# Attitude Parametrization, Kinematics, and Dynamics

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## 3.1 INTRODUCTION

This chapter provides a fundamental theory of attitude parametrization, kinematics and dynamics of a rigid body. Attitude parametrization of a rigid body is the mathematical representation of its orientation in the three-dimensional space. Usually, the orientation of a rigid body is given with respect to frame called *inertial fixed reference system*. Other reference systems will be defined and can be used to parametrize the attitude [1].
Various attitude parametrizations exist in the literature [12]. The natural representation of attitude is rotation matrices elements of the Special Orthogonal Group $SO(3)$. Each physical attitude position can be parametrized by one and only one rotation matrix, which means that this parametrization is unique. In the same time, all possible physical attitude positions and attitude motions can be parametrized using rotation matrix, which means that this parametrization is global. All other attitude parametrizations are either nonunique and global or nonunique and singular. Nonunique parametrization means that there exist at least two mathematical values representing only one physical attitude position, which is the case of quaternion parametrization. Singular parametrization means that there exists at least one physical attitude position or attitude motion that is undefined mathematically. Table 3.1 summarizes the properties of various parametrizations [1].

The rotational motions of a rigid body about the center of mass (CoM) are described using attitude motion. The study of attitude kinematics [9,13] is based on the time derivative of a vector in a rotating coordinate system. Depending on the used attitude parametrization, several formulations for attitude kinematics will be detailed. After, the attitude dynamics will be illustrated using the Newton’s law of motion.

### 3.2 REFERENCE SYSTEMS

#### 3.2.1 Mobile Reference System \( \{ B \} \)

This reference is associated with the vehicle (the specific reference to the mobile), with its origin at the CoM of the mobile. The axes of this frame are as follows:

- **x-axis**: Directed along the longitudinal axis oriented from the rear toward the front;
- **y-axis**: Directed along the transverse axis oriented from left to right;
- **z-axis**: Completes the direct Cartesian coordinate following the rule of the right hand (Figure 3.1).

<table>
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<th>Attitude Parametrization</th>
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**TABLE 3.1**

Global and Unique Properties of Attitude Parametrization

![FIGURE 3.1 Mobile frame.](image)
3.2.2 **The Navigation Reference System \( \{n\} \)**

Defined in the *local tangent plane* (LTP), its origin is always at the current position of the mobile and the \( xy \) plane is tangent to the surface of the Earth. Two conventions LTP systems are common in navigation:

- **NED**: North, East, Down (down in the direction of the gravity vector), (Figure 3.2);
- **ENU**: East, North, Up (up in the inverse direction of the gravity vector).

3.2.3 **The ECI Reference System \( \{I\} \)**

The *Earth-centered inertial* (ECI) is a system in which Newton's laws are applicable. It does not follow the rotation of the Earth and therefore do not rotate relative to the stars. The origin of this system is the center of the Earth. The corresponding coordinate system is a coordinate system with axes marked:

- **x-axis**: To the “Vernal Equinox” (distant star);
- **y-axis**: North Pole;
- **z-axis**: To complete the direct reference system.

3.2.4 **The ECEF Reference System \( \{e\} \)**

The *Earth-centered Earth-fixed* (ECEF) follows the rotation of the Earth and the origin of this system is the center of the Earth; therefore, this system coincides with the inertial system once a complete revolution of the Earth on itself.

- **x-axis**: To the Greenwich meridian (longitude = 0);
- **y-axis**: North Pole;
- **z-axis**: To complete the direct reference system, (Figure 3.2).

![ECEF, NED, and Geodetic Coordinates systems](image-url)
3.2.5 Geodetic Coordinates System (The WGS-84 Standard)

Several problems arise when we wish to get absolute position of an object on the globe.

- The Earth is not actually a volume of regular shape. It is usually treated as a geoid or ellipsoid.
- The geoid is an equipotential surface coinciding with the mean sea level and at each point perpendicular to the direction of the local vertical (direction of gravity).
- The ellipsoid is a mathematical surface coinciding as well as possible with the geoid and usually characterized by its semi-major axis and flattening. Depending on the location of the globe where there is some local ellipsoid models that are more accurate than others.

The WGS 84 is a three-dimensional terrestrial reference system expressing the position in terms of latitude, longitude, and altitude. These are based on a reference ellipsoid that is an approximation of the shape of the Earth.

*The latitude* \( \phi \): This is the angle between the equatorial plane and the normal to the surface of the Earth (ellipsoid) at the point in question. It is zero at the equator and is counted positive for the northern hemisphere and negative for the southern hemisphere.

*The longitude* \( \lambda \): This is the angle between the Greenwich meridian and the desired point. It is counted positively toward the east.

*The height* \( h \): ellipsoidal height—not to be confused with altitude: This is the difference in meters between that point and the reference ellipsoid measured normal to the ellipsoid. This value is set in a geodetic system and may differ from the altitude of several tens of meters. It should be noted that in general the satellite positioning systems provide ellipsoidal height and not an altitude.

The altitude of a point M of a topographic surface is an approximation of the distance between the point and the reference surface known as the geoid (Figure 3.2).

3.3 Attitude Representation

The orientation of a rigid body in space is often crucial, especially in aerospace applications. In this section, we provide a description of various attitude parametrizations. Especially, four types of attitude representations are detailed, as described in Table 3.1. The natural parametrization of rigid body attitude is the set of orthogonal matrices whose determinant is one \( [3] \), named rotation matrix or direction cosine matrix (DCM). It is a unique and global mathematical parametrization. All others are only either global, such as the axis-angle and unit quaternion representations, or singular, and not unique, such as Euler angles. Axis-angle and unit quaternions use four parameters to represent the attitude. Usually, the singularity is due to the fact that only three parameters are used, which is the case of Euler angles. In the literature \([9]\), two other singular minimal parametrizations are used: Rodrigues parameters and modified Rodrigues parameters.

