Boolean Algebras

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Abstract
Boolean algebra, named after the 19th century mathematician and logician, George Boole, has contributed to many aspects of computer science and information science. In information science, Boolean logic forms the basis of most end-user search systems, from searches in online databases and catalogs, to uses of search engines in information seeking on the World Wide Web.

INTRODUCTION
In this entry, attention is focused on mathematical models of proven utility in the area of information handling, namely, Boolean algebras. Following some general comments concerning mathematical models, particular examples of Boolean algebras, serving as motivation for the subsequent axiomatization, are presented. Some elementary theorems are cited, particularly the very important representation theorem that justifies, in some sense, the focusing of attention on a particular Boolean algebra, namely, the algebra of classes, and applications more directly related to the information sciences are given.

Running the risk of redundancy, attention will be called to an often-repeated observation, but one of extreme importance in applications of mathematics to physical problems. Referring to Fig. 1, it is important to realize that when one constructs a mathematical model as a representation of a physical phenomenon, one is abstracting and, as a consequence, the model formulated is doomed to imperfection. That is, one can never formally mirror the physical phenomenon, and must always be satisfied with an imperfect copy. However, following the initial commitment to a model, the logic that one appeals to dictates the resultant theorems derived within the framework of the model. Of course, the depth of the theorems realized is limited by the sophistication of the model as well as the ingenuity of those who attempt to formulate the propositions within it. After theorems are derived within the framework of the model, they are interpreted relative to the physical situation that motivated the model.

It is not necessary to go very deeply into mathematics before facing the necessity of examining, in some detail, this cycle and developing a feeling for its power as well as its limitations. By way of example, almost any student of calculus encounters, in one form or another, the following problem:

The deceleration of a ship in still water is proportional to its velocity. If the velocity is \( v_0 \) feet per second at the time the power is shut off, show that the distance \( S \) the ship travels in the next \( t \) seconds is \( S = \left( \frac{v_0}{k} \right) \left[ 1 - e^{-kt} \right] \), where \( k \) is the constant of proportionality.

HISTORY
The desired equation relating the distance traveled to the time is easily arrived at by means of the calculus. However, a close look at the solution reveals a few puzzling aspects. When does the ship stop? The conclusion is that it never stops. How far does it go? The conclusion is that it goes no further than \( \frac{v_0}{k} \), that is, the distance it travels is bounded. Sympathy is due the beginning student of calculus who is puzzled by these observations, but, too often, we neglect to focus our attention on the source of the puzzlement. It really has nothing to do with the limit process that plays such an integral role in analysis, nor must we drag poor Zeno into the picture. This disturbing conclusion is not the consequence of any faulty mathematics, but is more directly related to the naiveté of the original model. If we say that the deceleration of a ship in still water (an idealization in itself) is proportional only to its velocity, then the conclusion that asserts itself is that the ship never stops but only goes a finite distance.

The usual remedy applied in such cases as the ship problem is to construct a more sophisticated model, that is, a model that takes into account more of the phenomena observed. For example, in the ship problem, the assertion that the deceleration is proportional only to the velocity might be amended to include friction in some way, resulting in an equation of greater complexity, the formulation and solution of which require a more general mathematical model. We might extend the preceding model to look like Fig. 2.
The great power of mathematics lies in its ability to reflect several different phenomena at one time, and the theorems derived within the framework of a single axiomatization of these varied phenomena will, in turn, be applicable to each of them. However, the trade-off that exists between generalization and depth must constantly be kept in mind. That is, it should be remembered that it is difficult to prove deep theorems in very general models. But when axioms are added to the model, the phenomena that the model reflects begin to be delimited, and certainly one does not wish to undermine the real power of mathematics, that is, its ability to treat a variety of situations at the same time.

**Examples**

We now turn our attention to an examination of some of the particular examples of the model that is the principal concern of this entry, Boolean algebras. One must keep in mind that the common characteristics of these models are precisely those that will constitute the elements of our later axiomatization. To avoid infinite regress a certain level of sophistication on the part of the reader, if not actual mathematical experience, is assumed.

**Example 1 (a finite algebra)**

The system considered in this example consists of the two digits, 0 and 1, and two binary operations of multiplication, “·” and addition, “+.” The operations are defined by the multiplication and addition tables shown.

