Nonlinear Effects in Optical Fibers

OVERVIEW

In conventional optics, focused intense light only interacts with solid media at length scales of centimeters at most, limited by the size of the beam waist of the focused beam. High-intensity light in the small core of an optical fiber laser can interact with glass at length scales 3–6 orders of magnitude longer, i.e., from meters to many kilometers. The resulting nonlinear effects often present limitations for power scaling of fiber lasers. They, sometimes, do offer opportunities, too, as in the cases of optical solitons and super-continuum generation. Key nonlinear effects impacting fiber lasers will be covered in this chapter.

NONLINEAR EFFECTS IN OPTICAL FIBERS

At high optical intensity, the response of dielectric media is no longer linear under strong electromagnetic fields. Fundamentally, the nonlinear response originates from the anharmonic motion of bound electrons in the medium. The total polarization $P$ can be written as

$$P = \varepsilon_0 \left( \chi^{(1)} \cdot E + \chi^{(2)} \cdot EE + \chi^{(3)} \cdot EEE + \cdots \right)$$

(3.1)

where $\varepsilon_0$ is the vacuum permittivity and $\chi^{(j)}$ is $j$th-order susceptibility. The first-order susceptibility $\chi^{(1)}$ represents the dominant linear term included in the refractive index $n_0$ of a medium. The second-order susceptibility $\chi^{(2)}$ is responsible for second harmonic generation and sum-frequency generation and vanishes for a centro-symmetric medium such as silica. The lowest-order nonlinear effect in optical fibers comes from the third-order susceptibility $\chi^{(3)}$, which is responsible for third harmonic generation, four-wave mixing, and nonlinear refraction. The third-order susceptibility is also responsible for the intensity dependence of the refractive index, referred to as Kerr nonlinearity, and, in the simplest case, it can be written as

$$n = n_0 + n_2 |E|^2$$

(3.2)

where $n_0$ is the linear refractive index, $n_2$ is the nonlinear index coefficient, and $|E|^2$ is the optical intensity in the fiber. Two prominent effects arising from the intensity dependence of the refractive index are self-phase modulation (SPM) and cross-phase modulation (XPM). SPM describes the self-induced nonlinear phase change and XPM the nonlinear phase change induced by another optical field at a different frequency in the optical fiber. For SPM, the nonlinear index can be written as

$$n_{NL} = n_2 |E|^2$$

(3.3)

For XPM involving optical fields $E_1$ and $E_2$ at respective angular frequencies $\omega_1$ and $\omega_2$, the nonlinear index at $\omega_1$ can be written as

$$n_{NL} = n_2 \left( |E_1|^2 + 2|E_2|^2 \right)$$

(3.4)
The first term in parentheses is from SPM and the second term from XPM. For two optical fields of equal intensity, the nonlinear phase from XPM is twice that of SPM. This can be understood in the context that the total intensity of two different fields of equal intensity is twice that of a single field. XPM is always accompanied by SPM and can occur even when the frequencies of two optical fields are different. Because the group velocities are different for the two fields at different frequencies, temporal walk-off between two optical pulses can lead to complex interactions. XPM can also take place between two optical fields orthogonally polarized in a fiber. The resulting nonlinear birefringence can manifest itself as nonlinear polarization rotation. For two fields $E_x$ and $E_y$ polarized along the $x$ and $y$ birefringence axis, the nonlinear indexes are

$$n_{x}^{NL} = n_2 \left( |E_x|^2 + \frac{2}{3} |E_y|^2 \right)$$  \hspace{1cm} (3.5a)$$

$$n_{y}^{NL} = n_2 \left( |E_y|^2 + \frac{2}{3} |E_x|^2 \right)$$  \hspace{1cm} (3.5b)$$

NONLINEAR WAVE EQUATION

Starting from Maxwell’s equations, we can arrive at a wave equation containing just electric field $E$ and polarization $P$ in isotropic source-free media,

$$\nabla \times \nabla \times E + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E = -\frac{1}{\varepsilon_0 c^2} \frac{\partial^2}{\partial t^2} P$$  \hspace{1cm} (3.6)$$

where $c$ is the speed of light in a vacuum and $c^2 = \varepsilon_0 \mu_0$. $\varepsilon_0$ is vacuum permittivity and $\mu_0$ is vacuum permeability. This is the most general form of the nonlinear wave equation. We also have

$$\nabla \times \nabla \times E = \nabla (\nabla \cdot E) - \nabla^2 E$$  \hspace{1cm} (3.7)$$

The first term on the right side of the equation vanishes identically for an infinite plane wave and is generally very small for waves with slowly varying amplitude. By ignoring this term, the nonlinear wave equation is simplified to

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E = -\frac{1}{\varepsilon_0 c^2} \frac{\partial^2}{\partial t^2} P$$  \hspace{1cm} (3.8)$$

By splitting polarization $P$ into the sum of the linear and nonlinear polarization $P_{NL}$, we have

$$\nabla^2 E - \frac{n_0^2}{c^2} \frac{\partial^2}{\partial t^2} E = \frac{1}{\varepsilon_0 c^2} \frac{\partial^2}{\partial t^2} P_{NL}$$  \hspace{1cm} (3.9)$$

Without the nonlinear source term on the right side of the equation, this is just a wave equation in a homogeneous medium with a refractive index $n_0$. The nonlinear polarization term is typically much smaller than other terms in the equation and can, therefore, be treated as a perturbation. Over a short length in fiber lasers, we can also assume that the polarization is maintained. This allows us to use a scalar approximation. We will also assume the optical field has a relatively narrow spectrum centered around $\omega_0$ and that its amplitude $A(z, \omega)$ varies slowly with a transverse mode profile $\phi(x,y)$. Assuming a fiber loss $\alpha$, the electric field $E$ can be written as

$$E(x, y, z, \omega) = A(z, \omega) \phi(x, y) e^{\frac{-\alpha z}{2}} e^{i\omega_0}$$  \hspace{1cm} (3.10)$$
We will make a further assumption that the nonlinear response is instantaneous in the following analysis. The nonlinear response typically consists of contributions from both nonlinear motions of electrons and nuclei. The electronic response has a sub-femtosecond time scale and can be considered instantaneous for most pulses. The nuclear response in the form of inelastic Raman scattering is much slower and has a time scale of few tens of femtoseconds. The Raman scattering can lead to intra-pulse energy transfer from higher-frequency components to lower-frequency components for pulses shorter than 1 ps and is responsible for the Raman-induced frequency shift. By assuming an instantaneous nonlinear response, we have ignored Raman effects and limit ourselves to pulse durations above 1 ps. By applying perturbation theory to equation 3.9 in the frequency domain and then converting back to the time domain, we have

\[
\frac{\partial A}{\partial z} + i\beta_1 \frac{\partial A}{\partial t} + i\beta_2 \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A = i\gamma |A|^2 A \tag{3.11}
\]

\(\beta_1\) and \(\beta_2\) are from the Taylor expansion of the propagation constant \(\beta\) around frequency \(\omega_0\), as described in equations 2.33–2.35 in Chapter 2. We have ignored all terms above the second term in the expansion. The nonlinear coefficient \(\gamma\) at optical wavelength \(\lambda\) is given by

\[
\gamma = \frac{2\pi n_2}{\lambda A_{\text{eff}}} \tag{3.12}
\]

where \(n_2 = 2.2 - 3.4 \times 10^{-20}\) m\(^2\)/W for silica fibers depending on detailed composition, e.g., the GeO\(_2\) doping level. The effective mode area is given by

\[
A_{\text{eff}} = \frac{\int \int \int \left| \phi(x,y) \right|^2 \, dx \, dy}{\int \int \left| \phi(x,y) \right|^4 \, dx \, dy} \tag{3.13}
\]

The integration is over the cross section of the optical fiber. For an optical fiber mode with a Gaussian distribution \(\phi(x,y) = \exp\left(-\frac{x^2 + y^2}{w^2}\right)\), \(A_{\text{eff}} = \pi w^2\). The effective mode area can vary widely from a few \(\mu\)m\(^2\) in highly nonlinear fibers to several thousand \(\mu\)m\(^2\) in fibers for high-power lasers. In highly nonlinear silica fibers, \(\gamma\) can reach 100 W\(^{-1}\) km\(^{-1}\).

Equation 3.11 can be modified for sub-picosecond pulses. In this regime, it is no longer valid to assume an instantaneous nonlinear response. The spectrum of the pulses is also significantly broader. We need to account for the wavelength dependence of the nonlinear coefficient \(\gamma\) and the fiber loss \(\alpha\). We can expand \(\gamma\) and \(\alpha\) in a Taylor series,

\[
\gamma(\omega) = \gamma(\omega_0) + \gamma_1 (\omega - \omega_0) + \frac{1}{2} \gamma_2 (\omega - \omega_0)^2 + \cdots \tag{3.14}
\]

\[
\alpha(\omega) = \alpha(\omega_0) + \alpha_1 (\omega - \omega_0) + \frac{1}{2} \alpha_2 (\omega - \omega_0)^2 + \cdots \tag{3.15}
\]

It is sufficient to retain only the first terms in the expansion in most cases. Equation 3.11 can then be modified to take into consideration electronic and nuclear nonlinear contributions.

\[
\frac{\partial A}{\partial z} + \left[ \alpha(\omega_0) + i\alpha_1 \frac{\partial}{\partial t} \right] A + \left[ \beta_1 + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} - \frac{\beta_3}{6} \frac{\partial^3 A}{\partial t^3} \right] \tag{3.16}
\]

\[
= i \gamma(\omega_0) + i\gamma_1 \frac{\partial}{\partial t} A(z,t) \int_0^\infty R(t') |A(z,t - t')|^2 \, dt'
\]
The expansion of $\gamma$ in equation 3.14 takes into consideration frequency dependence of both $n_2$ and $A_{\text{eff}}$. $\gamma_1$ is responsible for the intensity dependence of the group velocity, leading to the phenomenon referred to as self-steepening. The nonlinear response $R(t)$ includes both the electronic and the nuclear response. The functional form can be written as

$$R(t) = (1 - f_R)\delta(t - t_c) + f_R h_R(t)$$

(3.17)

The first term is from the electronic contribution and the second from the nuclear. The fast electronic response is accounted for by $t_e (<1 \text{ fs})$. The fractional contribution of the slow Raman contribution is accounted for by $f_R$ and $h_R$ is the Raman response function. The fraction $f_R$ is found to be $\sim 0.18$ [1]. One useful form for $h_R(t)$ is given as

$$h_R(t) = \frac{\tau_1^2 + \tau_2^2}{\tau_1 \tau_2} \frac{1}{\tau_1 \sin \left( \frac{t}{\tau_1} \right)}$$

(3.18)

