{-v^2} \, dv \quad (3.52) $$

Tables of the error function and of the complementary error function can be found, for example, in Appendix 3A.

**Example 3.15**

Consider a substance diffusing at the boundary of a clay
layer with an effective diffusion coefficient $10^{-9} \text{m}^2/\text{s}$. Find
the time it will take to obtain a concentration of
15% of the initial concentration at a distance of 20 m.

**Solution.** Equation 3.50 yields $\text{erfc} \left( \frac{x}{2 \sqrt{D^* t}} \right) = 0.15$. Interpolating from a table of complementary
error functions, one finds $x/2 \sqrt{D^* t} = 1.02$. Thus

$$ t = \frac{400}{4 \times 10^{-10} (1.02)^2} = 9.61 \times 10^6 \text{ s} \approx 3047 \text{ year} $$

This shows that diffusion is a very slow mechanism but
over geological time it can become important.

The importance of diffusion increases as flow velocities decrease. Thus diffusion may be the governing transport mechanism in unfractured clays with low
hydraulic conductivities. Diffusion can generally be neglected in gravel aquifers with high flow velocities. It can also be significant in fractured porous aquifers.
Flow and transport processes in fractured rocks are discussed in Chapter 20.

### 3.13.3 Dispersion

Dispersion is the spreading of the plume that occurs along
and across the main flow direction due to aquifer heterogene-
ities at both the small scale (pore scale) and at the macroscale
(regional scale). Dispersion tends to increase the plume uni-
formity as it travels downstream. Factors that contribute to
dispersion include faster flow at the center of the pores than at
the edges, some pathways being longer than others, flow velocity
larger in smaller pores than in larger ones. This is known as mechanical dispersion. The spreading due to both mechanical
dispersion and molecular diffusion is known as hydrodynamic dispersion.

As a conceptual example, consider an aquifer with an abrupt
concentration front at $t = 0$. At that time there is a tracer, such
as sodium chloride, with a concentration $C = 1$ to the left of the
front and there is no tracer to the right of the front, that is $C = 0$,
as shown in Figure 3.23a. At a later time $t$, the center of the front
has moved through a distance $L = vt$, where $v$ is the pore flow
velocity. But, due to dispersion, the tracer has spread around the
center as shown in the lower part of Figure 3.23a. The plot of the
S-shaped curve of the concentration at time $t$, $C(t)$, is called the breakthrough curve. In a second experiment, a quantity of tracer is
injected at point $x = 0$ at time $t = 0$. At a later time $t_0$, the center of the plume has moved a distance $L_0 = vt_0$ but, due to dispersion,
the tracer has spread around the center with elliptical concentration contours as shown in Figure 3.23b. At time $t_0$, the spreading has extended further as shown in the figure.

If one considers a representative elementary volume REV
(see Chapter 4; Section 4.1.3), dispersion can be described by
Fick’s law (Equation 3.48). The dispersion coefficient $D_r$ replaces
the diffusion coefficient $D^*$ and becomes a phenomenological
coefficient that combines the effects of diffusion and dispersion.
As mechanical dispersion is more pronounced in the longitudi-
unal direction than in the transverse direction, a longitudinal dispersion coefficient $D_l$ and a transverse dispersion coefficient $D_t$.
are introduced. The longitudinal and the transverse dispersion–

diffusion coefficients are defined as

\[ D_L = \alpha_L v + D^* \]  \hspace{1cm} (3.53)

and

\[ D_T = \alpha_T v + D^* \]  \hspace{1cm} (3.54)

where

- \( \alpha_L \) is the longitudinal dispersivity (L)
- \( \alpha_T \) is the transverse dispersivity (L)
- \( v \) is the pore velocity (LT\(^{-1}\))

The use of the Fickian theory to describe dispersion requires that

dispersivity coefficients be travel–distance dependent or time

dependent. A rough approximation based on averaging published data is (Gelhar, 1986, Gelhar et al., 1992)

\[ \alpha_L \approx 0.1L \]  \hspace{1cm} (3.55)

where \( L \) is the length of the flow path. Another estimate for flow lengths between 100 and 3500 m was given by Neuman (1990) as

\[ \alpha_L \approx 0.0175L^{1.46} \]  \hspace{1cm} (3.56)

where \( \alpha_L \) and \( L \) are in meters. Xu and Eckstein (1995) gave an estimate of \( \alpha_L \) that does not have travel length limitation as

\[ \alpha_L \approx 0.83 (\log L)^{2.414} \]  \hspace{1cm} (3.57)

where \( \alpha_L \) and \( L \) are in meters. This regression equation is based on weights of 1:2:3 to points of low, medium, and high reliability, respectively. When the scale is greater than 1 km, then the increase in longitudinal dispersivity becomes so small that it can be ignored without significant error.