3.3.1 Rotation Matrices and Axis-Angle Representations

We call a rotation matrix or DCM \([13]\), denoted by \( R \) every rotation of the mobile frame \( \{B\} \) relative to the inertial fixed frame \( \{I\} \). Let \( e_1^B, e_2^B, \) and \( e_3^B \in \mathbb{R}^3 \) be the column vectors of the principal axis of \( \{B\} \) expressed in \( \{I\} \), then
\[ R = \begin{bmatrix} e_1^b & e_2^b & e_3^b \end{bmatrix}, \quad (3.1) \]

where \( e_1^b, e_2^b, e_3^b \) are column vectors forming the columns of \( R \).

**Remark 3.1**

Another definition of the rotation matrix is used where \( e_1^b, e_2^b, e_3^b \) are column vectors forming the rows of \( R \).

The correspondence between a vector \( b \in \mathbb{R}^3 \) expressed in \( \{B\} \) and its equivalent vector \( r \in \mathbb{R}^3 \) expressed in \( \{I\} \) can be written as

\[ r = Rb \quad (3.2) \]

Rotation matrices form a group under the operation of matrix multiplication called the *special orthogonal* group \( SO(3) \subset \mathbb{R}^{3 \times 3} \). The abbreviation \( SO \) refers to the properties of rotation matrices:

\[ SO(3) = \{ R \in \mathbb{R}^{3 \times 3} | R^T R = RR^T = I_d, \det(R) = 1 \} \quad (3.3) \]

where \( I_d \) is the 3 by 3 identity matrix.

### 3.3.1.1 Properties of the Rotation Matrix \( R \)

According to Euler’s theorem, for every rotation matrix \( R \), there exist an invariant vector \( a \) such that \( Ra = a \). This means that the vector \( a \) is an eigenvector of \( R \) corresponding to the eigenvalue \( \lambda = 1 \). In this case, a line \( \beta a \) is called rotation axis of \( R \) and the eigenvalues of \( R \) are \( \{1, e^{i\beta}, e^{-i\beta}\} = \{1, \cos(\beta) + \sin(\beta), \cos(\beta) - \sin(\beta)\} \), where \( \beta \) is the angle of Euler axis \( \beta a \) and \( i \) is the standard imaginary unit \( (i^2 = -1) \). Then,

1. The sum of eigenvalues of \( R \) defines its trace:

\[ \text{trace}(R) = 1 + 2\cos(\beta) \quad (3.4) \]

2. Since for every \( \beta \in \mathbb{R} \), we have \(-1 \leq \cos(\beta) \leq 1\). Therefore,

\[ -1 \leq \text{trace}(R) \leq 3 \quad (3.5) \]

3. The product of eigenvalues of \( R \) defines its determinant:

\[ \det(R) = 1, \quad (3.6) \]

Owing to this property, the group of rotation matrices is called *special*.

4. Since \( e_1^b, e_2^b, e_3^b \) form an orthonormal basis, then the matrix \( R \) is real and orthogonal matrix. This means that

\[ R^T R = RR^T = I_d, \quad (3.7) \]

Owing to this property, the group of rotation matrices is called *orthogonal*.

### 3.3.1.2 Lie Algebra of \( SO(3) \)

It will be shown that the rotation of a rigid body around a given unit vector \( a \in \mathbb{R}^3 \) with a given angle \( \theta \in \mathbb{R} \) conducts directly to the notion of the Lie algebra of \( SO(3) \). Consider a point \( p(t) \in \mathbb{R}^3 \) of a rigid
body rotating around the axis \( \mathbf{a} \) with an initial condition denoted \( \mathbf{p}(0) \in \mathbb{R}^3 \), the time derivative of \( \mathbf{p}(t) \) can be written as

\[
\frac{d}{dt}(\mathbf{p}(t)) = \mathbf{a} \times \mathbf{p}(t) = \mathcal{A} \mathbf{p}(t),
\]

(3.8)

where \( \times \) stands for the vector cross product and \( \mathcal{A} \in \mathbb{R}^{3 \times 3} \). Note that the cross product \( \mathbf{a} \times \mathbf{p}(t) \) can be written as a product of a matrix \( \mathcal{A} \) and a vector \( \mathbf{p}(t) \). The properties of the matrix \( \mathcal{A} \) will be detailed later.

The unique solution of differential equation (3.8) is well known to be \( \mathbf{p}(t) = e^{\mathcal{A}t} \mathbf{p}(0) \), which means that the matrix exponential \( e^{\mathcal{A}t} \) is nothing rather than a rotation of the point \( \mathbf{p} \) from the initial position \( \mathbf{p}(0) \) to a new position \( \mathbf{p}(t) \). Therefore, the rotation matrix \( \mathcal{R} \) can be given by

\[
\mathcal{R} = e^{\mathcal{A}t}
\]

(3.9)

**Remark 3.2**

The uniqueness of the existence of the matrix \( \mathcal{R} \) can be derived from the fact that \( \mathbf{p}(t) = \mathbf{p}(0) e^{\mathcal{A}t} \) is a unique solution of Equation 3.8. Also, this equation is verified for every physical point \( \mathbf{p}(t) \) which means that \( \mathcal{R} \) is global.