When $\tau_1 = 12.2 \text{ fs}$ and $\tau_2 = 32 \text{ fs}$, equation 3.18 provides a reasonable fit to the Raman gain spectrum [2]. In deriving the equation, the Raman gain spectrum shape is approximated to a single Lorentzian. It therefore fails to account for the fine details of Raman gain and underestimates the Raman-induced frequency shift. With these modifications, equation 3.16 can be used for pulses of a few optical cycles if sufficient higher-order dispersive terms are included. For pulses with duration longer than 100 fs, which contain many optical cycles, equation 3.16 can be further simplified to

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2} A + \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial t^2} - \frac{\beta_3}{6} \frac{\partial^3 A}{\partial t^3} = i \gamma \left( |A|^2 A + \frac{i}{\omega_0} \frac{\partial}{\partial T} (|A|^2 A) - T_R A \frac{\partial^2 |A|^2}{\partial T^2} \right)$$

(3.19)

where

$$T_R = \int_0^\infty t R(t) dt = f_R \int_0^\infty t h_R(t) dt = f_R \frac{d \ln(h_R)}{d \omega} \bigg|_{\omega = \omega_0}$$

(3.20)

The reference frame moves with the pulse at the group velocity $v_g$, referred to as retarded frame,

$$T = t - \frac{z}{v_g} = t - \beta_1 z$$

(3.21)

The third-order dispersion term $\beta_3$ is added in equation 3.19. It is responsible for third-order dispersion and becomes important for ultra-short pulses. For pulses longer than 1 ps, equation 3.19 can be further simplified to equation 3.22, i.e.,

$$\frac{1}{i} \frac{\partial A}{\partial z} + \frac{\alpha}{2} A - \frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \gamma |A|^2 A = 0$$

(3.22)

When $\alpha = 0$, equation 3.22 is referred to as a nonlinear Schrödinger (NLS) equation due to its resemblance to the Schrödinger equation with a nonlinear potential.

At very high peak powers, higher-order nonlinear terms need to be considered. One simple method is to use $\gamma = \gamma_0 (1 - b_4 |A|^2)$ when the peak intensity approaches 1 GW/cm². The resulting equation is often called a cubic-quintic nonlinear Schrödinger equation because it contains both $|A|^3$ and $|A|^5$ terms.
The NLS equation cannot be solved analytically except for some very special cases. A numerical method is generally used, often using a split-step Fourier method where the effects of dispersion and nonlinearity are accounted for in separate steps during propagation (for more details see Reference 3).

**SELF-PHASE MODULATION (SPM)**

SPM arises from the intensity dependence of the refractive index and can lead to both spectral and temporal changes in pulses during propagation in an optical fiber. It is the temporal equivalence of self-focus, which is the spatial-domain consequence of intensity-dependent refractive index (more on this later). SPM generates new frequency components due to the temporally varying phase induced by the temporally varying intensity profile of optical pulses. Since the original chirp of the pulse sets the initial relative position of the various frequency components of the pulse and the fiber group velocity dispersion (GVD) changes the relative position of the frequency components during propagation, the output pulse spectrum and shape are results of an interplay among SPM, initial chirp of the pulses, and the fiber GVD. It is useful to introduce the concepts of two length scales, dispersion length \(L_D\) and nonlinear length \(L_{NL}\), which will help us to identify the various regimes of operation. For optical pulses with an initial width \(T_0\) (defined by intensity drop to 1/e of peak intensity) and initial peak power \(P_0\),

\[ L_D = \frac{T_0^2}{|\beta_2|} \tag{3.23} \]

\[ L_{NL} = \frac{1}{\gamma P_0} \tag{3.24} \]

The dispersion length \(L_D\) and nonlinear length \(L_{NL}\) are the respective length scales over which effects of dispersion and nonlinear effects become important. For a fiber with length \(L\), when \(L << L_D\) and \(L << L_{NL}\), neither dispersion nor nonlinear effects play a significant role in pulse propagation. When \(L << L_D\) and \(L = L_{NL}\), nonlinear effects dominate. When \(L = L_D\) and \(L << L_{NL}\), dispersion dominates. When \(L = L_D\) and \(L \approx L_{NL}\), both effects need to be considered.

The effect of dispersion broadens a pulse with no initial chirp. Under the influence of second-order dispersion \(\beta_2\), a Gaussian pulse with intensity \(\exp(-T^2/T_0^2)\) will stay as Gaussian pulse while its width increases according to

\[ T(z) = T_0 \sqrt{1 + \left( \frac{z}{L_D} \right)^2} \tag{3.25} \]

When \(z >> L_D\),

\[ T(z) \approx T_0 \frac{z}{L_D} = \frac{|\beta_2|}{T_0} z \tag{3.26} \]

The pulse width at the output is, therefore, proportional to dispersion \(\beta_2\) and propagation distance \(z\). It is also inversely proportional to the initial pulse width \(T_0\) because the spectral width of the initial pulse is inversely proportional to \(T_0\). Higher-order dispersion terms in general distort pulses due to the nonlinear change in the relative temporal position of the frequency components during propagation.
The maximum nonlinear phase $\phi_{NL}$ happens at the intensity peak when $T = 0$ and can be more conveniently written in term of peak optical power $P_0$

$$\phi_{NL}(0) = \frac{L_{eff}}{L_{NL}} = \gamma P_0 L_{eff}$$  \hspace{1cm} (3.27)

Across the pulse, the nonlinear phase $\phi_{NL}(T)$ is dependent on the local intensity and is a function of the relative temporal position. The effective length for an amplifier with output power $P_{out}$, length $L$, and power distribution $P(z)$ is given by

$$L_{eff} = \int_0^L \frac{P(z)}{P_{out}} dz$$  \hspace{1cm} (3.28)

For a fiber of length $L$ and loss $\alpha$, the effective length $L_{eff}$ can be written as

$$L_{eff} = \frac{1 - e^{-\alpha L}}{\alpha}$$  \hspace{1cm} (3.29)

The local instantaneous optical frequency as a result of the time-varying nonlinear phase can be written as

$$\Delta\omega(T) = -\frac{\partial \phi_{NL}}{\partial T}$$  \hspace{1cm} (3.30)

The negative sign is due to our choice of using the notation $\exp[i(\beta z - \omega t)]$ for the phase term. The dependence of local instantaneous frequency on time across a pulse is commonly referred to as frequency chirp. Local frequencies are continuously changing across the pulse according to equation 3.30. The magnitude of the frequency change is proportional to the local rate of change of the nonlinear phase. The maximum frequency change is achieved at the leading and trailing edges of the pulse. The frequency at the leading edge is lowered, i.e., redshifted, and at the trailing edge, increased, i.e., blueshifted (see Figure 3.1). Continuous redshift at the leading edge and blueshift at the trailing edge takes place during propagation, resulting in a positive chirp (see Figure 3.1). For a Gaussian pulse, the SPM-induced chirp is given by

$$\Delta\omega(T) = \frac{2 L_{eff} T}{L_{NL} T_0^2} e^{-\left(\frac{T}{T_0}\right)^2}$$  \hspace{1cm} (3.31)

**FIGURE 3.1** SPM-induced chirp in an optical pulse.
The chirp is linear near the center of the Gaussian pulse. The effect of SPM is more pronounced in shorter pulses. The peak nonlinear phase is commonly referred to as the B-integral, which is an important parameter to consider when designing an ultrafast laser. The B-integral is typically kept below $\pi$ to minimize pulse distortion by SPM in a laser. New frequency generation as a result of SPM can lead to significant spectral broadening. The resulting power spectra as a consequence of SPM when dispersion is ignored are demonstrated in Figure 3.2 for an unchirped Gaussian input pulse with a FWHM (full-width half-maximum) pulse width of 1 ps for various maximum nonlinear phases. In general, the spectrum broadens with an increase of maximum nonlinear phase. The key spectral features are the two dominant peaks generated at the leading and trailing edges of the pulse. The number of oscillations within the spectrum increases with an increase of maximum nonlinear phase. The oscillation is a result of coherent interference among components of the same frequency generated at multiple locations within the pulse.

For longer pulses ($T_0 > 10$ ps), the dispersion length $L_D$ is much longer than most fiber lengths in fiber lasers and the dispersion effect can be ignored. For shorter pulses ($<1$ ps), the dispersion length can become comparable to the fiber length. It is then necessary to consider the effects of both SPM and GVD. SPM generates a positive chirp with low frequency at the leading edge of the pulse and high frequency at the trailing edge. In the normal dispersion regime, the low-frequency components at the leading edge travel faster than the high-frequency components at the trailing edge. This leads to a much accelerated pulse broadening. In the anomalous dispersion regime, the opposite is true where the high-frequency components at the trailing edge travel faster than the low-frequency components at the leading edge. The dispersion effectively compresses the pulse in this case. When the pulse has a hyperbolic secant shape, i.e., a sech pulse, a chirp-free pulse can propagate without changing its pulse shape. This pulse is referred to as a soliton and will be discussed in more detail in the next section.

In an optical fiber amplifier with normal dispersion, pulses evolve asymptotically to parabolic pulses while being amplified. The parabolic pulses are linearly chirped. The pulse width increases linearly with the pulse amplitude as it is amplified. This process is referred to as self-similar, as the pulse maintains its parabolic shape while its width and amplitude increase. The rate of intensity change is linear with time in a parabolic pulse, which leads to the linear chirp across the pulse. The amplified pulse in this regime depends only on energy and not on the shape of the input pulse.

**FIGURE 3.2** SPM-induced spectral broadening of an unchirped Gaussian pulse with FWHM = 1 ps at various peak $\phi_{NL}$. 
When the intensity dependence of GVD is considered, the pulse peak moves slower than its edges, leading to a slowdown of the pulse peak toward the trailing edge. This causes steepening of the trailing edge, commonly referred to as **self-steepening**. Self-steepening leads also to spectral asymmetry, where the redshifted peaks are more intense. The SPM-induced spectral broadening has a larger effect on the blue shifted light at the steeper trailing edge of the pulse.

### OPTICAL SOLITONS

A **soliton** is a pulse that can propagate without change in its shape for a significant distance. Its existence has been observed in many branches of physics and is certainly not limited to optics. John Scott Russell, a Scottish engineer, was the first to observe it, in a canal in 1844. He reported the observation of a heap of water, caused by the stoppage of a boat, traveling unchanged for 1–2 miles at 8–9 miles per hour. The phrase soliton is nowadays also used more generally to include a pulse that returns to its original shape periodically as it propagates down a fiber. This pulse is referred to as a **higher-order soliton**. If the pulse maintains a constant shape during the propagation it is referred to as a **fundamental soliton**.

As we have discussed previously, SPM leads to the generation of instantaneous frequencies that are lower at the leading edge of a pulse and higher at the trailing edge. This is equivalent to a positive chirp. In the anomalous dispersion regime, lower-frequency light travels slower than higher-frequency light. The effect of the dispersion is to create a negative chirp. It can, therefore, be used to cancel the effect of SPM. This balancing act between SPM and GVD is precise for the fundamental solitons regardless of propagation distance and is periodically achieved for higher-order solitons during propagation. The fundamental soliton remains chirp-free during propagation while the higher-order soliton is chirp-free only periodically when it returns to its original shape.