The transverse dispersivity \( \alpha_T \) is typically 1/10 to 1/100 of the

longitudinal dispersivity \( \alpha_L \).

The combined advection–dispersion equation in 1D is thus

\[ \frac{\partial C}{\partial t} = -v \frac{\partial C}{\partial x} + D_L \frac{\partial^2 C}{\partial x^2} \]  \hspace{1cm} (3.58)

The solution of Equation 3.58 subject to the fixed step conditions:

- Initial condition \( C(x, 0) = 0, x \geq 0 \)
- Boundary condition \( C(0, t) = C_0, t \geq 0 \)
- Boundary condition \( C(\infty, t) = 0, t \geq 0 \)

has been given by Ogata and Banks (1961) as

\[ C(x, t) = \frac{C_0}{2} \left[ \text{erfc} \left( \frac{x - vt}{2\sqrt{Dt}} \right) + \exp \left( \frac{vx}{Dt} \right) \text{erfc} \left( \frac{x + vt}{2\sqrt{Dt}} \right) \right] \]  \hspace{1cm} (3.59)
where \( x \) is the distance from the injection point. The argument of the exponential is the Peclet number \( (P_e = v x / D) \). The Peclet number is a measure of the ratio of the rate of transport by advection to the rate of transport by diffusion. For large Peclet numbers \( (P_e > 100) \), the advection dominates and the second term in the right-hand side becomes negligible. Figure 3.24 shows the behavior of the solution (3.59).

If instead of a fixed step function a line source with continuous injection into the aquifer is considered, for example, the leakage from a canal, Sauty (1978) found that in the solution (3.59) the + sign in the right hand side is replaced by a − sign. For large Peclet numbers the fixed step solution and the line source solution are essentially identical as the second term in the right hand side of Equation 3.59 is negligible. Further discussion and other boundary conditions are considered in Fetter (1999).

Example 3.16

An aquifer has a hydraulic conductivity of \( 2 \times 10^{-5} \) m/s, a hydraulic gradient of 0.003 m/m, an effective porosity \( n_e = 0.2 \), and an effective diffusion coefficient \( D^* = 0.5 \times 10^{-9} \) m²/s. A chloride solution with a concentration of 500 mg/L penetrates in the aquifer along a line source. Find the chloride concentration at a distance of 20 m from the point of entry, after a period of 2 years.

Solution. From Darcy’s law, the pore flow velocity is

\[
v = \frac{K}{n_e} \frac{dh}{dx} = 2 \times 10^{-5} \times 0.003 \times 0.2 = 3 \times 10^{-7} \text{ m/s}
\]

Using Neuman’s approximation (3.56 and 3.57), the longitudinal dispersivity is

\[
\alpha_L = 0.0175 \times 20 = 0.35 \text{ m}
\]

The coefficient of longitudinal dispersion-diffusion, from Equation 3.53 is

\[
D_L = 1.388 \times 0.5 \times 10^{-7} + 4.170 \times 10^{-7} = 5.558 \times 10^{-7} \text{ m}^2/\text{s}
\]

The Peclet number is \( P_e = v L / D_L = 3 \times 10^{-7} \times 20 / 5.558 \times 10^{-7} = 67.54 \). As \( P_e < 100 \), both terms in the right-hand side of Equation 3.59 need to be considered. Thus after 2 years = \( 2 \times 365 \times 24 \times 60 \times 60 = 6.3072 \times 10^7 \) s and at a distance of 20 m the concentration is obtained from Equation 3.59 as

\[
C = 500 \left[ \frac{1}{2} \left( \text{erfc} \left( \frac{20 - 3 \times 10^{-7} \times 6.3072 \times 10^7}{2 \sqrt{4.170 \times 10^{-7} \times 6.3072 \times 10^7}} \right) \right) 
+ \exp(14.384) \times \text{erfc} \left( \frac{20 + 3 \times 10^{-7} \times 6.3072 \times 10^7}{2 \sqrt{4.170 \times 10^{-7} \times 6.3072 \times 10^7}} \right) \right]
\]

\[
= 250(\text{erfc} (0.105) + 14.384 \times \text{erfc} (3.795)) = 250 \times 0.0881984 = 220.5 \text{ mg/L}
\]

as the last erfc (·) is negligible as seen in Appendix 3A.

3.13.4 Sorption

Sorption refers to the exchange of molecules and ions between the solid phase and the liquid phase. It includes adsorption and desorption. Adsorption is the attachment of molecules and
ions from the solute to the rock material. Adsorption produces a decrease of the concentration of the solute or, equivalently, causes a retardation of the contaminant transport compared with water movement. Desorption is the release of molecules and ions from the solid phase to the solute.