The matrix \( \mathcal{A} \) is formed by the elements of the vector \( \mathbf{a} \). In general, for any two vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \), we can denote \( \mathbf{x} \times \mathbf{y} = S(\mathbf{x})\mathbf{y} \), where \( S(\mathbf{x}) \) is a skew-symmetric matrix given by

\[
S(\mathbf{x}) = \begin{bmatrix}
0 & -x_z & x_y \\
x_z & 0 & -x_x \\
-x_y & x_x & 0
\end{bmatrix}
\]

(3.10)

With this notation, the relation between the unit vector \( \mathbf{a} \) (which specify the direction of the rotation), the angle of rotation \( \theta \) and the rotation matrix \( \mathcal{R} \) can be given by

\[
\mathcal{R}(\mathbf{a}, \theta) = e^{S(\mathbf{a})\theta}
\]

(3.11)

The set of skew-symmetric matrices \( S(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3 \) is called the Lie algebra of \( SO(3) \) and denoted \( \mathfrak{so}(3) \), defined by

\[
\mathfrak{so}(3) = \{ \mathbf{A} \in \mathbb{R}^{3 \times 3} | \mathbf{A}^T = -\mathbf{A} \}
\]

and \( S \) is the Lie algebra isomorphism from \( \mathbb{R}^3 \to \mathfrak{so}(3) \) that associates to \( \mathbf{x} \in \mathbb{R}^3 \) the skew-symmetric matrix \( S(\mathbf{x}) \).

For every \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^3 \) and a given \( \mathcal{R} \in SO(3) \), the following identities can be verified:

\[
S(\mathbf{x})\mathbf{y} = -S(\mathbf{y})\mathbf{x}
\]

(3.12)

\[
S(\mathbf{x})\mathbf{x} = \mathbf{0}
\]

(3.13)

\[
S(\mathbf{x})S(\mathbf{y}) = \mathbf{y}\mathbf{x}^T - \mathbf{x}^T \mathbf{y} I_d
\]

(3.14)

\[
S^2(\mathbf{x}) = \mathbf{x}\mathbf{x}^T - \mathbf{x}^T \mathbf{x} I_d
\]

(3.15)
\[ S^1(x) = -x^T x S(x) \] (3.16)

\[ S(S(x)y) = S(x)S(y) - S(y)S(x) \] (3.17)

\[ S(Rx) = RS(x)R^T \] (3.18)

\[ S(x)^T = -S(x) \] (3.19)

The eigenvalues of \( S(x) \) are \( \{0, i \|x\|, -i \|x\|\} \).

For every \( x, y \in \mathbb{R}^3 \) and any constant matrix \( A \in \mathbb{R}^{3 \times 3} \), the following partial derivatives can be verified:

\[
\frac{\partial}{\partial x}[S(x)y] = S(y)
\] (3.20)

\[
\frac{\partial}{\partial x}[x^T Ax] = (A + A^T)x
\] (3.21)

\[
\frac{\partial}{\partial x}[S(x)Ax] = S(x)A + S(Ax)
\] (3.22)

\[
\frac{\partial}{\partial x}[S^2(x)y] = S(y)S(x) - 2S(x)S(y) = x^T y I_d + xy^T - 2yx^T
\] (3.23)

\[
\frac{\partial}{\partial x}[S(x)^2 Ax] = S(x)^2 A + x(Ax)^T - 2(Ax)x^T + x^T (Ax)I_d
\] (3.24)

### 3.3.1.3 The Exponential Map \( \mathfrak{so}(3) \to SO(3) \)

Let \( a \) be a unit vector representing the direction of a rotation with an angle \( \theta \), corresponding to a rotation matrix \( R \). Therefore, the matrix \( R \) can be expressed in function of \( (a, \theta) \) by Equation 3.11. Using Taylor expansion, one can get

\[
R(a, \theta) = e^{S(a)\theta} = I_d + S(a)\theta + \frac{S^2(a)}{2!}\theta^2 + \frac{S^3(a)}{3!}\theta^3 + \ldots
\] (3.25)

At first time, Equation 3.25 seems to be unusable since it is an infinite series. In what follows, we show that it is possible to get a closed form of Equation 3.25. Before, let us state the following lemma.

**Lemma 3.1**

Let \( x \in (\mathbb{R}^3)^* \) and \( S(x) \in \mathfrak{so}(3) \). Then, for any integer \( n \geq 3 \), the following identities can be verified

\[
S^n(x) = \begin{cases} 
\frac{1}{n!} (-1)^k \|x\|^k S(x) & \text{if } n = 2k + 1 \text{ and } k \geq 1 \\
\frac{1}{n!} (-1)^k \|x\|^k S^2(x) & \text{if } n = 2k \text{ and } k \geq 2 
\end{cases}
\] (3.26)