If \( x, y, \) and \( z \) are any variables that are allowed to assume one of the two values 0 or 1, then the structure defined earlier has an algebra possessing (among others) the following properties:

\[
\begin{array}{c|c|c}
0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]

**Scheme 1** Multiplication and addition tables.

The elements in this example are propositions, that is, statements to which it is possible to assign one of the truth values “true” or “false.” Two propositions \( p \) and \( q \) are defined to be equal if and only if they have the same truth value. We consider the two logical binary operations of conjunction and disjunction as well as the unary (operating on a single proposition as contrasted with a binary operation, which operates on pairs of propositions) operation of negation. The conjunction of the propositions \( p \) and \( q \) is denoted by \( pq \) and is the proposition corresponding to that obtained by applying the logical connective “and.” The conjunction is defined to be true only if both \( p \) and \( q \) are true. Otherwise it is false. The disjunction of \( p \) and \( q \), denoted by \( p + q \), is the proposition corresponding to that obtained by applying the logical connective “or.” The proposition \( p + q \) is false if and only if both \( p \) and \( q \) are false. The negation of \( p \), denoted by \( \bar{p} \), is the proposition having truth values opposite those of \( p \). It corresponds to the logical statement, “It is false that \( p \).”

\[
\begin{array}{c|c|c}
0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

**Example 2 (algebra of propositions)**

Each of the aforementioned can be verified by a consideration of all the possible values of the variables.

If \( B \) is a collection consisting of the elements 0 and 1, and for each \( x \) in \( B \) is defined by \( x' = 1 \) if \( x = 0 \), and \( x' = 0 \) if \( x = 1 \), then

\[
\begin{align*}
x' \cdot x &= 0 & x + x' &= 1 \\
0 \cdot x &= 0 & 1 \cdot x &= x & 0 + x &= x & 1 + x &= x
\end{align*}
\]
All of the aforementioned can be summarized well by employing “truth tables” that give the truth values of compound statements, realized by applying the operations discussed to the truth values of the component propositions.

One can verify that, in view of the definitions given, if \( p, q, \) and \( r \) are any propositions, the following statements hold:

\[
\begin{align*}
pp &= p & p + p &= p \\
pq &= qp & p + q &= q + p \\
p(qr) &= (pq)r & p + (q + r) &= (p + q) + r \\
(qp) + p &= p & (p + q)p &= p \\
p(q + r) &= pq + pr & p + (qr) &= (p + q)(q + r) \\
\end{align*}
\]

The structure described in Example 2 is denoted in the sequel by \([\mathcal{G}, +, -]\).

**Example 3 (algebra of sets)**

The term set is taken as undefined and used synonymously with class, aggregate, and collection. The objects that constitute a set \( E \) are called the elements of \( E \). To denote the logical relation of “being an element of \( E \)” we use the notation \( x \in E \). This is read: “\( x \) is an element of \( E \)”.

The denial of this relation is symbolized by \( x \notin E \).

The notation of \( E = \{x \in P(x)\} \) denotes the set \( E \) consisting of all \( x \) for which the proposition \( P(x) \) is true. The notation of \( E = \{x \in P(x)\} \) and there are no elements that satisfy the proposition \( P(x) \). \( E \) is said to be the empty set. The empty (or null) set is denoted by \( \emptyset \).

If the sets \( E \) and \( F \) have the property that every element of \( E \) is an element of \( F \), \( E \) is called a subset of \( F \); this is denoted by \( E \subseteq F \). If the set \( E \) is a subset of \( F \), but \( F \) is not a subset of \( E \), then \( E \) is said to be a proper subset of \( F \), or \( F \) properly contains \( E \). The empty set \( \emptyset \) is a subset of every set.

Two sets \( E \) and \( F \) are equal, written \( E = F \), if and only if \( E \subseteq F \) and \( F \subseteq E \).

Given two sets \( E \) and \( F \), we define the union, denoted by \( E \cup F \), by the set equation

\[
E \cup F = \{x | x \in E \lor x \in F\}
\]

Similarly, the intersection of \( E \) and \( F \), denoted by \( E \cap F \), is defined by

\[
E \cap F = \{x | x \in E \land x \in F\}
\]

In general, consideration centers on subsets of a fixed set often referred to as the universal set. In particular, if \( X \) is the universal set, we let \( \mathcal{G}(X) \) denote the set of all subsets of \( X \). The set \( \mathcal{G}(X) \) is often called the power set of \( X \). If \( E \in \mathcal{G}(X) \), then the complement of \( E \), denoted by \( E' \), is defined as \( E' = \{x | x \in X \land x \notin E\} \).

If \( E \) and \( F \) are elements of \( \mathcal{G}(X) \), that is, subsets of \( X \), then the difference of the sets \( E \) and \( F \), denoted by \( E - F \), is the set defined by

\[
E - F = \{x | x \in E \land x \notin F\}
\]

It should be noted that \( E - F = E \cap F' \).