The relationship between the solute concentration in the adsorbed phase and in the water phase is called a sorption isotherm. The simplest expression is the linear isotherm

\[ C_a = K_d C \]  \hspace{1cm} (3.60)

where

- \( C_a \) is the sorbed concentration as mass of contaminant per mass of dry rock matrix (dimensionless)
- \( C \) is the dissolved concentration in mass of contaminant per volume of water (ML\(^{-3}\))
- \( K_d \) is the distribution coefficient (L\(^3\)M\(^{-1}\))

This expression implies that there is equilibrium between the adsorbed concentration and the dissolved concentration. This can be assumed when the adsorption process is fast compared with advection of contaminant.

The adsorption causes retardation in the migration of contaminants compared with advection. The contaminant transport gets more retarded as the fraction adsorbed increases. This effect can be described by a retardation factor, \( R_a \), which for a linear isotherm, is

\[ R_a = 1 + \frac{(1-n)p_s}{n} K_d \]  \hspace{1cm} (3.61)

where

- \( n \) is the porosity
- \( p_s \) is the density of the solids

The retardation coefficient may take values from 1 to 10,000. The velocity of the solute front \( v \) (where the concentration is half that of the original concentration) is given by

\[ v_a = \frac{v}{R_a} \]  \hspace{1cm} (3.62)

The 1D advection–dispersion equation then becomes

\[ \frac{\partial C}{\partial t} = -v \frac{\partial C}{\partial x} + \frac{D_L}{R_a} \frac{\partial^2 C}{\partial x^2} \]  \hspace{1cm} (3.63)

where the term on the left side represents the time rate of change in storage of contaminant in the control volume, the first term on the right-hand side represents the retarded advective inflow–outflow, and the last term represents the retarded diffusion and dispersion. Solutions of the advection/dispersion/adsorption equation have been given by van Genuchten (1981).

The linear isotherm also applies to the case of organic compounds dissolved in groundwater. If the aquifer or soil contains at least 1% of organic carbon, then

\[ K_d = K_{oc} f_{oc} \]  \hspace{1cm} (3.64)

where

- \( K_{oc} \) is the organic carbon partition coefficient for the organic solute
- \( f_{oc} \) is the fraction of organic carbon by weight in the soil

Methods for estimating \( K_{oc} \) from solubility data and from the octanol–water partition coefficient can be found, for example, in Fetter (1999) and in Spitz and Moreno (1996) (see Table 5.9 in Chapter 5 on "Groundwater Contaminants"; see also Section 3.15).

In this section it was assumed that the adsorption was fast compared with the advection of the contaminant. If instead the reaction is slow compared to the travel time and if chemical equilibrium cannot be attained, then it is necessary to describe the kinetics of the reaction. The reader is referred to Chapter 19 and to de Marsily (1986) or Fetter (1999).

### 3.13.5 Radioactive Decay and Degradation

The simplest model for decay of contaminants without transport is first order model:

\[ \frac{\partial C}{\partial t} = -\lambda C \]  \hspace{1cm} (3.65)

where \( \lambda \) is the first order decay rate constant (T\(^{-1}\)). This relation also applies to radioactive decay and degradation processes. Equation 3.65 integrates as

\[ C = C_0 e^{-\lambda t} \]  \hspace{1cm} (3.66)

where \( C_0 \) is the concentration at time \( t = 0 \) and

\[ \lambda = \frac{\ln 2}{T_{1/2}} \]  \hspace{1cm} (3.67)

where \( T_{1/2} \) is the half-life (T) of the radioactive isotope or of the degraded contaminant. Some values of radioactive and organic half-lives can be found in Spitz and Moreno (1996). The degradation causes a mass \( \lambda C \) to disappear per unit volume per unit of time. The transport equation including decay is obtained by appending the quantity \(-\lambda C\) to the right-hand side of Equation 3.63.
3.14 Monitoring, Site Remediation, and Landfills

Monitoring wells are necessary to measure the elevations of the water table or of the piezometric level, to collect samples of water for chemical analyses, to collect samples of nonaqueous phase liquids, to provide access for geophysical instruments, etc. The techniques used in groundwater monitoring and soil sampling are discussed in detail in Chapter 35 and the geophysical exploration techniques requiring wells and of other types are presented in Chapters 14 and 15.