**Proof.** The proof is trivial and left as an exercise to the reader (indication: use Cayley–Hamilton theorem). 

\[ \Box \]
Using Equation 3.26, the Taylor expansion (Equation 3.25) can be rewritten as
\[
R(a, \theta) = I_d + \frac{S(a)}{\|a\|} \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\|a\| \theta)^{2n+1} \right) + \frac{S^2(a)}{\|a\|^2} \left( 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\|a\| \theta)^{2n} \right),
\]
using expansion Taylor theorem for \(sin\) and \(cosine\) functions, one can get
\[
R(a, \theta) = I_d + \frac{S(a)}{\|a\|} \sin(\|a\| \theta) + \frac{S^2(a)}{\|a\|^2} (1 - \cos(\|a\| \theta)),
\]
finally, using the fact that \(a\) is a unit vector, one can obtain Rodrigues formula given by
\[
R(a, \theta) = I_d + S(a)\sin(\theta) + S^2(a)(1 - \cos(\theta)),
\]
3.3.1.4 Angle-Axis \((a, \theta)\) from Rotation Matrix \(R\)
As mentioned before, angle-axis representation is global but not unique. To show these properties, it suffices to express the angle-axis parametrization in function of rotation matrix. Given a rotation matrix \(R \in SO(3)\) as follows:
\[
R = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix},
\]
and using the properties of rotation matrices detailed before, one can verify that for every matrix \(R(a, \theta) = e^{S(a)\theta}\) we have
\[
\theta = \arccos \left( \frac{1}{2} (\text{trace}(R)) - 1 \right),
\]
and
\[
a = \frac{1}{2\sin(\theta)} \begin{bmatrix}
    r_{32} - r_{23} \\
    r_{13} - r_{31} \\
    r_{21} - r_{12}
\end{bmatrix}, \quad \text{if } \theta \neq 0
\]
where \(\arccos = \cos^{-1}\). Note that if \(\theta = 0\) and using Equation 3.28, one can get that \(R(a, 0) = I_d\) and \(a\) can be chosen arbitrary. Observing Equation 3.30, one can conclude that for one value of \(R\) there are two corresponding values of \(\theta(\theta + 2k\pi, -\theta + 2k\pi)\), which give us two directions of rotations \((a, -a)\). This means that the couple \((a, \theta)\) can represent \(R\) globally, but not uniquely. The proof of Equations 3.30 and 3.31 can be found in Chapter 2 of [13].

3.3.2 Quaternion Parametrization
Generally, the Euler axis-angle attitude representation is not trivial for the mathematical manipulation point of view. For this and to give another global parametrization using only four parameters [9,12,13]
(against nine in rotation matrix parametrization), Euler extend his theorem of angle-axis representation by introducing a rotation around a unit vector \( \mathbf{a} \) considered as imaginary complex part with an angle \( \theta \) considered as scalar part, which gives

\[
Q = e^{\frac{\mathbf{a} \theta}{2}} = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})(\mathbf{a}i + \mathbf{a}j + \mathbf{a}k),
\]

(3.32)

where \( i^2 = j^2 = k^2 = ijk = -1 \). This is a generalization of complex numbers. Using \( 1, i, j, k \) as a basis, we can note \( Q = (q_0, \mathbf{q}) \), thus

\[
Q = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})\mathbf{a} \end{bmatrix},
\]

(3.33)

where \( q_0 \in \mathbb{R} \) and \( \mathbf{q} \in \mathbb{R}^3 \). This notation conducts us to the fact that in general \( Q \in \mathbb{R}^4 \), but since \( \mathbf{a} \) is a unit vector, therefore

\[
\|Q\| = Q^T Q = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1,
\]

(3.34)

where

\[
\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix},
\]

which means that the set of unit quaternions defines the unit sphere \( S^3 \) such that:

\[
S^3 = \{Q \in \mathbb{R}^4 \mid Q^T Q = 1\}
\]

(3.35)

Using Equation 3.32, one can conclude that the multiplication of two quaternions \( P = (p_0, \mathbf{p}) \) and \( Q = (q_0, \mathbf{q}) \) is a quaternion and if we denoted it by “\( \odot \),” then

\[
P \odot Q = \begin{bmatrix} p_0q_0 - \mathbf{p}^T \mathbf{q} \\ p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q} \end{bmatrix},
\]

(3.36)

and the inverse of a quaternion \( Q = (q_0, \mathbf{q}) \) is also a quaternion defined by \( Q^{-1} = (q_0, -\mathbf{q}) \).

### 3.3.2.1 Angle-Axis \((\mathbf{a}, \theta)\) from Unit Quaternion \(Q\)

Using Equation 3.33 and given a unit quaternion \( Q \in S^3 \), one can get the angle-axis representation \((\mathbf{a}(Q), \theta(Q))\) corresponding to \( Q \) as follow

\[
\theta = 2\arccos(q_0),
\]

(3.37)

and

\[
\mathbf{a} = \begin{cases} \frac{q}{\sin(\frac{\theta}{2})} & \text{if } \theta \neq 0, \\ 0 & \text{otherwise}, \end{cases}
\]

(3.38)
3.3.2.2 Rotation Matrix $R$ from Unit Quaternion $Q$

Denote the mapping from $S^3$ to $SO(3)$ as $\mathcal{R}: S^3 \rightarrow SO(3)$. Given a quaternion $Q \in S^3$, the goal is to find the corresponding rotation matrix $R$, such that $R = \mathcal{R}(Q)$. Using the fact that $1 - \cos(\theta) = 2\sin\left(\frac{\theta}{2}\right)^2$ and $\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ together with Equations 3.37 and 3.38, one can get from Rodrigues formula (Equation 3.28) the Euler–Rodrigues rotation formula as

$$R = \mathcal{R}(Q) = I_d + 2q_0 S(q) + 2S^2(q),$$

(3.39)

where $Q \in S^3$. It is easy to verify that $\mathcal{R}(Q) = \mathcal{R}(-Q)$, where $-Q = (-q_0, -q)$, which means that $\mathcal{R}$ defines a double covering map of $SO(3)$ by $S^3$, that is, for every $R \in SO(3)$ the equation $\mathcal{R}(Q) = R$ admits exactly two solutions $Q_R$ and $-Q_R$. As a consequence, a vector field $f$ of $S^3$ projects onto a vector field of $SO(3)$ if and only if, for every $Q \in S^3$, $f(-Q) = -f(Q)$ (where we have made the obvious identification between $T_Q S^3$ the tangent space of $S^3$ at $Q$ and $T_{-Q} S^3$ the tangent space of $S^3$ at $-Q$) (for more details see [6,10]).