It is often helpful to employ the schematics shown in Fig. 3 in visualizing the set-theoretic relations and operations defined earlier. The rectangular area represents the universal set \( X \); subsets of \( X \) are denoted by areas within the rectangle.
Focusing our attention on a particular universal set \( X \) and its power set \( \mathcal{P}(X) \), it is easy to verify the following properties (in no sense exhaustive) of the algebra of sets, where \( E, F, \) and \( G \) are arbitrary subsets of \( X \).

\[
\begin{align*}
E \cap F &= E \quad E \cup F = F \cup E \\
E \cap (F \cap G) &= (E \cap F) \cap G \\
E \cup (F \cup G) &= (E \cup F) \cup G \\
E \cap (F \cup G) &= (E \cap F) \cup (E \cap G) \\
E \cup (F \cap G) &= (E \cup F) \cap (E \cup G) \\
E \cap E &= \phi \quad E \cup E = X \\
\phi \cap E &= \phi \quad X \cap E = E \\
\phi \cup E &= E \quad X \cup E = E
\end{align*}
\]

In the sequel we denote the preceding algebra of sets by \( \mathcal{P}(X); \cap, \cup, ^\prime \).

**Axiomatization**

In the previous section we examined three structures possessing some common properties, namely, \((1i), (2i), \) and \((3i)\), where \( i = 1, 2, 3, 4, 5, 6 \). We abstract to construct the important mathematical (see Fig. 1) model called a **Boolean algebra**, named in honor of G. Boole who first studied it in 1847.\(^{[1,2]}\)

**Table 1** Boolean algebra and examples

<table>
<thead>
<tr>
<th>Boolean algebra</th>
<th>Example 1</th>
<th>Example 2</th>
<th>Example 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B ) Set ( {0, 1} )</td>
<td>Set of all propositions</td>
<td>( \mathcal{P}(X) ) of all subsets of a fixed set ( X )</td>
<td></td>
</tr>
<tr>
<td>( \land ) Conjunction</td>
<td>Intersection</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \lor ) Disjunction</td>
<td>Union</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \neg ) Negation</td>
<td>Complementation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 ( \neg \neg ) Empty set ( \phi )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 ( \lor \neg ) Universal set ( X )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**A Boolean algebra** is a set \( B \) with two binary operations \( \land \) (cap) and \( \lor \) (cup) and a unary operation (complementation) satisfying the following axioms:

\[
\begin{align*}
x \land x &= x \quad x \lor x &= x & (4a) \\
x \land (y \land z) &= (x \land y) \land z \quad x \lor (y \lor z) &= (x \lor y) \lor z & (4b) \\
x \land (y \lor z) &= (x \land y) \lor (x \land z) \quad x \lor (y \land z) &= (x \lor y) \land (x \lor z) & (4c)
\end{align*}
\]

\( B \) contains distinct elements 0 and 1 such that

\[
\begin{align*}
x \land x' &= 0 \quad x \lor x' &= 1 \\
0 \land x &= 0 \quad 1 \land x &= x \quad 0 \lor x &= x \quad 1 \lor x &= 1
\end{align*}
\]

This is by no means the only axiomatization possible,\(^{[3]}\) but it is probably the one that is most commonly used.

To emphasize the relationship between the aforementioned axiomatization and the preceding particularizations, (Examples 1, 2, and 3) we identify in a tabular form the corresponding structural elements (Table 1).

We now prove a particular theorem to illustrate the generation of results within the framework of the model and their subsequent application.

**Theorem.** If \( (B; \land, \lor, ^\prime) \) is a Boolean algebra, then for any \( x \) and \( y \) in \( B \) we have

\[
\begin{align*}
x'' &= x & (i) \\
(x \land y)' &= x' \lor y' & (ii)
\end{align*}
\]

**Proof.** First of all we prove that every element has only one complement. Suppose \( x \) is an element such that \( x \land \overline{x} = 0 \) and \( x \lor \overline{x} = 0 \). Then

\[
\begin{align*}
\overline{x} &= \overline{x} \land 1 = \overline{x} \land (x \lor x') = (\overline{x} \land x) \lor (\overline{x} \land x') \\
&= 0 \lor (\overline{x} \land x') = \overline{x} \land x'
\end{align*}
\]

but

\[
\begin{align*}
x' &= x' \land 1 = x' \land (x \lor \overline{x}) = (x' \land x) \lor (x' \land \overline{x}) \\
&= 0 \lor (x' \land \overline{x}) = (x' \land \overline{x})
\end{align*}
\]

and since \( \overline{x} \land x' = x' \land \overline{x} \), we have \( \overline{x} \land x' \). We then apply the preceding by demonstrating that \((x \land y) \land (x' \lor y') = 0 \) and \((x \land y) \land (x' \lor y') = 1 \), so that \((x \land y)' = x' \lor y'\).
Fig. 4 Complement of intersection of two sets equals union of their complements.