Site remediation must consider at least source control and treatment of contaminated water or soil or both. Source control is necessary to prevent continuing discharge of contaminants to the subsurface or the groundwater. Treatment may be necessary to remove or substantially decrease the concentration of contaminants. Techniques for the proper design of landfills have been developed to eliminate or minimize the leakage to the vadose zone or the groundwater. These activities are regulated by legislation such as the Resource Conservation and Recovery Act (RCRA) and the Comprehensive Environmental Response, Compensation and Liability Act (CERCLA).

These topics of regulations, landfills, and site remediation are the subject of Chapters 32, 33, and 36, respectively.

3.15 Parameter Values

In the previous sections, a number of equations have been presented that describe the flow of groundwater and the transport of contaminants. The application of these equations and of the models that are based upon them requires the use of a number of parameters. In general, the hydrogeological parameters and the pollutant characteristics exhibit great variability and hence uncertainty exists in the model predictions. In many applications data may not be readily available. It may then be useful to have access to a database that brings together most of the previous experience, at least until field experiments can be conducted.


Appendix

Values of the Error Function and Complementary Error Function

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<th>erfc x</th>
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\[
\text{erf}(x) = \left( \frac{2}{\sqrt{\pi}} \right) \int_{0}^{x} e^{-t^2} dt
\]

Glossary

**Adsorption**: Adhesion of solute molecules and ions to the rock or soil material.

**Advection**: Mass transport of solute by the gross movement of groundwater.

**Conservative solute**: Solute that remains stable and does not react with rock or soil material.

**Contaminant**: Substance that causes contamination.

**Contamination**: Degradation of groundwater quality that renders it unfit for domestic consumption.

**Darcy’s law**: An equation that relates the gross flow velocity (i.e., the discharge divided by the gross cross-section of an aquifer segment) to the product of the hydraulic conductivity and the gradient of the total head.

**Datum**: An arbitrary reference elevation from which the hydraulic heads are measured.

**Desorption**: Removal of molecules or ions from the rock or the soil.

[Values of the Error Function and Complementary Error Function are presented in the table above.]

[The Handbook of Groundwater Engineering]
Diffusion: The flux of solute from areas of higher concentration to areas of lower concentration due to random molecular motion.

Dispersion: The spread of solute due to heterogeneities of the pore sizes and shapes (mechanical dispersion) and heterogeneities in the aquifer (macrodispersion).

Dispersion coefficient: A coefficient in Fick’s law that relates mass flux to concentration gradient.

Dispersivity: A constant of dispersion that, when multiplied by the pore flow velocity, yields the dispersion coefficient.

Dupuit–Forchheimer assumption: The assumption of primarily horizontal flow, neglecting unimportant vertical flow.

Effective porosity: The part of the porosity that is available for the fluid flow.

Elevation head: Difference in elevation between a point in a flow field and an arbitrary reference datum, the latter often being taken as the mean sea level.

Equipotential line: Line of equal potential, used in the flow net.

Fick’s law: Equation that describes the diffusion or dispersion of solutes.

Force potential: The product of the acceleration of gravity and the total head; it represents the total energy per unit of mass.

Flow net: A network of streamlines and equipotential lines that intersect at right angles.

Half-life: The time required for the concentration of a solute to be reduced to half its initial value by radioactive decay or biodegradation.

Head: See elevation head, pressure head, velocity head, and total head.

Hydraulic gradient: The rate of change of the piezometric head with displacement in a given direction.

Karst: Geological formation characterized by features associated with dissolution and collapse of carbonate rocks such as underground drainage, caves, sinkholes, and deep gullies. Named after the Karst plateau, a barren limestone plateau in West Slovenia.

Leakage: Seepage of water through a semipermeable layer called an aquitard.

Leaky aquifer: An aquifer into which there is seepage from an overlying formation.

Linear isotherm: A chemical equilibrium relationship in which the concentration of the adsorbed solute in the solid phase is assumed to be proportional to the concentration in the water phase.

Longitudinal dispersion coefficient: Dispersion coefficient in the flow direction.

Monitoring well: A nonpumping well used to measure water levels or to obtain water samples for chemical analysis.

Organic: Pertaining to carbon compound usually in rings or chains and with other elements, not necessarily derived from a living organism.

Peclet number: A dimensionless quantity that expresses the relative importance of convection and dispersion of solutes.

Piezometric head: The sum of the elevation head and the pressure head.

Pollutant: Substance that causes pollution.

Pollution: Excessive contamination of the environment, in this chapter specifically contamination of groundwater that renders it unfit for human consumption as a result of human or natural activities.

Pressure head: The ratio of the fluid pressure intensity to the fluid specific weight; it has the dimension of length.

Retardation factor: Ratio of transport velocity of nonreacting solute to transport velocity of solute reacting with the solid phase.