3.3.2.3 Unit Quaternion $Q$ from Rotation Matrix $R$

Given a rotation matrix $R \in SO(3)$ defined by Equation 3.29 and using the Euler–Rodrigues rotation formula (Equation 3.39), one can have

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 2(q_0^2 + q_2^2) - 1 & 2(q_0 q_2 - q_3) & 2(q_0 q_3 + q_1) \\ 2(q_0 q_3 + q_1) & 2(q_0^2 + q_3^2) - 1 & 2(q_1 q_3 - q_2) \\ 2(q_0 q_1 - q_3) & 2(q_1 q_3 + q_2) & 2(q_0^2 + q_1^2) - 1 \end{bmatrix},$$

(3.40)

where

$$Q = (q_0, q) = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}. $$

Note that from Equation 3.40 one can get $Q$ from $R$ in different ways, depending on which elements of $R$ we want to use. The easiest one is to use only the diagonal of $R$. Let us first determine the value of $q_0$. Using the first property of $R$ (Equation 3.4) and the fact that $\cos(\theta) = 2\cos\left(\frac{\theta}{2}\right)^2 - 1$, it is straightforward to obtain

$$q_0 = \pm \frac{1}{2}\sqrt{1 + \text{trace}(R)},$$

(3.41)

Thus, using Equations 3.40, 3.41, and 3.34, one can get

$$q_1 = \pm \sqrt{\frac{1}{2}(r_{11} + 1) - q_0^2} = \pm \frac{1}{2}\sqrt{2r_{11} + 1 - \text{trace}(R)},$$
$$q_2 = \pm \sqrt{\frac{1}{2}(r_{22} + 1) - q_0^2} = \pm \frac{1}{2}\sqrt{2r_{22} + 1 - \text{trace}(R)},$$
$$q_3 = \pm \sqrt{\frac{1}{2}(r_{33} + 1) - q_0^2} = \pm \frac{1}{2}\sqrt{2r_{33} + 1 - \text{trace}(R)},$$

note that if $R = I_d$, then $Q = (\pm 1, 0)$. 
3.3.2.4 Vector Rotation Using Unit Quaternions

Let \( \mathbf{b} \in \mathbb{R}^3 \), be a vector expressed in \( \{B \} \) corresponding to the vector \( \mathbf{r} \in \mathbb{R}^3 \) expressed in \( \{I \} \). The rotation of a vector \( \mathbf{b} \) needs the quaternion \( Q \) and its inverse \( Q^{-1} \) as follows:

$$ \mathbf{r} = Q \circ \mathbf{b} \circ Q^{-1}, $$

(3.42)

where \( \mathbf{r} \) and \( \mathbf{b} \) are the pure quaternions of \( \mathbf{r} \) and \( \mathbf{b} \) such that \( \mathbf{r} = (0, \mathbf{r}) \) and \( \mathbf{b} = (0, \mathbf{b}) \). Let us prove that Equation 3.42 is equivalent to Equation 3.2. Using Equations 3.42 and 3.36 the vector part of \( \mathbf{r} \) can be written as

$$ r = qq^T b + q_0^2 b + 2q_0 S(q)b - S(q)b, $$

where the property (Equation 3.17) was used. Now, using Equations 3.15 and 3.34, one can get

$$ qq^T = S^2(q) + (1 - q_0^2)I. $$

Replacing this last expression in the above equality and after some manipulations, it is straightforward to obtain

$$ r = (I_d + 2q_0 S(q) + 2S^2(q))b, $$

which is equivalent to Equation 3.2.

3.3.3 Euler Angles Parametrization

In Euler angles representation, the rotation from frame \( \{I\} \) to frame \( \{B\} \) is formed by three successive rotations using the right hand rule. A rotation around the axis \( e_3 \) with an angle \( \psi \) is called yaw angle, which transforms the basis \( \{e_1, e_2, e_3\} \) to \( \{e'_1, e'_2, e'_3\} \). The corresponding rotation matrix is

$$ R_z(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

(3.43)

A rotation around the axis \( e'_2 \) with an angle \( \theta \) is called pitch angle, which transforms the basis \( \{e'_1, e'_2, e'_3\} \) to \( \{e''_1, e''_2, e''_3\} \). The corresponding rotation matrix is

$$ R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} $$

(3.44)

A rotation around the axis \( e'_1 \) with an angle \( \phi \) is called roll angle, which transforms the basis \( \{e''_1, e''_2, e''_3\} \) to \( \{e''''_1, e''''_2, e''''_3\} \). The corresponding rotation matrix is

$$ R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} $$

(3.45)
The total rotation $R_{\beta\gamma\delta}(\psi, \theta, \phi) = R_{\beta}(\phi)R_{\gamma}(\theta)R_{\delta}(\psi)$, which transforms $\{I\}$ (defined by $\{e^i_1, e^i_2, e^i_3\}$) to $\{B\}$ (defined by $\{e^b_1, e^b_2, e^b_3\}$) is given by
\[
R(\psi, \theta, \phi) = \begin{bmatrix}
\cos\theta \cos\psi & -\cos\theta \sin\psi & \sin\theta \\
\cos\psi \sin\theta \sin\psi + \cos\theta \sin\phi & \cos\psi \cos\theta \sin\psi - \sin\phi \sin\theta & \cos\phi \cos\theta \\
-\sin\psi \sin\theta & \cos\theta \sin\phi & \cos\phi \cos\theta
\end{bmatrix},
\]
(3.46)

where $c(\cdot) = \cos(\cdot)$ and $s(\cdot) = \sin(\cdot)$. The angles $(\phi, \theta, \psi)$ are called Euler angles (Figure 3.3).