\[(x \land y) \land (x' \lor y') = ([x \land y] \land x') \lor [(x \land y) \land y']
\]

\[= [(y \land x) \land x'] \lor [(x \land y) \land y']
\]

\[= [y \land (x \land x')] \lor [x \land (y \land y')]
\]

\[= [y \land 0] \lor [x \land 0] = 0 \lor 0 = 0
\]

\[(x \land y) \lor (x' \lor y') = [(x \land y) \lor x'] \lor y'
\]

\[= ([x \lor x'] \land (y \lor x')) \lor y' = [1 \land (y \lor x')] \lor y'
\]

\[= (y \lor x') \lor y' = (x' \lor y') \lor y'
\]

\[= x' \lor (y \lor y') = x' \lor 1 = 1
\]

An interpretation (application) of this theorem in Example 2 yields the fact that the negation of the conjunction of two propositions is the disjunction of the negations of each of them. For example, “it is false that \(x\) is a positive integer and \(x\) is greater than or equal to 5” is logically equivalent to the proposition “\(x\) is not a positive integer or \(x\) is less than 5.”

An interpretation of the above in Example 2 yields the set-theoretic equation

\[(E \land F)' = E' \cup F'
\]

that is, the complement of the intersection of two sets is equal to the union of their complements, as shown in Fig. 4.

We describe very briefly some of the more significant results and developments in the theory of Boolean algebras. For a comprehensive treatment of the subject, see Birkhoff and Halmos and their bibliographies.

Every Boolean algebra can be made into a ring with identity in which every element is multiplicative idempotent; that is, \(x^2 = x\) for every \(x\). This is accomplished by defining addition and multiplication as follows:

\[x + y = (x \land y') \lor (x' \land y)\]

\[xy = x \land y\]

Because rings are more familiar and more carefully studied, many of the useful concepts can be translated into the context of Boolean algebras.

Conversely, if one starts with a ring with identity in which every element is idempotent (usually called a Boolean ring), defining \(\land\) and \(\lor\) by

\[x \land y = xy\]

\[x \lor y = x + y + xy\]

the Boolean ring is converted into a Boolean algebra.

Two Boolean algebras, \(B_1\) and \(B_2\), are said to be isomorphic if there exists a function \(h: B_1 \rightarrow B_2\) that maps \(B_1\) onto \(B_2\) in such a way that distinct elements of \(B_1\) are mapped onto distinct elements of \(B_2\), and \(h\) preserves the operations; that is,

\[h(x \land y) = h(x) \land h(y); \quad h(x \lor y) = h(x) \lor h(y); \quad \text{and} \quad h(x') = h(x)'.\]

If \(X\) is a compact Hausdorff space, then the class of sets that are both open and closed forms a Boolean algebra. A topological space is totally disconnected if the only components (maximal connected sets) are points. There is a very important representation theorem in the theory of Boolean algebras, the Stone Representation Theorem (M. H. Stone[5]). If \(B\) is a Boolean algebra, then a compact totally disconnected Hausdorff space \(S\) exists such that \(B\) is isomorphic to the Boolean algebra of all open–closed subsets of \(S\).

An Application

We consider here one modest application of the preceding to switching theory. Switching theory is concerned with circuits composed of elements that can assume a finite number of discrete states, most commonly two states. These circuits are modeled as described earlier, and the models are analyzed. This is an idealization; the models neglect such characteristics as stability, temperature effects, and transition times. The theory of Boolean algebras has played an important role in the analysis of these models for circuits made of binary (two-state) devices.

A switching function is a rule by which the output of a composite circuit can be ascertained from the states of its components. If the variables \(x\), \(y\), and \(z\) denote switches and each switch can assume one of the states, open or closed (0 or 1), then the function \(w = x \land y\) describes the output of a series circuit containing the switches \(x\) and \(y\). Similarly, \(t = x \lor y\) is a function describing a parallel circuit containing the switches \(x\) and \(y\). These components, along with the negation function \((x')\) which is open whenever \(x\) is closed and closed whenever \(x\) is open), allow the construction and analysis of complex circuits. This analysis can be carried out by the use of truth tables, and the circuits can be indicated by a diagram, as shown in Fig. 5.

With this interpretation it is readily seen that the above is a Boolean algebra. For example, the verification of axiom (Id) involves the observation that the circuits in Fig. 6 are equivalent.
After observing that it is indeed a Boolean algebra, the machinery of that algebra may be used to synthesize circuits, consider questions of realizability, minimize circuitry, and so forth. We can only hint at the possible applications.\cite{6}

**REFERENCES**

1. Boole, G. *The Mathematical Analysis of Logic*; Cambridge, 1847.