Solitons can be found by solving the nonlinear wave equation described earlier. Fundamental solitons can be found by directly solving the NLS equation. Higher-order solitons have also been found using an inverse scattering method [4]. The constant pulse shape and chirp-free properties of fundamental solitons led to strong interest from the optical communications community in the late 1980s and early 1990s. Higher-order solitons are also regularly observed experimentally and are fundamental to the understanding of such effects as super-continuum generation in optical fibers [3].

Solitons are found to be very stable in both numerical simulations and experiments. A chirp-free pulse would evolve asymptotically into a soliton of the nearest order while adjusting its shape and width as it propagates. Pulse width can either increase or decrease during propagation depending on whether its nearest-order soliton is a higher- or lower-order soliton. Part of the pulse energy, which cannot be converted into a soliton, is shed away as a broad background radiation in a process known as **continuum generation**. Even pulses with small levels of chirp can evolve into solitons, given that the chirp is below a certain threshold level. The amount of energy shed into the continuum radiation can be substantial at large chirps. Stable solitons are critical for some practical applications. Higher-order solitons, however, are weakly bound lower-order solitons without any binding energy. They can readily break up into their constituents in a process referred to as **soliton fission** in the presence of higher-order dispersion, self-steepening, or intra-pulse Raman scattering.

Under the influence of gain or loss in a fiber, both fundamental and higher-order solitons will decrease or increase their width respectively in response to the change in power. Abrupt gain and loss can lead to energy loss to continuum radiation. The amplification process also adds noise from **amplified spontaneous emission** (ASE). The amplitude noise can lead to phase noise through SPM, which in turn leads to frequency fluctuations in solitons. This causes jitters in the arrival time of solitons, commonly referred to as **Gordon-Haus timing jitter**. Solitons will also interact with each other when they get close enough during propagation. The overall effect is an attractive force between two solitons with the same amplitude and phase, leading eventually to a collision.
the phase difference between two solitons is large (>π/8), the overall effect of the interaction is a repulsive force. Such interactions lead to additional uncertainties in the arrival time of solitons.

Intra-pulse Raman scattering can cause energy transfer from higher-frequency components of the pulse to lower-frequency components, leading to a continuous redshift, referred to as soliton self-frequency shift. This is only significant when a pulse is very short, where the spectral width of the pulse is broad (more on this in Chapter 15). Over a short distance, the frequency shift grows linearly with distance while the pulse remains largely chirp-free. At larger distances, the pulse is progressively more chirped, leading to a saturation of the frequency shift.

By using dimensionless variables the NLS equation can be written as

\[ i \frac{\partial U}{\partial \xi} - \text{sign}(\beta_2) \frac{1}{2} \frac{\partial^2 U}{\partial \tau^2} + N^2 |U|^2 U = 0 \]  

(3.32)

where

\[ N^2 = \frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|\beta_2|} \]  

(3.33)

\[ U = \frac{A}{\sqrt{P_0}} \]  

(3.34)

\[ \xi = \frac{z}{L_0} \]  

(3.35)

\[ \tau = \frac{T}{T_0} \]  

(3.36)

where \( P_0 \) is the peak power the dispersion length \( L_D \) and nonlinear length \( L_{NL} \) were defined previously. Loss, higher-order dispersion, and higher-order nonlinear effects are ignored. It is important to note from equation 3.32 that the normalized pulse shape is determined only by parameter \( N \) and the sign of \( \beta_2 \). By rescaling the pulse shape \( U \) by

\[ u = NU = \sqrt{\gamma L_D} A \]  

(3.37)

equation 3.32 can be more generalized and, in an anomalous dispersion regime, it becomes

\[ i \frac{\partial u}{\partial \xi} + \frac{1}{2} \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0 \]  

(3.38)

This can be solved to obtain the solution for a fundamental soliton when \( N = 1 \),

\[ u(\xi, \tau) = \text{sech}(\tau) e^{\frac{\xi}{2}} \]  

(3.39)

Its pulse shape is a hyperbolic secant and independent of the distance of propagation (see Figure 3.3). The effect of the propagation is only the accumulation of phase \( \xi / 2 = \gamma P_0 z / 2 \) (note \( L_D = L_{NL} \) for \( N = 1 \)). Peak power can be obtained by setting \( N = 1 \) in equation 3.33.

\[ P_0 = \frac{|\beta_2|}{\gamma T_0^2} = \frac{3.11|\beta_2|}{\gamma T_0^2 \text{FWHM}} \]  

(3.40)
Please note that FWHM pulse width is given by \( T_{\text{FWHM}} = 1.763 T_0 \). The pulse energy can be obtained by integrating the pulse power,

\[
E_s = 2T_0P_0 \frac{2|\beta_2|}{\gamma T_0} \frac{3.53|\beta_2|}{\gamma T_{\text{FWHM}}}
\]

(3.41)

The second-order soliton has a peak of \( N^2 = 4 \) times of that of the fundamental soliton and is given by \([5]\)

\[
u(\xi, \tau) = 4 \frac{\cosh(3\tau) + 3\cosh(\tau)e^{i\xi}}{\cosh(4\tau) + 4\cosh(2\tau) + 3\cosh(4\xi)} e^{i\xi/2}
\]

(3.42)

The pulse shape \(|\nu(\xi, \tau)|^2\) in this case is a function of normalized propagation distance \(\xi\) and repeats itself with a period of \(\xi = \pi/2\). It is also worth noting that the phase of the second-order soliton also varies periodically with pulse shape in addition to the propagation phase \(\xi/2 = \gamma P_0 z / (2N^2)\). This periodic behavior is true for all higher-order solitons. The temporal and spectral evolutions of the second- and third-order solitons are illustrated in Figures 3.4 and 3.5, respectively. The period of the higher-order solitons can be written in real units as

\[
z_0 = \frac{\pi}{2} L_0 = \frac{\pi}{2} \frac{T_0^2}{|\beta_2|}
\]

(3.43)
MODULATION INSTABILITY

The interaction of SPM and GVD can also lead to the breakup of CW (continuous wave) light into a train of pulses in an anomalous dispersion regime. For a perfect CW light, SPM only leads to a time-invariant additional nonlinear phase without any new frequency generation. Any power fluctuations can, however, lead to ripples in the CW light and new frequency generation with SPM-induced red and blue shifts at the leading and trailing edges of the ripples, respectively. In the anomalous dispersion regime, the effect of GVD causes the redshifted leading edge to slow down and the blueshifted trailing edge to speed up. This effectively leads to a power transfer to the peak of the ripples from around the ripples, resulting in an increase in the amplitudes of the fluctuations. This positive feedback process eventually causes CW light to break up into a train of pulses in a process referred to as modulation instability.

If we start with NLS equation by setting loss term to zero in equation 3.22,

\[
\frac{i}{A} \frac{\partial A}{\partial z} - \frac{\beta_2}{2} \frac{\partial^2 A}{\partial T^2} + \gamma |A|^2 A = 0
\]  

(3.44)

For CW light, the solution is simply

\[
A = \sqrt{P_0} e^{i\gamma P_0 z}
\]  

(3.45)

This is simply CW light with an additional nonlinear phase \( \gamma P_0 z \). We can check the stability of the CW light by adding a small perturbation in the amplitude.

\[
A = (\sqrt{P_0} + a) e^{i\gamma P_0 z}
\]  

(3.46)

Substituting equation 3.46 into equation 3.44, we have

\[
i \frac{\partial a}{\partial z} - \frac{\beta_2}{2} \frac{\partial^2 a}{\partial T^2} + \gamma P_0 \left(a + a^*\right) = 0
\]  

(3.47)

The solution takes the form

\[
a(z, T) = a_0 e^{-i(Kz - \Omega T)} + a_2 e^{i(Kz - \Omega T)}
\]  

(3.48)

\( K \) and \( \Omega \) are the wave number and frequency of the perturbation, respectively. Solutions to equations 3.47 and 3.48 are nontrivial only when

\[
K = \pm \frac{1}{2} \beta_2 \Omega \sqrt{\Omega^2 + \text{sign}(\beta_2) \Omega_c^2}
\]  

(3.49)

where

\[
\Omega_c^2 = \frac{4 \gamma P_0}{|\beta_2|}
\]  

(3.50)

In the normal dispersion regime where \( \beta_2 > 0 \), \( K \) is always real. SPM only affects the phase but not the amplitude of the perturbation. The perturbation will not grow. In the anomalous dispersion regime where \( \beta_2 < 0 \), \( K \) is imaginary when \( |\Omega| < \Omega_c \). The amplitude of the perturbation will, therefore, grow exponentially. It is inherently unstable for a CW in the anomalous dispersion regime. The gain is determined by

\[
g(\Omega) = |\beta_2 \Omega| \sqrt{\Omega_c^2 - \Omega^2}
\]  

(3.51)
The normalized gain spectrum $g/P_0$ is shown in Figure 3.6. It is symmetrical around $\Omega = 0$. The maximum gain is achieved at frequencies determined by

$$\Omega_{\text{max}} = \pm \frac{\Omega_c}{\sqrt{2}} = \pm \frac{2\gamma P_0}{|\beta_2|}$$

(3.52)

The maximum gain is

$$g_{\text{max}} = \frac{1}{2} |\beta_2| \Omega_c^2 = 2\gamma P_0$$

(3.53)

**FOUR-WAVE MIXING**

Nonlinear effects in silica optical fibers are dominated by third-order susceptibility $\chi^{(3)}$, which can lead to nonlinear interactions among four optical waves. The strength of these interactions is strongly dependent on the conservation of both energy and momentum. Momentum conservation in quantum mechanics is commonly referred to as *phase matching* in wave optics, i.e., the matching of wave vectors. Some of these nonlinear interactions, e.g., third harmonic generation, involve optical frequencies significantly different from each other. Phase matching is hard to achieve in these cases. The strongest of the nonlinear interactions involves the simultaneous annihilation of two pump photons and the creation of two new photons. This is commonly referred to as *four-wave mixing* (FWM). FWM is a parametric process, describing the fact that it is an elastic process where there is no net transfer of energy to the medium. The passive participation of the medium is directly responsible for the requirement of precise phase matching. This is markedly different from *stimulated Brillouin scattering* (SBS) and *stimulated Raman scattering* (SRS), which will be discussed later in this chapter. These are caused by inelastic scattering of the medium, resulting in a net energy transfer to the medium. Phase matching in both cases is automatically satisfied by the additional wave vector provided by the medium, as evidenced by the fact that both SBS and SRS are naturally occurring phenomena. The requirements of energy and momentum conservation, ignoring the nonlinear effects on phase matching for now, lead to

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

(3.54)

$$\Delta k = \beta_3 + \beta_4 - \beta_1 - \beta_2 = 0$$

(3.55)
Generally, the frequencies of the two pump waves do not need to be the same, \(\omega_1 \neq \omega_2\). In the more common degenerate case where \(\omega_1 = \omega_2\), only one pump is required. In degenerate FWM, higher- and lower-frequency waves generated around the pump frequency are respectively referred to as anti-Stokes and Stokes waves. In parametric terms, the higher and lower frequency waves are referred to as the signal and idler, respectively. The signal and idler waves can be generated from noise once phase matching is achieved. If a seed at either signal or idler frequency is launched with the pump, it can be amplified by the parametric gain accompanied by a growth of the respective idler or signal.