Note that there exist other possible ways to define Euler angles, depending on the order of the successive rotations and the selected rotation axis, which gives in total 12 possibilities. Five interesting possibilities are illustrated in Table 3.2.

### 3.3.3.1 Euler Angles $(\phi, \theta, \psi)$ from Rotation Matrix $R$

Given a rotation matrix $R \in SO(3)$ defined by Equation 3.29 and using Equation 3.46 one can have

\[
\theta = \arcsin(r_{13}),
\]
(3.47)
\[
\phi = -\arctan2(r_{12}, r_{33}),
\]
(3.48)
\[
\psi = -\arctan2(r_{11}, r_{11}).
\]
(3.49)

where $\arctan2(y, x)$ computes $\tan^{-1}(y/x)$ but uses the sign of both $x$ and $y$ to determine the quadrant in which the resulting angle lies (for more details, see Appendix A of [13]).

### Table 3.2 Euler Angles Configurations

<table>
<thead>
<tr>
<th>Rotations Order</th>
<th>Resulting Rotation Matrix</th>
</tr>
</thead>
</table>
| $R_{\alpha\beta\gamma}(\phi, \theta, \psi)$ | $\begin{bmatrix}
\cos\phi \cos\theta & -\sin\phi \cos\theta \sin\psi & \cos\phi \sin\theta \\
\sin\phi \cos\theta \sin\psi + \cos\phi \sin\theta & \sin\phi \cos\phi \sin\theta - \cos\theta \sin\psi & \sin\phi \cos\psi \\
-\sin\theta \sin\psi & \cos\theta \sin\psi & \cos\theta
\end{bmatrix}$ |
| $R_{\alpha\beta\gamma}(\psi, \theta, \phi)$ | $\begin{bmatrix}
\cos\theta \cos\psi & -\cos\theta \sin\psi & \sin\theta \\
\cos\psi \sin\theta \sin\psi + \cos\theta \sin\phi & \cos\psi \cos\theta \sin\psi - \sin\phi \sin\theta & \cos\phi \cos\theta \\
-\sin\psi \sin\theta & \cos\theta \sin\phi & \cos\phi \cos\theta
\end{bmatrix}$ |
| $R_{\alpha\beta\gamma}(\psi, \theta, \phi)$ | $\begin{bmatrix}
\cos\theta \cos\psi & -\cos\theta \sin\psi & \sin\theta \\
-\sin\psi \sin\theta & \cos\theta \sin\phi & \cos\phi \cos\theta \\
\cos\phi \cos\theta \sin\psi - \sin\phi \sin\theta & \cos\phi \cos\theta \cos\theta - \sin\phi \sin\theta & \cos\phi \cos\theta
\end{bmatrix}$ |
| $R_{\alpha\beta\gamma}(\phi, \theta, \psi)$ | $\begin{bmatrix}
\cos\phi \cos\theta & -\sin\phi \cos\theta \sin\psi & \cos\phi \sin\theta \\
\sin\phi \cos\theta \sin\psi + \cos\phi \sin\theta & \sin\phi \cos\phi \sin\theta - \cos\theta \sin\psi & \sin\phi \cos\psi \\
-\sin\theta \sin\psi & \cos\theta \sin\psi & \cos\theta
\end{bmatrix}$ |
| $R_{\alpha\beta\gamma}(\phi, \theta, \psi)$ | $\begin{bmatrix}
\cos\phi \cos\theta & -\sin\phi \cos\theta \sin\psi & \cos\phi \sin\theta \\
-\sin\psi \sin\theta & \cos\theta \sin\phi & \cos\phi \cos\theta \\
\cos\phi \cos\theta \sin\psi - \sin\phi \sin\theta & \cos\phi \cos\theta \cos\theta - \sin\phi \sin\theta & \cos\phi \cos\theta
\end{bmatrix}$ |
3.4 ATTITUDE KINEMATICS AND DYNAMICS

3.4.1 ATTITUDE KINEMATICS

The attitude kinematics of a rigid body is based on the time derivative of a vector in a rotating coordinate system. Let us rewrite Equation 3.8 for a derivative of the principal axis \( e_i^b(t), e_j^b(t), e_k^b(t) \in \mathbb{R}^3 \) of \( \{B\} \) expressed in \( \{I\} \)

\[
\frac{d}{dt} \mathbf{e}_i^b(t) = S(\omega_i(t)) \mathbf{e}_i^b(t),
\]

(3.50)

\[
\frac{d}{dt} \mathbf{e}_j^b(t) = S(\omega_j(t)) \mathbf{e}_j^b(t),
\]

(3.51)

\[
\frac{d}{dt} \mathbf{e}_k^b(t) = S(\omega_k(t)) \mathbf{e}_k^b(t),
\]

(3.52)

where \( \omega_i \) is the vector of angular velocity of \( \{B\} \) expressed in \( \{I\} \) and \( S(\omega_i(t)) \) is the skew-symmetric matrix defined by Equation 3.10.