Reynolds number: A dimensionless quantity that expresses the relative importance of inertia forces and viscous forces in a flow system. A small Reynolds number is associated with laminar flow; a large Reynolds number is associated with turbulent flow.

Salt-water interface: A surface forming a common boundary between adjacent salt water and fresh water.

Sorption: Includes adsorption and desorption.

Specific discharge: The flow rate through a cross-section of an aquifer divided by the area of that cross-section.

Streamline: A line that is everywhere tangent to the flow velocity vector, also called the flow line. It is used in the construction of flow nets.

Suction: A negative pressure head found in the unsaturated zone.

Total head: The sum of the elevation head, pressure head, and velocity head. In flow through porous media, the velocity head is small and often neglected in the calculation of the total head.

Tracer: A conservative solute that is used to track the path of groundwater movement.

Transverse dispersion coefficient: Dispersion coefficient in the direction perpendicular to the flow direction.

Velocity head: The kinetic energy of the flow per unit weight of fluid; it has the dimension of length.

Velocity potential: The product of the hydraulic conductivity and the total head. By virtue of Darcy’s law, the negative of the gradient of the velocity potential is a flow velocity (specific discharge or Darcy velocity).

Water balance: An accounting of the inflows and outflows in a fluid control volume.

References


Further Information

This chapter deals with elementary groundwater flow and transport problems. More advanced aspects are deferred to subsequent chapters. The following books give an introductory treatment of groundwater problems:

- Hudak P.F. (2005) gives introductory principles and applications of hydrogeology for students with little or no background in hydrogeology. The principal topics covered are aquifers, monitoring wells, groundwater flow, wells and aquifer tests, groundwater quality and solute transport, and management of groundwater pollution.
- Lowman S.W. (1972) gives a good introductory treatment of hydrologic properties of water-bearing materials, flowing wells, aquifer tests by well methods and by areal methods.
- National Research Council (1984) provides a well-documented, nonmathematical introduction to groundwater contamination, including case studies.
- Palmer C.M. (1996) gives a nonmathematical introduction to contaminant hydrogeology. A practical approach to completing investigations and the basics of collecting data are presented.

There are a number of excellent textbooks on the subject of hydrogeology The principal topics covered are aquifers, contaminant transport, and groundwater management and remediation.

Fitts C.R. (2002) has written a good undergraduate textbook. The mathematical modeling is treated in more detail than in the other chapters.

Freeze R.A. and Cherry J.A. (1979) have written a classical text that covers the fundamentals of groundwater geology, the flow and chemical evolution of groundwater, and groundwater contamination. It is still one of the favorite textbooks and a standard reference.

Schwartz F.W. and Zhang H. (2003) provide an excellent up-to-date textbook that integrates the concepts of hydrology with geology. Its organization is similar to that of Freeze R.A. and Cherry J.A. (1979) but includes many results developed in the interim time. It includes a section on groundwater modeling centered on MODFLOW and an excellent chapter on groundwater management. Transport processes are very well presented. There are some similarities to Domenico P.A. and Schwartz F.W. (1990) discussed below.

Todd D.K. (1980) wrote a good practical text on groundwater hydrology. This has recently been updated and extended as Todd D.K. and Mays L.W. (2005). It contains a good listing of journals, web sites, and organizations with interests in subsurface flows and contaminants issues.

At a more advanced level there are also excellent textbooks among which we have selected the following:

Bear J. (1979) gives an excellent mathematical treatment of the laws and equations that describe the flow of groundwater and the management of groundwater.

Bear J. and Verruijt A. (1987) are concerned with the movement and accumulation of groundwater and pollutants in aquifers, and the construction of conceptual and mathematical models. A number of computer programs written in BASIC are included.

de Marsily G. (1986) presents a very lucid and classical treatment of the mathematics of groundwater flow and contaminant transport.


Zheng C. and Bennett G.D. (2002) provide an excellent graduate level text on modeling of contaminant transport that is useful for students and practitioners. The first part includes advective transport, diffusion, dispersion, chemical reactions, and numerical formulation. Part 2 is concerned with field applications and part 3 is on advanced topics that include variable density, unsaturated flow and transport, and groundwater management.

If one needs a handy bibliographical reference of publications in the groundwater field there is Van der Leeden F. (1991). He lists approximately 5600 references on general bibliographies, journals, texts, handbooks, and dictionaries, and references by subjects such as history, environment, geophysical exploration, well logging, hydraulics of soils and aquifers, theory of groundwater flow, pumping, well maintenance, pumping equipment, tracers, water quality, contamination, salt water intrusion, models, laws and regulations, management, etc.