Starting from the nonlinear wave equation, making substitution with the summation of four co-polarized fields and collecting only terms at the same frequencies, we can obtain four coupled equations for FWM,

\[
\frac{\partial A_1}{\partial z} = \frac{in_2 \omega_1}{c} \left[ \left( f_{I1} A_1^2 + 2 \sum_{m \neq 1} f_{Im} |A_m|^2 \right) A_1 + 2 f_{I234} A_2^* A_3 A_4 e^{i\Delta k z} \right] \quad (3.56a)
\]

\[
\frac{\partial A_2}{\partial z} = \frac{in_2 \omega_2}{c} \left[ \left( f_{I2} A_2^2 + 2 \sum_{m \neq 2} f_{Im} |A_m|^2 \right) A_2 + 2 f_{I234} A_1 A_3^* A_4 e^{i\Delta k z} \right] \quad (3.56b)
\]

\[
\frac{\partial A_3}{\partial z} = \frac{in_2 \omega_3}{c} \left[ \left( f_{I3} A_3^2 + 2 \sum_{m \neq 3} f_{Im} |A_m|^2 \right) A_3 + 2 f_{I312} A_1 A_2 A_4^* e^{i\Delta k z} \right] \quad (3.56c)
\]

\[
\frac{\partial A_4}{\partial z} = \frac{in_2 \omega_4}{c} \left[ \left( f_{I4} A_4^2 + 2 \sum_{m \neq 4} f_{Im} |A_m|^2 \right) A_4 + 2 f_{I312} A_1 A_2 A_3^* e^{i\Delta k z} \right] \quad (3.56d)
\]

where

\[
f_{mn} = \frac{\int \int_{-\infty}^{\infty} \phi_m(x,y) \phi_n(x,y) \phi_n^*(x,y) \phi_n(x,y) dxdy}{\int \int_{-\infty}^{\infty} \phi_n(x,y) \phi_n(x,y) dxdy} \quad (3.57)
\]

\[
f_{mpq} = \sqrt{\int \int_{-\infty}^{\infty} \phi_m(x,y) \phi_p(x,y) \phi_p(x,y) \phi_q(x,y) dxdy} \quad (3.58)
\]

where \(m, n, p\), and \(q = (1, 2, 3, 4)\). The four coupled-amplitude equations can only be solved numerically in general. Under the approximations that there is no pump depletion and that there are relatively small differences in optical frequency, we can make the following approximations

\[
f_{mn} = f_{mpq} = \frac{1}{A_{ef}} \quad (3.59)
\]

\[
\gamma_m = \frac{n_2 \omega_m}{c A_{ef}} = \gamma \quad (3.60)
\]

Under these approximations, analytical solutions can be obtained,

\[
A_i(z) = \sqrt{P_i} e^{i\gamma(n_i + 2P_i)z} \quad (3.61a)
\]
\[ A_2(z) = \sqrt{P_e} e^{i(\Delta k + 2\gamma P_0)z} \]  
\[ A_3(z) = \left( a_3 e^{i\Delta k} + b_3 e^{-i\Delta k} \right) e^{-1/2 \frac{\kappa^2 z}{4}} e^{i(\Delta k + 2\gamma P_0)z} \]  
\[ A_4(z) = \left( a_4 e^{i\Delta k} + b_4 e^{-i\Delta k} \right) e^{1/2 \frac{\kappa^2 z}{4}} e^{i(\Delta k + 2\gamma P_0)z} \]

where \( a_3, b_3, a_4, \) and \( b_4 \) need to be determined from launch conditions. The effective phase mismatch is given by

\[ \kappa = \Delta k + \gamma(P_1 + P_2) \]  

The parametric gain is determined by

\[ g = \sqrt{4\gamma^2 P_1 P_2 - \frac{\kappa^2}{4}} \]  

In the more common degenerate FWM case, we have \( P_1 = P_2 = P_0/2 \). In this case,

\[ \kappa = \Delta k + 2\gamma P_0 \]  

\[ g = \sqrt{\gamma^2 P_0^2 - \frac{\kappa^2}{4}} \]  

The parametric gain described above is plotted in Figure 3.7. The maximum gain of \( \gamma P_0 \) is reached when \( \Delta k = -2\gamma P_0 \). The phase matching is achieved away from \( \Delta = 0 \) due to the influences of SPM and XPM. The gain exists only when \( -4\gamma P_0 < \Delta k < 0 \). The parametric gain is typically larger than the peak Raman gain and, therefore, has a lower threshold in optical fibers. SRS however dominates in long fibers due to the difficulty in maintaining phase matching for FWM over long fiber lengths. Efficient FWM for optical pulses requires both phase matching and matching of group velocities.

![FIGURE 3.7 Parametric gain for degenerate FWM.](image)
By expanding the $\beta_3$ and $\beta_4$ around the pump frequency in a degenerate FWM process and noting that $\omega_3$ and $\omega_4$ are symmetrically located at two sides of $\omega_1 = \omega_2 = \omega$, the phase mismatch can be written as in terms of frequency shift $\Omega_s = |\omega_1 - \omega_3| = |\omega_4 - \omega_2|$, 

$$\kappa = \beta_2 \Omega_s^2 + \frac{1}{12} \beta_4 \Omega_s^4 + 2\gamma P_0$$ (3.66)

All the odd-order terms in the Taylor expansion are canceled out in the summation. When higher-order dispersion terms can be ignored, i.e., $|\beta_2| \gg |\beta_m|$ where $m > 2$, phase matching can be achieved in the anomalous dispersion regime where $\beta_2 < 0$. The frequency shift can be written as

$$\Omega_s = \frac{2\gamma P_0}{|\beta_2|}$$ (3.67)

Phase matching can also be achieved in the normal dispersion regime where $\beta_2 > 0$ only when $\beta_4 < 0$. In this case, the frequency shift is given by

$$\Omega_s^2 = -\frac{6}{\beta_4} \left( \beta_2 + \frac{2}{3} \beta_4 \gamma P_0 \right)$$ (3.68)

**NONLINEAR SELF-FOCUSBING**

The refractive index dependence on intensity from the Kerr nonlinearity can act as a positive lens for an optical beam. This can cause the beam to collapse, leading to optical damage in a process referred to as *nonlinear self-focus* (NSF). Typically a small beam (<5 mm) would collapse uniformly in a bulk medium and leave a damage track. For a larger beam, the beam would break up into many collapsed filaments in a process referred to as *filamentation*. Nonlinear self-focus is the result of the instable nature of Kerr media. It is the spatial equivalence of SPM. A high-intensity ripple in a large beam can form a positive lens and draws more power to enhance the ripple, leading to a runaway process and, eventually, a total collapse. A small beam is likely to have smooth spatial intensity and NSF can lead to a total collapse of the beam. For a larger beam, any spatial intensity fluctuations can initiate the filamentation process. Once started, the small filaments collapse a lot faster and determine the NSF threshold.

In a bulk medium, diffraction causes a beam to expand and the Kerr effect causes a beam to contract. A stable spatial beam, referred to as a *Townes soliton*, can propagate when the two effects precisely balance each other out. The power at which this balance is achieved is referred to as the critical power for NSF. It is a critical power, not intensity. This is due to the fact that both contraction from the Kerr effect and expansion from diffraction are inversely proportional to the beam size for a fixed power. The critical power can be estimated by balancing diffraction and the Kerr effect. We will follow a procedure first performed by Chiao et al. [6]. Considering a beam with a flattop intensity $I$, the Kerr nonlinearity will cause a refractive index increase of $n_2 I$ from the surrounding medium index of $n_0$. The critical angle for total internal reflection at this boundary can be determined,

$$\theta_r \approx \sqrt{\frac{2n_2 I}{n_0}}$$ (3.69)

Assuming a beam diameter $d$, the diffraction half angle of the beam will be

$$\theta_D \approx 1.22 \frac{\lambda}{d}$$ (3.70)
If $\theta_T = \theta_D$, most of the beam will be trapped by total internal reflection. This leads to

$$P_{cr} = \frac{7.345 \lambda^2}{4 \pi n_1 n_2}$$  \hspace{1cm} (3.71)

Equation 3.71 overestimates the critical power. There have been many studies on the critical power for NSF. The general form for the critical power is given as

$$P_{cr} = \alpha \frac{\lambda^2}{4 \pi n_1 n_2}$$  \hspace{1cm} (3.72)

Using a paraxial approximation in a hollow-core waveguide, Fibich and Gaeta obtained $\alpha = 1.86225$ as a lower bound [7]. Using a perturbation approach in optical fibers with Gaussian modes, $\alpha = 3$ was obtained in Reference 8. Using $n_2 = \sim 2.3 \times 10^{-30} \text{m}^2/\text{W}$, the critical power is estimated to be $\sim 5-8 \text{MW}$ at $\sim 1 \mu\text{m}$. In optical fibers, mode size is reduced under the influence of the Kerr nonlinearity when operating at high powers but below the critical power for NSF. The steady-state mode size versus power is illustrated for fibers with different $V$ values in Figure 3.8a. The reduction of mode size is larger in fibers with smaller $V$ values (i.e., weaker guidance). At and above the critical power, the mode totally collapses. The collapse distance $z_{sf}$ scales with $(P_{cr}/P)^{1/2}$. The adiabatic evolution of mode size along a fiber with $V = 2.5$ is illustrated in Figure 3.8b for various powers both below and above the critical power, where normalized distance $\delta = z/z_R$ ($z_R = \pi \omega_0^2/\lambda$) is Rayleigh range.

**STIMULATED BRILLOUIN SCATTERING**

There are three waves involved in SBS, two counter-propagating optical waves at frequencies $\omega_1$ and $\omega_2$ and a traveling acoustic wave at frequency $\Omega$. This situation is illustrated in Figure 3.9. The forward propagating optical wave at $\omega_1$ traveling from left to right in the figure is the input beam in the system. The acoustic wave is typically initiated from thermal fluctuations. The counter-propagating wave can be generated from the interaction of the forward-propagating optical wave and the traveling acoustic wave in a similar way to that described for the coupling of counter-propagating waves in Chapter 2. This is the first part of the SBS process. In the second part, the interference of the two counter-propagating optical waves generates further acoustic power through the electrostrictive effect, where high optical intensity leads to a higher density in the medium. This forms a positive
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feedback loop where the phase-matching condition is automatically met in both parts of the process. Due to the intrinsically self-optimized positive feedback loop, SBS is a powerful naturally occurring phenomenon with extremely high gain allowing buildup from quantum noise. SBS can lead to a significant fraction of the original optical wave being reflected in a Stokes wave when the input power is above certain threshold power.