3.4.1.1 Attitude Kinematics on \( \text{SO}(3) \)

To get attitude kinematics using rotation matrix as parametrization of attitude it suffices to calculate the expression of the derivative of \( R \) using the elementary definition (Equation 3.1). Thus, the derivative of Equation 3.1 in view of Equations 3.50 through 3.52 gives

\[
\dot{R}(t) = \begin{bmatrix}
\dot{e}_i^b(t) & \dot{e}_j^b(t) & \dot{e}_k^b(t)
\end{bmatrix},
\]

\[
= S(\omega_i(t)) \begin{bmatrix}
e_i^b(t) & e_j^b(t) & e_k^b(t)
\end{bmatrix},
\]

(3.53)
using the fact that the corresponding angular velocity vector \( \omega_i(t) \) to \( \omega(t) \) (where \( \omega(t) \) is expressed in \( \mathcal{B} \) and \( \omega_i(t) \) is expressed in \( \mathcal{I} \)) can be written using Equation 3.2 as \( \omega_i(t) = R(t) \omega(t) \). Therefore, one can get from Equation 3.53

\[
\dot{R}(t) = R(t) S(\omega(t))
\] (3.54)

where property (Equation 3.18) was used.

### 3.4.1.2 Reduced Attitude Kinematics on \( \mathbb{S}^2 \)

Consider a vector \( \mathbf{b}_i(t) \in \mathbb{R}^3 \) expressed in \( \mathcal{B} \) and its corresponding vector \( \mathbf{r}_i \in \mathbb{R}^3 \) expressed in \( \mathcal{I} \), \( i = 1, \ldots, m \), then from Equation 3.2 one can write

\[
\mathbf{b}_i(t) = R(t) \mathbf{r}_i,
\] (3.55)

where \( R \) is the rotation matrix of the mobile frame \( \mathcal{B} \) relative to the inertial fixed frame \( \mathcal{I} \). Since \( R \) preserves distances, therefore if \( \mathbf{r}_i \) is a unit vector, then \( \mathbf{b}_i \) is also a unit vector. In this case, \( \mathbf{b}_i(t) \in \mathbb{S}^2 \) such that

\[
\mathbb{S}^2 = \{ \mathbf{x} \in \mathbb{R}^3 | \mathbf{x}^T \mathbf{x} = 1 \}
\] (3.56)

Using the fact that \( \mathbf{r}_i \) is constant, differentiating Equation 3.55 with respect to time in view of Equation 3.54 gives

\[
\dot{\mathbf{b}}_i(t) = -S(\omega(t)) \mathbf{b}_i(t),
\] (3.57)

It is clear from Equation 3.57 that the reduced attitude vector evolves in \( \mathbb{S}^2 \), which can be verified by the evaluation of the time derivative of \( \mathbf{b}_i^T(t) \mathbf{b}_i(t) \); see [3].

### 3.4.1.3 Attitude Kinematics on \( \mathbb{S}^3 \)

The evaluation of the equivalent attitude kinematics (Equation 3.54) using unit quaternions can be done by using the basic definition of a function derivative as described in page 71 of [9]. The unit quaternion kinematics is given by

\[
\dot{Q}(t) = \frac{1}{2} Q(t) \odot \ddot{\omega}(t) = \frac{1}{2} \left[ -Q^T(t) \omega(t) \right] \omega(t),
\] (3.58)

where \( \ddot{\omega} \) is the pure quaternion of \( \dot{\omega} \), such that \( \ddot{\omega} = (0, \omega) \). It is possible to write the quaternion kinematics (Equation 3.58) as a product of a matrix and the quaternion. Consider a matrix \( M(\omega(t)) \) defined by

\[
M(\omega(t)) = \begin{bmatrix} 0 & -\omega^T(t) \\ \omega(t) & -S(\omega(t)) \end{bmatrix},
\] (3.59)

then,

\[
\dot{Q}(t) = \frac{1}{2} M(\omega(t)) Q(t)
\] (3.60)
3.4.1.4 Euler Angles Kinematics on $\mathbb{R}^3$

Consider a rotation matrix $R(t) \in SO(3)$ defined by Equation 3.29 and using Equation 3.54, one can get

$$
\dot{R}(t) = \begin{bmatrix}
\dot{r}_{11} & \dot{r}_{12} & \dot{r}_{13} \\
\dot{r}_{21} & \dot{r}_{22} & \dot{r}_{23} \\
\dot{r}_{31} & \dot{r}_{32} & \dot{r}_{33}
\end{bmatrix}
$$

where $\mathbf{\omega}(t) = [\omega_x \quad \omega_y \quad \omega_z] \in \mathbb{R}^3$ is the angular velocity vector expressed in $\{B\}$.

Now, using Equations 3.46, 3.61, 3.47, 3.48, and 3.49, together with the fact that $(d/dt)(\arcsin(x)) = \left(\frac{x}{\sqrt{1-x^2}}\right)$ and $(d/dt)(\arctan(x)) = \left(\frac{1}{1+x^2}\right)$, and after some manipulations, one can have

$$
\dot{\theta}(t) = \cos(\phi)\omega_y - \sin(\phi)\omega_z,
$$

$$
\dot{\phi}(t) = \omega_x + \sin(\phi)\tan(\theta)\omega_y + \cos(\phi)\tan(\theta)\omega_z,
$$

$$
\dot{\psi}(t) = \frac{\sin(\phi)}{\cos(\theta)}\omega_y + \frac{\cos(\phi)}{\cos(\theta)}\omega_z,
$$

which means that the attitude kinematics using the minimal Euler angles parametrization is not always defined, as can be verified when $\theta = \pm \frac{\pi}{2} + 2k\pi$.