Assuming the two optical waves can be written as

\[ E_1 = A_1(z,t) \varphi(r,\theta)e^{j(\beta_1 z - \omega_1 t)} \]  
\[ E_2 = A_2(z,t) \varphi(r,\theta)e^{j(\beta_2 z + \omega_2 t)} \]

where \( \varphi(r, \theta) \) describes transverse distribution of the optical mode. The density change in the optical fiber can be written as

\[ \rho_t = \rho_0 + \Delta \rho \]  
\[ \Delta \rho = \rho(z,t) \xi(r,\theta)e^{j(qz - \Omega t)} \]

where \( \rho_t \) is density, \( \rho_0 \) is the mean density, \( \rho \) is the amplitude of density modulation at acoustic frequency \( \Omega \), and \( q \) is the wave vector of the acoustic wave. Typically, the core and cladding of an optical fiber have different acoustic velocities, i.e., acoustic indexes, and form an acoustical waveguide. \( \xi(r, \theta) \) represents the transverse acoustical mode distribution in the acoustic waveguide. Combining equations 3.73, 3.74, and 3.75 with the nonlinear wave equation and applying a slowly varying amplitude approximation, we can obtain a pair of coupled equations for SBS in an optical fiber perturbed by a density wave (see Reference 9 for more details for the following derivations),

\[ \frac{\partial A_1}{\partial z} + \frac{\alpha}{2} A_1 + \frac{n}{c} \frac{\partial A_1}{\partial t} = i k_1 \frac{\rho}{2 n \rho_0} A_2 \int \int \gamma e \xi \varphi^* dS \]  
\[ \frac{\partial A_2}{\partial z} + \frac{\alpha}{2} A_2 + \frac{n}{c} \frac{\partial A_2}{\partial t} = i k_1 \frac{\rho^*}{2 n \rho_0} A_1 \int \int \gamma e \xi \varphi^* dS \]

where \( k_1 = \omega_1/c \), \( k_2 = \omega_2/c \), \( \alpha \) is the loss coefficient of the optical power, \( n \) is refractive index, \( c \) is the speed of light, \( \gamma_e = \rho_0 (\partial \varepsilon / \partial \rho) \) is known as the electrostrictive constant, and \( \varepsilon \) is the relative dielectric constant. From energy and momentum conservation, we also have, respectively,

\[ \Omega = \omega_1 - \omega_2 \]  
\[ q = \beta_1 + \beta_2 \]
The density fluctuation $\Delta \rho$, driven by a pair of interfering counter-propagating optical waves is governed by

$$\frac{\partial^2 \Delta \rho}{\partial t^2} - \Gamma' \nabla^2 \Delta \rho_t - \upsilon^2 \nabla^2 \Delta \rho = \varepsilon_0' \gamma q^2 A_1 A_2 \phi \phi^* e^{i(qz - \Omega t)}$$

where $\Gamma'$ is the acoustic damping parameter and $\upsilon$ is the acoustic velocity. The Brillouin linewidth is given by $\Gamma_B = q^2 \Gamma'$. The term on the right side is the driving term from two counter-propagating optical waves. The driving term does not have $\theta$ dependence if we are only interested in the optical power in the fundamental mode in a circular optical fiber. In this case, the acoustic mode field distribution $\xi$ is only a function of $r$, indicating that only longitudinal acoustic modes are of consequence in an SBS process in an optical fiber operating in the fundamental mode. We are interested in steady-state solutions over a short section of fiber where field amplitudes $A_1$ and $A_2$, along with density modulation amplitude $\rho$, are approximately constant without $z$-dependence. In this case, inserting equation 3.75 in equation 3.79 and introducing cylindrical coordinates, the transverse acoustic field distribution should satisfy the following equation:

$$\left\{ \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} + \left[ k_a^2 n_a^2 \left( 1 + i \frac{\Gamma_B}{\Omega} \right) - q_m^2 \right] \right\} \rho = \varepsilon_0' \gamma q^2 A_1 A_2 \phi \phi^*$

(3.80)

where $n_a = \upsilon / \upsilon$ is the equivalent acoustic index and $k_a$ is the acoustic vacuum wave vector. We can now expand to include more than one acoustic mode with each mode denoted by a mode number $m$. $\rho$ and $\xi$ in equation 3.75 now become mode-specific $\rho_m$ and $\xi_m$. The total density fluctuation is a sum of longitudinal acoustic modes $\xi_m(r)$

$$\Delta \rho = \sum_m \rho_m z(r, \theta) e^{i(q_m z - \Omega t)}$$

(3.81)

with each acoustic mode satisfying the acoustical mode equation

$$\frac{\partial^2 \xi_m}{\partial r^2} + \frac{1}{r} \frac{\partial \xi_m}{\partial r} + \left[ k_a^2 n_a^2 \left( 1 + i \frac{\Gamma_B}{\Omega} \right) - q_m^2 \right] \xi_m = 0$$

(3.82)

Extending equation 3.79 for more than one acoustic mode and using equation 3.82 to simplify the result, with the assumption of sufficiently small relative changes in $n_a(r)$ and $\upsilon$ across the fiber, we have

$$\rho_m = \frac{\varepsilon_0' q^2 A_1 A_2^*}{q_m^2 - q^2} \int \int \frac{\gamma_c}{v^2} \phi \phi^* \xi_m^* dS$$

(3.83)

When the small relative variation of $\gamma_c$ and $\upsilon$ across the fiber can be ignored

$$\rho_m = \frac{\varepsilon_0' \gamma q^2 A_1 A_2^*}{(q_m^2 - q^2) v^2} \int \int \phi \phi^* \xi_m^* dS$$

(3.84)

Note that to account for acoustic mode loss

$$q_m = q_m^* + i q_m^i$$

(3.85)
and the excited acoustic mode amplitude driven by the counter-propagating optical waves is

\[ \rho_m = \frac{\varepsilon_0 q^2 A_1 A_2^*}{q_m r^2 - q_m^2 - q_0^2 + 2q_m q_i} \int \int \frac{\gamma \phi \xi \xi^*}{\nu \nu^*} \, dS \int \int \xi \xi^* \, dS \]  

(3.86)

It is worth noting that this process of acoustic mode excitation is still governed by the conservation of energy and momentum given in equations 3.77 and 3.78. In fact, the \( \rho_m \), a function of \( \Omega \), has a maximum where these conservation conditions are met. From the coupled equations of the counter-propagating optical waves, driven by a density fluctuation given in equation 3.76, considering the multiple acoustic modes described in equation 3.86, we have

\[ \frac{\partial A_1}{\partial z} + \frac{\alpha}{2} A_1 + \frac{n \partial A_1}{c \partial t} = \frac{i k_1}{2 n p_0} A_2 \sum_m \rho_m \int \int \frac{\gamma \phi \xi \xi^*}{\nu \nu^*} \, dS \int \int \phi \phi^* \, dS \]  

(3.87a)

\[ - \frac{\partial A_2}{\partial z} + \frac{\alpha}{2} A_2 + \frac{n \partial A_2}{c \partial t} = \frac{i k_2}{2 n p_0} A_1 \sum_m \rho_m^* \int \int \frac{\gamma \phi \xi \xi^*}{\nu \nu^*} \, dS \int \int \phi \phi^* \, dS \]  

(3.87b)

SBS is typically seeded by thermally excited spontaneous phonons at the output end of an optical fiber. The driving acoustic waves on the right hand of equation 3.87 couple the optical power between the counter-propagating optical waves. This generates back-propagating optical power from nothing at the output end of the fiber. Once there is back-propagating optical power, more acoustic phonons are generated by the counter-propagating optical waves. Since optical power is the most relevant term to use in optical fibers, we can rewrite equations 3.87 governing the SBS process in an optical fiber using optical power:

\[ \frac{\partial P_1}{\partial z} + \alpha P_1 = g P_1 P_2 \]  

(3.88a)

\[ - \frac{\partial P_2}{\partial z} + \alpha P_2 = g P_1 P_2 \]  

(3.88b)

The SBS gain is given by

\[ g = g_0 \sum_m \left( \frac{1}{A_m} \right) \left( \frac{\Gamma_m B}{2} \right)^2 \left( \Omega_m - \Omega \right)^2 + \left( \frac{\Gamma_m B}{2} \right)^2 \]  

(3.89)

where

\[ \Omega_m = q_m \nu_m \]  

(3.90)

\( \nu_m \) is the effective acoustic velocity of the acoustic mode. We have assumed that both optical waves are in the same mode. This may not be strictly true in highly multimode fibers; however, maximum
SBS gain is always achieved when there is optimal overlap of optical waves. This is achieved when both optical waves are in the same mode. In our case, the momentum conservation described in equations 3.77 and 3.78 can be written for the \( m \)-th acoustic mode at an optical wavelength \( \lambda_{\text{opt}} \) in optical fibers as

\[
\Omega_m = \frac{4n\pi v_m}{\lambda_{\text{opt}}} \tag{3.91}
\]

If we consider the case of optical waves at a single optical wavelength \( \lambda_{\text{opt}} \), the SBS gain peak frequency \( \Omega_m \) for each acoustic mode will be different unless the acoustic modes involved are degenerate. There is no such degeneracy if we consider only the longitudinal acoustic modes. The total SBS gain described in equation 3.89 consists of many peaks from each of the acoustic modes involved. Material peak gain \( g_0 \) is largely governed by material properties and, using \( k_1 \approx k_2 \approx k \), \( \omega_1 \approx \omega_2 \approx \omega \), and \( q \approx 2nk \) from equation 3.78, is given by

\[
g_0 = \frac{\gamma^2 \omega^2}{2n^2 c \Omega_0 \Gamma_B} = \frac{\gamma^2 \omega^2}{n^2 c \rho_0 \Gamma_B} \tag{3.92}
\]

It is important to note that \( \gamma, n, \nu, \rho_0, \) and \( \Gamma_B \) are the properties of the bulk materials. The relative strength of each peak is, therefore, only a function of mode property and is determined by the coefficient term in brackets in the summation in equation 3.89. The last part in the summation describes a normalized Lorentzian line-shape function. The acousto-optic effective area at acoustic velocity \( v_m \) for mode \( m \) in equation 3.89 is given by

\[
A_m^{ao} = \left( \frac{1}{v_m} \right)^2 \frac{\iint \phi \phi^* dS \iint \xi_m \xi_m^* dS}{\iint \frac{1}{v_m^2} \phi \xi_m \phi^* dS \iint \phi \xi_m^* \phi^* dS} \tag{3.93}
\]

and the Brillouin linewidth for each peak in equation 3.89 is

\[
\Gamma_B^m = \frac{2q_m^* q_m}{q_m^* + q_m} v_m \tag{3.94}
\]

It is worth noting that the effective acoustic-optic area defined in equation 3.93 is more general than that given in Reference 11. The Brillouin linewidth is largely determined by the total acoustic mode loss \( q_m^* \). When there is no waveguide loss, \( q_m^* \) arises entirely from the acoustic loss of the material and \( \Gamma_B^m \) equals the material Brillouin linewidth \( \Gamma_B \). Typically, we have \( q_m^* \gg q_m^* \). Equation 3.94 can be rewritten as

\[
\Gamma_B^m = 2q_m^* v_m \tag{3.95}
\]

Once acoustic modes can be found, equation 3.89 describes Brillouin gain very well. Two examples are given in Figures 3.10–3.13 [9]. The fiber in Figures 3.10 and 3.11 has a germanium-doped core and the fiber in Figures 3.12 and 3.13 has a silica core and a fluorine-doped cladding. In both cases, leaky longitudinal acoustic modes (labeled LL in Figures 3.10–3.13) in addition to guided longitudinal modes (labeled GL in Figures 3.10–3.13) need to be included. These leaky acoustic modes have finite waveguide losses and significant field strength in the region with low acoustic index. The acoustic frequency involved in SBS in optical fibers is typically in the multi-GHz region where acoustical waves are heavily damped by the material loss. These material losses are far higher than...
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The waveguide losses of the leaky acoustic modes and, therefore, dominate. These leaky acoustic modes must, therefore, be considered in modeling SBS in optical fibers.

The dispersion relation of acoustic modes for a fiber with a germanium-doped core is shown in Figure 3.10. The vertical axis is the effective longitudinal acoustic velocity of the acoustic modes and the horizontal axis is the frequency of the acoustic waves. Horizontal lines indicate longitudinal acoustic velocities in the cladding and at the core center. Solid lines show the guided acoustic modes with an effective acoustic index between that of the core center and the cladding. As frequency increases, the effective mode velocity moves toward the velocity at the core center, indicating an increasing concentration of the mode power in the core. The number of guided modes also increases at higher frequencies. A total of four guided longitudinal modes, GL1, GL2, GL3, and GL4, are plotted in this case. There are also leaky acoustic modes in this acoustic waveguide with

FIGURE 3.10 Dispersion relations of guided and leaky acoustic modes along with the momentum-conservation condition for a fiber with a germanium-doped core (for details see Reference 9). GL stands for guided longitudinal acoustic modes and LL, leaky longitudinal acoustic modes. ©2010 IEEE. Reprinted, with permission from Dong, IEEE J. Lightwave Technol. 28:3162–75.

FIGURE 3.11 Measured and simulated SBS gain of the fiber in Figure 3.10. ©2010 IEEE. Reprinted, with permission from Dong, IEEE J. Lightwave Technol. 28:3162–75.
their acoustic index below that of the cladding, i.e., the lowest acoustic index of any glasses in the fiber. Three leaky modes, LL2, LL3, and LL4, are plotted. At the low-frequency end, they start off near the velocity of the cladding and their effective velocity increases with frequency, the opposite of the guided acoustic modes. The momentum conservation condition described in equation 3.78 is also plotted. The crossing points give the positions of peaks in the SBS spectrum. There are four crossings in this case, three for the guided acoustic modes and one for the leaky acoustic modes.
The dispersion relationships are shown with the momentum-conservation condition for the fiber with a silica core and fluorine-doped cladding in Figure 3.12. The longitudinal acoustic velocities of the cladding and at the center of core are also shown. The low acoustic index of the core essentially makes the cladding an acoustic waveguide, capable of supporting a large number of guided acoustic longitudinal modes. GL1 to GL5 are shown in Figure 3.12. The effective velocities of these guided modes move toward the velocity of the cladding at higher frequencies, indicating an increasing concentration of mode power in the cladding. Corresponding leaky acoustic modes, LL1 to LL5, are also shown with their effective velocity above that at the core center, which has the lowest acoustic velocity of any glass in the fiber, and moving away from the velocity at the core center with increasing frequency. There are four intersections with the momentum-conservation condition line for the guided modes and two for the leaky modes.

The fiber with a germanium-doped core, shown in Figures 3.10 and 3.11, has a high acoustic index in the core. The guided longitudinal acoustic modes are mostly in the core and therefore have better overlap with the optical modes. The Brillouin gain is dominated by peaks associated with the guided longitudinal acoustic modes and the leaky longitudinal acoustic mode, mostly in the cladding with the low acoustic index, only contributes a weak shoulder at the high-frequency side. In the fiber with a silica core and fluorine-doped cladding in Figures 3.12 and 3.13 the core has a low acoustic index. Leaky longitudinal acoustic modes, being mostly in the core, have better overlap with the optical mode in this case and, therefore, dominate Brillouin gain. The guided longitudinal acoustic modes, being mostly in the cladding in this case, contribute only weak peaks on the low-frequency end of the Brillouin gain spectrum. The simulated results provide a good fit to the measured Brillouin spectra in both cases.

Recently, much effort has been devoted toward SBS suppression by tailoring the guided longitudinal acoustic modes to minimize their overlap with the optical modes. The need to consider leaky longitudinal acoustic modes requires that the overlap between optical modes and these leaky modes also needs to be minimized. Many designs call for a lower acoustic index in the core where leaky longitudinal modes can dominate the Brillouin gain. The need to consider both types of acoustic modes limits the effectiveness of this approach [13].

The acoustic frequency involved in SBS is ~16GHz at an optical wavelength of ~1 μm in silica fibers with a spectral bandwidth of ~50 MHz. Any back-propagating optical wave with a frequency separation from the Stokes wave less than the SBS gain bandwidth can experience SBS gain. The SBS gain bandwidth is determined by the acoustic damping of the silica glass. The SBS threshold is defined as the input optical power when the backward-propagating Stokes power equals the input optical power at the input end of the fiber. It can be determined by injecting a photon per mode into the output end of a fiber. It was determined for an optical wave with a spectral width significantly less than the SBS spectral width [14],

\[ P_{\text{SBS}} = 21 \frac{A_{m}^{ao}}{g_{B} L_{\text{eff}}} \]  

(3.96)

where \( g_{B} \) is peak Brillouin gain (3–5 x 10^{-11} m/W in a silica fiber, mostly wavelength independent); \( A_{m}^{ao} \) is acousto-optic effective area; and \( L_{\text{eff}} \) is effective nonlinear length.

SBS typically has the lowest threshold among all nonlinear effects for an optical wave with narrow spectral width and can also be a limit in optical fiber telecommunication systems. Since it is a highly coherent process, SBS can be significantly suppressed by spectral broadening to beyond the SBS gain bandwidth. When the bandwidth of a laser \( \Delta \nu \) is significantly larger than Brillouin spectral bandwidth \( \Delta \nu_{\text{SBS}} \), the SBS threshold is increased roughly by a factor \( \Delta \nu/\Delta \nu_{\text{SBS}} \). In telecommunications, the optical carrier wave is modulated at high speed, leading to a significant broadening of the spectral bandwidth, which effectively suppresses SBS. For the same reason, SBS is usually not the major nonlinear limit for pulses with sub-nanosecond durations. In recent years, the drive to significantly scale up power in single-frequency fiber lasers for direct
energy weapons in defense has led to a strong interest in SBS suppression techniques. SBS suppression by bandwidth broadening to many GHz has been used in demonstrating kW-level single-mode fiber lasers.

**STIMULATED RAMAN SCATTERING**

Raman scattering is an inelastic light scattering process involving the excitation of optical phonons via direct interaction between photons and the medium. Light can also be scattered elastically; this is commonly referred to as Rayleigh scattering. Inelastic Raman scattering results in a frequency shift in the scattered light, unlike elastic Rayleigh scattering. The scattered light can have a lower-frequency Stokes or higher-frequency anti-Stokes wave, depending on the populations of the optical phonons in the media. Raman and Krishnan in India [15] and Landsberg and Mandelstam in Russia [16] discovered this effect almost simultaneously in 1928. The scattered Stokes wave can be in the forward as well as backward direction in contrast to SBS. The optical phonons involved have a much higher frequency than the acoustic phonons involved in the SBS process. The dispersion relations of the acoustic branch of phonons involved in SBS and the optical branch of phonons involved in SRS are illustrated in Figures 3.14a and 3.14b along with the wave vectors and frequencies involved in SBS and SRS, respectively. The dispersion relation of the acoustic branch starts from the origin and is almost linear at the small wave vector. It allows a unique acoustic frequency to correspond to an acoustic wave vector. This restricts SBS to a narrow spectral bandwidth. Being almost horizontal, the optical branch, on the other hand, can provide a wide range of wave vectors for very similar frequencies. Phase matching can be met for a much broader spectrum and even for co-propagating and counter-propagating waves (illustrated in Figure 3.14b). The gain bandwidth in SRS, therefore, is significantly larger than that of SBS and is over 40 THz with a peak at ~13 THz in silica fibers. Raman gain coefficients in three types of optical fibers are shown in Figure 3.15 and can be significantly larger in highly germanium-doped fibers.

Raman gain is strongly dependent on the polarization of the pump and Stokes light. Raman gain for a co-polarized Stokes is an order of magnitude higher than for orthogonally polarized Stokes at the Raman gain peak. For a CW, co-polarized and co-propagating pump and Stokes waves at angular frequencies \( \omega_p \) and \( \omega_s \), the evolution of pump power \( P_p(z) \) and Stokes power \( P_s(z) \) are governed by

\[
\frac{\partial P_p(z)}{\partial z} = \frac{g_R}{A_{\text{eff}}} P_p(z) P_s(z) - \alpha_s P_s(z)
\]

\[
\frac{\partial P_s(z)}{\partial z} = -\omega_p \frac{g_R}{\omega_s A_{\text{eff}}} P_p(z) P_s(z) - \alpha_p P_p(z)
\]

\( g_R \) is Raman gain coefficient, \( \alpha_s \) is loss at \( \omega_s \), \( \alpha_p \) is loss at \( \omega_p \), and the effective mode area is defined as

\[
A_{\text{eff}} = \frac{\int \int |E_p(x,y)|^2 dxdy \int \int |E_s(x,y)|^2 dxdy}{\int \int |E_p(x,y)|^2 |E_s(x,y)|^2 dxdy}
\]

where \( E_p(x,y) \) and \( E_s(x,y) \) are the electric field distributions for the pump and Stokes waves, respectively, and the integration is performed over the entire fiber cross section. If pump depletion can be ignored, evolution of the Stokes wave can be obtained,

\[
P_s(z) = P_s(0) e^{-\frac{g_R}{A_{\text{eff}}} \beta_{\text{eff}} - \alpha_s z}
\]
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FIGURE 3.14 Dispersion relations and phase matching for (a) SBS and (b) SRS.
When there is no input at the Stokes frequency in a fiber amplifier, all the Stokes input is from amplified spontaneous emission. The effective input in this case is one photon per mode at the Stokes frequency. The SRS threshold can be determined for the case when Stokes power equals pump power,

\[ P_{\text{SRS}} = 16 \frac{A_{\text{eff}}}{g_R L_{\text{eff}}} \]  

(3.100)

This analysis is very similar to that used by R. G. Smith in his study of the SRS threshold in 1972 [14]. SRS was first observed in optical fibers by Stolen in the same year [18]. In addition to being a detrimental effect in high-power fiber lasers, SRS can also be used to build Raman amplifiers. Stolen and Ippen observed Raman amplification in optical fibers in 1973 [19]. Significant interest emerged in the late 1990s. The development of compact fiber lasers and diode lasers as pumps along with the need for ultra-long-haul high-bandwidth wavelength-division-multiplexing networks provided the necessary impetus. Raman gain exists in almost any fiber, allowing distributed Raman amplifiers to be implemented in transmission fibers by providing the necessary pumps. The resulting distributed amplifiers provide significant noise figure improvement compared to that obtained using a discrete amplifier over the same span. The high Raman gain in dispersion compensating fibers (DCFs), which are already used in conventional systems, can be exploited, resulting in a significant improvement in signal-to-noise ratio. These benefits have led to the deployment of Raman amplifiers in the last decade.