3.4.1.5 The Discrete-Time Unit Quaternion Propagation

For implementation purpose and to avoid the integration of the continuous quaternion kinematics (Equation 3.58 or 3.60), the discrete-time quaternion propagation [9] can be used, and it is given by

$$
Q_{t+1} = \exp\left(\frac{T_s}{2} M(\omega_k)\right)Q_t,
$$

where $M(\omega_k)$ is defined by Equation 3.59, $T_s$ is the sample time, and

$$
\exp\left(\frac{T_s}{2} M(\omega_k)\right) = \cos\left(\frac{T_s}{2} \|\omega_k\|\right) I_4 + \sin\left(\frac{T_s}{2} \|\omega_k\|\right) \frac{\omega_k}{\|\omega_k\|} M(\omega_k),
$$

where $I_4 \in \mathbb{R}^{4 \times 4}$ is the identity matrix.

3.4.2 Attitude Dynamics

Consider a rigid body moving in 3D space with orthonormal body-frame $\{B\}$ fixed to its $COM$ and denote by $\{I\}$ the inertial fixed reference. The angular momentum [9] can be expressed in $\{I\}$ as

$$
L(t) = R(t)J_a(t),
$$

(3.63)
where:

- \( R(t) \) is the matrix of rotation defined by Equation 3.1
- \( J \) is the moment of inertia or inertia matrix of the rigid body expressed in \( \mathcal{B} \)
- \( \omega(t) \) is the vector of angular velocity expressed in \( \mathcal{B} \)

Using the Newton’s law of motion, one can get

\[
\frac{d}{dt} (L(t)) = R(t)(\tau(t) + \tau_{\text{ext}}(t)),
\]

where \( \tau(t) \) is the torque generated by actuators and \( \tau_{\text{ext}}(t) \) all others external torques (including gyroscopic torque) applied about the CoM of the rigid body expressed in \( \mathcal{B} \).

Using Equations 3.63, 3.54, and 3.64, one can get

\[
\frac{d}{dt} (R(t)J(t)\omega(t)) = R(t)(\tau(t) + \tau_{\text{ext}}(t)),
\]

\[
\dot{R}(t)J(t)\omega(t) + R(t)J(t)\dot{\omega}(t) = R(t)(\tau(t) + \tau_{\text{ext}}(t)),
\]

\[
R(t)S(\omega(t))J(t)\omega(t) + R(t)J(t)\dot{\omega}(t) = R(t)(\tau(t) + \tau_{\text{ext}}(t)),
\]

\[
S(\omega(t))J(t)\omega(t) + J(t)\dot{\omega}(t) = \tau(t) + \tau_{\text{ext}}(t).
\]

Finally, the attitude dynamics can be expressed as

\[
J(t)\dot{\omega}(t) = -S(\omega(t))J(t)\omega(t) + \tau(t) + \tau_{\text{ext}}(t).
\]

### 3.5 ATTITUDE QUADROTOR MODELING

The quadrotor modeling is well studied in the literature [2,4,7,8,11] (Figure 3.4). In this section, a brief resume of the different quadrotor modeling levels are given, especially for attitude.

The quadrotor motion consists of translation and rotation. The rotation kinematics and dynamics were detailed in Section 3.4. Neglecting all external torques and using Equations 3.54 and 3.65, the general simplified attitude model of quadrotor considered as rigid body is given by

\[
\begin{align*}
\dot{R}(t) &= R(t)S(\omega(t)), \\
J(t)\dot{\omega}(t) &= -S(\omega(t))J(t)\omega(t) + \tau(t).
\end{align*}
\]

Consider a quadrotor equipped with four actuators with propellers, each propeller generates a thrust \( T_i \) and a torque \( \tau_i \) (Figure 3.5), and rotates at an angular velocity \( \omega_i \). Propellers attached to actuators \( M_1 \) and \( M_2 \) are rotating in counterclockwise direction, while those attached to actuators \( M_3 \) and \( M_4 \) are rotating in clockwise direction. The mobile reference attached to the quadrotor is chosen to be in \( X \) configuration. The total thrust \( T(t) \) is the sum of all thrusts generated by each propeller [8] and given by

\[
T(t) = |T_1(t)| + |T_2(t)| + |T_3(t)| + |T_4(t)| = c_T \sum_{i=1}^{4} \omega_i^2(t),
\]

where \( c_T > 0 \) is the thrust constant and the assumption of the proportionality of propeller thrust and the square of rotor angular velocity \( \omega_i \) is considered. The reaction torque generated by each propeller [8] can be modeled by

\[
\tau_i(t) = c_T \omega_i^2(t),
\]
where \( c \tau > 0 \) is the torque constant. More details about propeller aerodynamics can be found in [7,8]. Denote the distance

\[
d = \frac{\sqrt{2}}{2} l,
\]  

(3.69)

which represents the vertical distance from the principal axis \( e_1^b \) to the point of application of thrusts \( T_i \), where \( l \) is the length of the quadrotor legs. Using Equations 3.67 through 3.69, one can write the total thrust \( T(t) \) and torque \( \tau(t) = \begin{bmatrix} \tau_x & \tau_y & \tau_z \end{bmatrix}^T \) as
where the constant matrix $\mathcal{P}$ can be determined experimentally, as detailed in [5].

Usually, autopilot of quadrotor generates $PW M_i(t)$ signals to drive the motor electronic system control (ESC), which itself generates the desired voltage $u_i$ to drive the motor, depending on the duty cycle of the $PW M_i(t)$ signals. The relation between $u_i(t)$ and $PW M_i(t)$ can be given by [7]

$$u_i(t) = k \sqrt{PW M_i(t)},$$

where $k \in \mathbb{R}^+$. 

BIBLIOGRAPHY