**TRANSVERSE MODE INSTABILITY**

The first reported observation of mode instability, where the single-mode output from a fiber amplifier becomes dynamically unstable at high output powers, showing a clear sign of the presence of higher-order modes, was reported in 2010 by Eidam et al. [20]. Since commercial fiber laser manufacturers had developed and supplied single-mode fiber lasers at multiple kilowatt levels well before this, it is highly likely this was not the first observation of mode instability. Since average power scaling from fiber amplifiers based on photonic crystal fibers was clearly limited by mode instability to just a few hundred watts, a conventional step-index fiber amplifier with a mode field diameter of 27 μm had to be used in this demonstration of femtosecond fiber chirped pulse amplifier
to an average power of 950 W and $M^2 \leq 1.3$ at the fiber amplifier output before the compressor [20]. Further increase in output power was limited by mode instability. In a follow-on paper from the same group [21], clear threshold onset of the mode instability was observed, while the total average output power continued to increase with pump power throughout the transition across the threshold. In fact, it is impossible to see the mode instability just by observing the dependence of the output power on the pump power. The output mode pattern was observed to fluctuate at a frequency of a few kHz just above the threshold and to become more chaotic at increasingly higher powers. A clear degradation of $M^2$ above the mode instability threshold was also observed as expected. Higher seed power, i.e., lower total gain, was seen to increase the mode instability threshold.

If only a small portion of the beam is detected, the fluctuations in mode pattern can be more accurately observed from the power fluctuations at the detector. This is more sensitive to any change at the output and is far more accurate for detecting the mode instability than simply observing the mode pattern change on a camera with limited speed and sensitivity or by measuring $M^2$ [22, 23]. Just above the threshold, the RF spectrum of the detected power sometimes shows clear peaks at a few kHz with related harmonics. Far above the threshold in the chaotic regime, the RF spectrum no longer shows any clear discrete features [22]. It was also shown that a seed linewidth change of a few nanometers has very little effect on the threshold [22].

The dynamics of mode instability were studied using a pulsed pump in [24]. An Yb-doped photonic crystal fiber with a core diameter of 42 μm (MFD = 33 μm) and pump core diameter of 500 μm was used. The rise time of the pump pulse was ~1 ms. It was observed there is a delay (buildup time) before mode instability appears after the pump is turned on. This buildup time was found to be dependent on the pump power varying from 18.3 ms at 800 W to 1.7 ms at 1200 W in this case. It was also found that a shorter delay between the pump pulses can lower the buildup time. At longer delays of >90 ms, there is no longer any influence on buildup time from the earlier pulse. It was also found that the buildup time could be lowered by having a higher pre-pulse pump power below the threshold power. A post-pulse pump power below the threshold was found to sustain the mode instability a bit longer. These observed time scales are clearly consistent with thermal buildup and diffusion times in fibers of these dimensions. It is also worth noting the observation of mode instability with 800 ps pulses at a 78 MHz repetition rate in Reference 20. In this case, the buildup time must have been determined mostly by the average power due to the much smaller delay between pulses (~128 μs in this case).

A number of theoretical studies have since been reported. Power coupling between the fundamental mode and higher-order mode clearly takes place in the fiber amplifier. It is also relatively easy to identify that this may be enabled by a refractive index perturbation arising from mode interference that can produce the correct pitch and spatial distribution for the coupling [25]. In addition, it is also easy to see that the refractive index perturbation is dominated by the thermal optic effect through the spatial variation in quantum defect heating arising from the intensity variation due to mode interference since this is the strongest physical effect. A more subtle aspect is the requirement of a frequency difference in the two interfering modes. This is required so that the coupled light to the higher-order mode from the refractive index perturbation is in phase with the original light in the higher-order mode. This would require the refractive index perturbation to be in the form of a traveling wave, very similar to that of the acoustic wave involved in SBS. Since the periods of mode interference are in the range of millimeters to centimeters in these fibers, longitudinal heat diffusion is too slow to allow such traveling wave at kHz frequencies. The intensity variation as the result of the interfering optical modes can travel longitudinally at this frequency. This can lead to variations in deposited heat by quantum defect heating along the fiber. When combined with a submillisecond transverse thermal diffusion rate in these fibers, this can lead to a traveling temperature wave despite the fact that there is very little longitudinal heat flow in the scale of the interference period during one time period of the traveling wave.

All this physics was first captured in a numerical model based on the assumption of steady-periodic heating [26]. More details and analysis from this model were later reported [27–29].
A slower dynamic model was also later introduced in Reference 22 and a more refined version of this was detailed in Reference 30. Using coupled mode equations based on a steady-periodic heating and the Green’s function temperature solver, a much simpler formulation capturing all the key physics was reported in Reference 31 and a more detailed version was reported later in Reference 32. This model was later expanded to include thermal lensing [33]. Using nonlinear coupled mode equations based on steady-periodic heating and spatial temperature modes to solve the heat transportation equation for a cylindrical boundary, a much simpler analysis with clear physical insights can also be obtained [34]. A detailed review of the models can be found in Reference 35.

Mode instability in optical fibers is a manifestation of stimulated thermal Rayleigh scattering (STRS), first observed in the 1960s in absorbing liquids using giant-pulse Ruby lasers [36–46]. Interference between pumped and scattered light leads to a traveling temperature wave via absorptive heating, which in turn stimulates further power coupling to the scattered light [36]. The basic physics was quickly understood [37, 38], including the nature of the traveling wave and frequency shift in the scattered light. In the following few years, the effect was thoroughly studied and well understood [39–43]. In the early 1980s, there was a resurgence of interest in STRS for phase conjugation and four-wave mixing [44–46]. The similarity between stimulated Rayleigh scattering (SRS) and SBS was noted in Reference 10. In fact, a simultaneous theoretical treatment of the two effects together is given in Reference 10 considering both thermal and electrostrictive effects. The analysis in Reference 10 provides most of the basic physics for understanding mode instability in optical fibers. The simple analytical solutions in Reference 10 are, however, obtained by ignoring any transverse dependence of the electric field and thermal gradient. This may be appropriate for bulk media but is not valid for optical waveguides. The analysis in Reference 10 is also more appropriate for liquids, for which it was developed. There are some very important differences between liquid and solid media.

Like most nonlinear coupling processes, the coupling between the fundamental mode LP_{01} and a higher-order mode LP_{mn} is governed by a set of coupled nonlinear equations [34]:

\[ \frac{\partial P_{01}(z)}{\partial z} = -g_{01} \chi_{mn} P_{01}(z)P_{mn}(z) + g_{01} P_{01}(z) \]  
\[ \frac{\partial P_{mn}(z)}{\partial z} = g_{01} \chi_{mn} P_{01}(z)P_{mn}(z) + (g_{mn} - \alpha_{mn}) P_{mn}(z) \]

(3.101a, b)

where \( g_{01} \) and \( g_{mn} \) are respective linear gains for the LP_{01} and LP_{mn} modes, \( \alpha_{mn} \) is the loss of the LP_{mn} mode, and \( \chi_{mn} \) is the nonlinear coupling coefficient for STRS, which can be written as

\[ g_{01} \chi_{mn} = \frac{2\pi k r}{\rho C} \left( \frac{\lambda_{s}}{\lambda_{p}} - 1 \right) \sum_{l=1}^{\infty} \frac{4 \Omega}{\Gamma_{ml} r^2} \int_{0}^{a} g(r) f_{01}(r) f_{mn}(r) T_{ml}(r) r dr \int_{0}^{b} f_{01}(r) f_{mn}(r) T_{ml}(r) r dr \]

(3.102)

where \( k \) is the vacuum wave vector, \( d \) is the radius of the active region, \( a \) is the core radius, \( b \) is the cladding radius, \( \Omega \) is the angular frequency difference between the two modes \( \Omega = \omega_{mn} - \omega_{01} \), \( \rho \) is density, \( C \) is specific heat, \( k_{r} = d ln DT \) is the thermal optics coefficient, \( \lambda_{s} \) and \( \lambda_{p} \) are the respective signal and pump wavelengths, \( g \) is the spatial gain distribution, \( f_{01} \) and \( f_{mn} \) are the respective electric fields of the LP_{01} and LP_{mn} modes, \( m \) and \( n \) are the respective optical mode numbers, \( l \) is the spatial temperature mode number of the heat transportation equation (for more details see Reference 34),
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\(N_{01}\) and \(N_{mn}\) are the respective mode normalization factors for the \(LP_{01}\) and \(LP_{mn}\) modes, \(T_{ml}\) is the spatial temperature mode, and \(\Gamma_{ml}\) is the heat damping factor. \(N_{01}, \, N_{mn}, \, T_{ml},\) and \(\Gamma_{ml}\) are given by

\[
N_{01} = \int_0^{2\pi} \int_0^\infty r \, dr \, f_{01}^2(r) \, d\phi = 2\pi \int_0^{\infty} f_{01}^2(r) \, r \, dr
\]

(3.103)

\[
N_{mn} = \int_0^{2\pi} \int_0^\infty r \, dr \, f_{mn}^2(r) \cos^2(m\phi) \, d\phi = \pi \int_0^{\infty} f_{mn}^2(r) \, dr
\]

(3.104)

\[
T_{ml}(r) = J_m \left[ \frac{\pi}{4b} \left(4l-1+2m\right) r \right]
\]

(3.105)

\[
\Gamma_{ml} = \frac{2\kappa}{\rho C} \left[ q^2 + \frac{\pi^2}{16b^2}(4l-1+2m)^2 \right]
\]

(3.106)

where \(q\) is the difference in the propagation constants of the \(LP_{01}\) and \(LP_{mn}\) modes and \(\kappa\) is the thermal conductivity.

The nonlinear gain coefficient in equation 3.102 consists of four parts. The first part at the front consists of mainly materials constants and the vacuum wave number. The second part in the bracket describes the quantum defect heating. The third part immediately after the summation sign describes the frequency dependence. The last part describes the two overlap integrals involved in the three-wave interaction. The first integral in the numerator describes the process where the interference of the two modes deposits heat in the active region to create the traveling temperature wave. The second integral in the numerator describes the process whereby the traveling temperature wave leads to further nonlinear coupling of the two modes. The terms in the denominator in the last part are all normalization factors except the damping factor \(\Gamma_{ml}\).

The third part describing the frequency dependence deserves some more comments. The frequency dependence of the nonlinear coupling can be clearly seen by examining this third term, which achieves a maximum of 1 when the frequency difference between the two modes \(\Omega = \Gamma_{ml}/2\). It is easy to see that the there is no coupling at \(\Omega = 0\) rad/s.

The STRS nonlinear coupling coefficient \(\chi\) can be easily evaluated from the equations. The results for four step-index fibers with NA = 0.06, cladding diameter 2\(b = 400\)\,\(\mu m\), and core diameters and doped-area diameters \(2a = 2d = 30\)\,\(\mu m\), 25\,\(\mu m\), 20\,\(\mu m\), and 15\,\(\mu m\) are summarized in Figure 3.16. These fibers are good representations of LMA fibers commonly found in high-power fiber amplifiers. The variations in core diameter lead to different \(V\) values at 5.3348, 4.4456, 3.5565, and 2.6674, respectively, for the four fibers. The \(LP_{mn}\) mode experiences gain when it has a higher frequency and loss when it has a lower frequency than that of the fundamental mode. This is opposite to other familiar nonlinear processes such as SRS and SBS. Energy conservation still applies. The pump adds additional energy to the system. The move of the peak toward higher frequency for smaller-core diameters is a reflection of a higher damping factor, i.e., faster transverse diffusion. It will be shown later on that the smaller nonlinear coupling coefficient at the peak for smaller-core diameters in this case is largely a reflection of the smaller \(V\) value of these fibers.

For the 30-\(\mu m\) core step-index fiber, the dependence of peak \(\chi\) and its corresponding frequency \(f_{\text{max}}\) on \(V\) value, obtained by varying the NA while keeping the rest of parameters unchanged, is shown in Figure 3.17. When \(V\) is reduced from 5.5, the peak nonlinear coupling coefficient decreases initially very slowly and this decrease then accelerates near \(LP_{11}\) mode cut-off at around 2.405. The absolute value of the peak frequency \(f_{\text{max}}\) decreases at smaller \(V\), reflecting the increasing
delocalization of the LP₁₁ mode while moving toward its cut-off. A reduction of the doped area can also reduce the peak nonlinear coupling coefficient [see Figure 3.17(b)]. This reduction is, however, small when \( \frac{d}{a} \) is near 1. Reducing the nonlinear coefficient by 50% requires \( \frac{d}{a} \approx 0.45 \), i.e., a doped area reduction of \( \sim 80\% \). This would require a significant increase in doping level to maintain the same level of gain/absorption per unit length. Considering doping levels are already near their upper limits in many fibers, this may not be possible. The peak frequency \( f_{\text{max}} \) increases with a reduction in core diameter as expected. The nonlinear coupling coefficient remains almost constant at various core diameters.

The overlap integrals in equation 3.102 are essentially dependent only on \( V \).

In cases where the power in the higher-order mode is smaller than that in the fundamental mode, the power in the LP_{mn} mode at the end of the amplifier can be written as

\[
P_{mn}(L) = P_{mn}(0) e^{\int_{0}^{L} \chi_{mn}(z) P_{01}(z) dz} \]

(3.107)
The first term in the exponent represents the linear gain in the amplifier and the second term the nonlinear gain. The local nature of gain \( g_{01}(z) \) is emphasized in the integration. This gain is dependent on inversion, which can vary along the fiber, leading the nonlinear coefficient \( \chi_{mn} \) to also be dependent on \( z \). The local nonlinear gain is driven by the product \( g_{01}(z) P_{01}(z) \). This is different from other nonlinear processes, reflecting the fact that the gain is essential in this case and the process is fundamentally driven by quantum defect heating, which is manifested in the gain. When the local gain is zero, there is no nonlinear gain for STRS.

In an unsaturated fiber amplifier where \( P_{01}(z) \) has an approximately exponential growth and in the high gain regime (in this case, \( \chi_{mn} \) and \( g_{01} \) are independent of \( z \)), equation 3.107 can be simplified to

\[
P_{mn}(L) = P_{mn}(0)e^{(\gamma_{mn} - \alpha_{mn})L + \chi_{mn} g_{01}(L)}
\]  

(3.108)

In this case, the nonlinear gain is only dependent on the nonlinear coupling coefficient \( \chi_{mn} \) and the output power in the \( LP_{01} \) mode. Assuming the threshold is defined as occurring when the power in the \( LP_{mn} \) mode reaches a certain fraction of the \( LP_{01} \) mode at the output, i.e., \( x = P_{mn}(L)/P_{01}(L) \), the threshold can be written as

\[
P_{01}^{th} = \frac{1}{\chi_{mn}} \left[ \ln \left( \frac{P_{01}(0)}{P_{mn}(0)} \right) - (g_{mn} - g_{01} - \alpha_{mn})L \right]
\]  

(3.109)

The second term in the bracket is typically much smaller than the first term. In this case of high gain in an unsaturated fiber amplifier, the threshold is largely determined by the nonlinear coefficient \( \chi \) and the relative input power \( LP_{01}(0)/LP_{mn}(0) \) and is independent of gain.

An amplifier based on an LMA fiber with 30 \( \mu \)m core diameter and NA of 0.06 was studied numerically by solving the coupled mode equations for threshold powers for a range of input conditions. The fiber parameters are \( NA = 0.06, 2b = 400 \mu m, 2a = 2d = 30 \mu m, V = 5.3348, \) and \( \alpha_{11} = 0 \). Both \( g_{01} \) and \( g_{11} \) are considered. The results versus total gain factor \( g_{01}L \) (plotted in dBs) are summarized in Figure 3.19. The predicted threshold powers from equation 3.109 are plotted as dashed lines. The threshold power is independent of gain when \( g_{01}L > 4 \), i.e., \( \sim 17 \) dB. Below this, the threshold power increases with a reduction in gain. For \( g_{01}L > 4 \), equation 3.109 fits the numerical data.
The threshold at $g_0 L = 0$ is expected to be infinite due to the total absence of quantum defect heating. The solid lines in Figure 3.19 are from

$$P_{th}^{01} = \frac{1}{\chi_{nm}} \ln \left[ x \frac{P_{01}(0)}{P_{mn}(0)} \right] e^{1.25i \theta_{01}^L}$$

(3.110)

The mode instability threshold is clearly dependent on the input conditions of the higher-order mode. It can be seeded by quantum noise in the absence of any other seeding sources. Since most seed input to an amplifier has a linewidth significantly wider than a few kHz, an amplifier seed input at the right frequency can seed the higher-order mode. In the case of single-frequency seeds with very narrow linewidths, it can also be seeded by the amplifier seed and pump noise [27].

In practice, most seed lasers can have much broader spectra than that of the STRS gain. In cases where the input power spectrum is broader than that of the STRS gain spectrum, if the power in the higher-order mode at a given frequency is seeded by an input signal at a very similar frequency (less than a few kHz frequency separation for a core diameter over 30 $\mu$m), it is reasonable to assume that the phases of the fields of $LP_{01}$ and $LP_{11}$ modes, with a small frequency separation of $\Omega$, are identical at the amplifier input. This may in fact be true across the entire input signal spectrum even for the case where the seed is an ASE source as in Reference 22. In this case, the input signal power at any given frequency can interfere with the power in the $LP_{mn}$ mode at an adjacent frequency $\Omega$ to produce an intensity traveling wave described by $\exp(i(qz - \Omega t))$, if the dispersive effects can be ignored. In other words, power at any frequency within the input power spectrum can interfere with its corresponding power in the $LP_{mn}$ mode to add to the intensity of the traveling temperature wave described by $\exp(i(qz - \Omega t))$. This is only true when the phase difference between the two fields in the interfering modes is constant across the power spectrum and the dispersive effects are small. This is not the case, for example, for SBS, where the counterpropagating wave is seeded from quantum noise without any fixed phase relationship to the input signal. This collaborative effect can lead to the possibility that the total power of the input signal contributes toward nonlinear coupling at any local frequency within the input signal spectrum in STRS despite the fact that the power spectrum of the signal is much larger than the STRS gain spectrum as long as the dispersive effects are small enough. This effect can lead to the mode instability threshold being independent of input signal bandwidth but more dependent on the total

---

**FIGURE 3.19** Simulated threshold powers (a) at $x = 1\%$ and various input conditions with $P_{01}(0)/P_{01}(0) = 10^{-5}, 10^{-10}, 10^{-15}, 10^{-20}, 10^{-25},$ and $10^{-30}$ and (b) $x = 0.0526$, i.e., 5% of total power in $LP_{11}$ mode, for $P_{11}(0)/P_{01}(0) = 10^{-2}, 10^{-3},$ and $10^{-4}$. The fiber parameters are $NA = 0.06$, $2b = 400\mu$m, $2a = 2d = 30\mu$m, $V = 5.3348$, and $\alpha_{11} = 0$. The dashed lines in (a) are obtained from equation 3.109 and solid lines are from equation 3.110.
power of the input signal as experimentally observed in Reference 22. This effect is unique only to STRS due to the fact that the interfering fields involved originate from the same source and can, therefore, have identical phase.

The mode instability threshold has been observed to increase when operating in the low-gain regime [20] and by gain profile tailoring as demonstrated in Reference 47. Gain profile tailoring can reduce the first overlap integral in equation 3.102 where the temperature wave is driven by quantum defect heating and mode interference. Lowering the quantum defect, i.e., the second term in equation 3.102, can also mitigate mode instability. A more effective method is going to smaller-core diameters with lower $V$. This, however, lowers thresholds of other nonlinear effects. Current mode instability thresholds in LMA fibers are in the 2–3 kW range [21, 48] and depend on the details of the thermal load on the fiber including the inversion of the amplifier.

Degradation of the mode instability threshold over repeated operation has also been observed [49]. This is probably due to additional photo darkening of the fiber, which has the potential to increase the thermal load in the amplifier. This paper also suggested using dynamic mode control at the input for the mitigation of the mode instability. This is not likely to be practical. The reported results still show an unacceptable level of higher-order modes even under dynamic control.

CONCLUSIONS

We have covered most of the important nonlinear effects in optical fibers in this chapter. The objective is to give readers a well-grounded perspective on the physics and features of the numerous nonlinear processes that can take place in optical fibers. Most of these nonlinear effects present limitations to power scaling of fiber lasers. Many of them can also offer opportunities for wavelength conversion, lasers, and amplifiers. We have not covered many of the wide range of applications based on nonlinear effects in optical fibers. Readers are encouraged to consult other books [3, 10] for further reading.

REFERENCES
